lunedì 12 ottobre 2020 11:06

$f \in C^2(A)$, A aperto di $\mathbb{R}^n$ $\qquad \frac{\partial^2 f}{\partial x_i \partial x_j}(\underline{x}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\underline{x}) \quad \forall i,j = 1 \dots n$

$H_f(\underline{x}) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\underline{x}) \right)_{i,j=1 \dots n}$ è simmetrica

$$f(\underline{x}_0 + \underline{h}) = f(\underline{x}_0) + \nabla f(\underline{x}_0) \cdot \underline{h} + \frac{1}{2} \underline{h}^t H_f(\underline{x}_0) \underline{h} + o(\|\underline{h}\|^2)$$

Teo di Fermat: Se $\underline{x}_0$ è un pto di estremo libero per f e f è derivabile in $\underline{x}_0 \Rightarrow \underline{\nabla f(\underline{x}_0) = 0}$

Se $x_0 \in A$ è stazionario / e $f \in C^2(A)$ $\Rightarrow$ $f(\underline{x}_0 + \underline{h}) = f(\underline{x}_0) + \frac{1}{2} \underline{h}^t H_f(\underline{x}_0) + o(\|\underline{h}\|^2)$

M matrice $n \times n$ simmetrica $\quad q: \underline{h} \in \mathbb{R}^n \to \underline{h}^t M \underline{h} \in \mathbb{R}$

TEOREMA (no dim) Sia $M \in \mathbb{R}^{n \times n}$ matrice simmetrica e sia $q(\underline{h}) = \underline{h}^t M \underline{h}$

Indico con M_k la sottomatrice di M data dall'intersezione tra le prime k righe e le prime k colonne $\quad M_k = (m_{ij})_{i,j=1,\dots k}$

Allora 1) q è una forma quadratica definita positiva SSE $\det M_k > 0 \quad \forall k = 1 \dots n$

2) q è una forma quadratica definita negativa SSE $(-1)^k \det M_k > 0 \quad \forall k = 1 \dots n$

$M \in \mathbb{R}^{n \times n}$ matrice simmetrica $\Rightarrow \exists S \in \mathbb{R}^{n \times n}$ t.c. $S^t = S^{-1}$ (MATRICE ORTOGONALE)

t.c. $S^t M S = \mathrm{diag}(\lambda_1 \dots \lambda_n)$ dove $\lambda_1 \dots \lambda_n \in \mathbb{R}$ sono gli autovalori di M

Pongo $\Lambda := \mathrm{diag}(\lambda_1 \dots \lambda_n)$

$S^t M S = \Lambda \qquad \underbrace{S S^t}_{} M \underbrace{S S^t}_{} = S \Lambda S^t$

$\Rightarrow M = S \Lambda S^t$

$$\begin{aligned} q(\underline{h}) &= \underline{h}^t M \underline{h} = \\ &= \underline{h}^t S \Lambda S^t \underline{h} = \\ &= \underline{h}^t (S^t)^t \Lambda S^t \underline{h} = \\ &= (S^t \underline{h})^t \Lambda (S^t \underline{h}) \\ &= \underline{k}^t \Lambda \underline{k} = \\ &= (k_1 \dots k_n) \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} = \\ &= (k_1 \dots k_n) \begin{pmatrix} \lambda_1 k_1 \\ \vdots \\ \lambda_n k_n \end{pmatrix} = \sum_{i=1}^n \lambda_i k_i^2 \end{aligned}$$

$S = (S^t)^t$

$\underline{h}^t (S^t)^t = (S^t \underline{h})^t$

$\underline{k} := S^t \underline{h}$

S ortogonale $\Rightarrow \|\underline{k}\| = \|\underline{h}\|$

$$q(\underline{h}) = \sum_{i=1}^n \lambda_i k_i^2$$ con $\lambda_1 \dots \lambda_n$ autovalori di M e $\underline{k} = S^t \underline{h}$

$\Rightarrow$ la forma q è definita positiva SSE $\lambda_i > 0 \quad \forall i = 1 \ldots n$

semidefinita positiva SSE $\lambda_i \geq 0 \quad \forall i = 1 \ldots n$

definita negativa SSE $\lambda_i < 0 \quad \forall i = 1 \ldots n$

semidefinita negativa SSE $\lambda_i \leq 0 \quad \forall i = 1 \ldots n$

indefinita SSE $\exists \lambda_i < 0$ ed $\exists \lambda_j > 0$

$\lambda_{min} := \min\{\lambda_1 \ldots \lambda_n\} \qquad \lambda_{MAX} := \max\{\lambda_1 \ldots \lambda_n\}$

$$\underset{\lambda_{min} \|\underline{h}\|^2}{\overset{\|}{\lambda_{min}\|\underline{K}\|^2}} = \sum_{i=1}^{n} \lambda_{min} K_i^2 \leq q(\underline{h}) = \sum_{i=1}^{n} \lambda_i K_i^2 \leq \sum_{i=1}^{n} \lambda_{MAX} K_i^2 = \lambda_{MAX}\|\underline{K}\|^2 = \lambda_{MAX}\|\underline{h}\|^2$$

cioè $\forall \underline{h} \in \mathbb{R}^n \qquad \lambda_{min}\|\underline{h}\|^2 \leq q(\underline{h}) \leq \lambda_{MAX}\|\underline{h}\|^2$

TEOREMA Sia $A \subseteq \mathbb{R}^n$ aperto, $f: A \to \mathbb{R}$, $f \in C^2(A)$ e sia $\underline{x}_0 \in A$ pto critico. Sia $f(\underline{x}_0 + \underline{h}) = f(\underline{x}_0) + \frac{1}{2} \underline{h}^t H_f(\underline{x}_0)\underline{h} + o(\|\underline{h}\|^2)$ lo sviluppo di Taylor al 2° ordine con resto di Peano nel pto e sia $q(\underline{h}) := \underline{h}^t H_f(\underline{x}_0)\underline{h}$.

Allora
1. Se q è definita positiva $\Rightarrow$ $\underline{x}_0$ è un pto di minimo locale stretto (*)
2. Se q è definita negativa $\Rightarrow$ $\underline{x}_0$ è un pto di massimo locale stretto
3. Se q è indefinita $\Rightarrow$ $\underline{x}_0$ è un pto di sella

(*) $\exists r > 0$ t.c. $f(\underline{x}) \geq f(\underline{x}_0) \quad \forall \underline{x} \in B_r(\underline{x}_0)$
$f(\underline{x}) > f(\underline{x}_0) \quad \forall \underline{x} \in B_r(\underline{x}_0) \setminus \{\underline{x}_0\}$

DIM (1) q definita positiva. So che
$\lambda_{min}\|\underline{h}\|^2 \leq q(\underline{h}) \leq \lambda_{MAX}\|\underline{h}\|^2$
con λ_{min} e λ_{MAX} autovalori minimo e massimo di $H_f(\underline{x}_0)$

$$f(\underline{x}_0 + \underline{h}) = f(\underline{x}_0) + \frac{1}{2} q(\underline{h}) + o(\|\underline{h}\|^2) \geq$$

$$\geq f(\underline{x}_0) + \frac{1}{2}\lambda_{min}\|\underline{h}\|^2 + o(\|\underline{h}\|^2)$$

$$= f(\underline{x}_0) + \frac{1}{2}\lambda_{min}\|\underline{h}\|^2 + \varepsilon(\underline{h})\|\underline{h}\|^2 \qquad \lim_{\underline{h} \to \underline{0}} \varepsilon(\underline{h}) = 0$$

$\forall \theta > 0 \quad \exists \delta > 0$ t.c. $\forall \underline{h} \in B_\delta(\underline{0}) \qquad |\varepsilon(\underline{h})| < \theta$

Scelgo $\theta = \frac{1}{4}\lambda_{min}$: $\exists \delta > 0$ t.c. $\forall \underline{h} \in B_\delta(\underline{0}) \quad -\frac{1}{4}\lambda_{min} < \varepsilon(\underline{h}) < \frac{1}{4}\lambda_{min}$

$\exists \delta > 0$ t.c. $\forall \underline{h} \in B_\delta(\underline{0}) \quad f(\underline{x}_0 + \underline{h}) > f(\underline{x}_0) + \frac{1}{2}\lambda_{min}\|\underline{h}\|^2 - \frac{1}{4}\lambda_{min}\|\underline{h}\|^2$

$$f(\underline{x}_0 + \underline{h}) > f(\underline{x}_0) + \frac{1}{4}\lambda_{min}\|\underline{h}\|^2 > f(\underline{x}_0) \text{ se } \underline{h} \neq \underline{0}$$

$$f(x_0+h) > f(x_0) + \frac{1}{4}\lambda_{min}\|h\|^2 > f(x_0) \quad \text{se } h \neq 0$$

cioè se $x_0+h \neq x_0$

(2) Supponiamo q sia una forma quadratica definita negativa

$$f(x_0+h) = f(x_0) + \frac{1}{2} q(h) + o(\|h\|^2)$$

$$= f(x_0) + \underbrace{\frac{1}{2} q(h)} + \|h\|^2 \varepsilon(h) \qquad \lim_{h\to 0} \varepsilon(h) = 0$$

$\forall \theta > 0 \ \exists \delta > 0$ t.c. $\forall h \in B_\delta(0)$ $\quad |\varepsilon(h)| < \theta$

$$f(x_0+h) \leq f(x_0) + \frac{1}{2}\lambda_{MAX}\|h\|^2 + \|h\|^2 \varepsilon(h) \qquad \varepsilon(h) < -\frac{1}{4}\lambda_{MAX}$$

Scelgo $\theta = -\frac{1}{4}\lambda_{MAX}$ perché q definita negativa $\Rightarrow \lambda_{MAX} < 0$

$\exists \delta > 0$ t.c. $\forall h \in B_\delta(0)$ $\quad f(x_0+h) \leq f(x_0) + \frac{1}{2}\lambda_{MAX}\|h\|^2 - \frac{1}{4}\lambda_{MAX}\|h\|^2$

$$\leq f(x_0) + \frac{1}{4}\lambda_{MAX}\|h\|^2 \qquad \text{se } h \neq 0$$

$$< f(x_0)$$

(3) Suppongo che q sia indefinita $\Rightarrow \lambda_{min} < 0 < \lambda_{MAX}$

Sia v autovettore relativo a λ_{min} $\quad \|v\| = 1$ $\qquad v^t H_f(x_0) v = v^t \cdot \lambda_{min} v$

$$= \lambda_{min} \underbrace{v^t \cdot v}_{\|v\|^2 = 1}$$

$$f(x_0 + tv) = f(x_0) + \frac{1}{2}\underbrace{(tv)^t H_f(x_0)(tv)} + o(\|tv\|^2) =$$

$$(tv)^t H_f(x_0)(tv) = t^2 \ v^t H_f(x_0) v \qquad H_f(x_0) v = \lambda_{min} v$$

$$= t^2 \ v^t \cdot \lambda_{min} v =$$

$$= t^2 \lambda_{min} \ v^t \cdot v = t^2 \lambda_{min} \|v\|^2 = t^2 \lambda_{min}$$

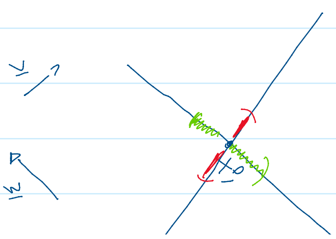
$$f(x_0 + tv) = f(x_0) + \frac{t^2}{2}\lambda_{min} + o(t^2)$$

$$= f(x_0) + \frac{t^2}{2}\lambda_{min} + \varepsilon(t) t^2 \qquad \lim_{t \to 0} \varepsilon(t) = 0$$

$\forall \theta > 0 \ \exists \delta > 0$ t.c. $|t| < \delta \Rightarrow |\varepsilon(t)| < \theta$

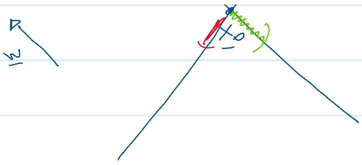
$\theta = -\frac{1}{4}\lambda_{min}$ $\qquad \exists \delta > 0$ t.c. $|t| < \delta \Rightarrow \frac{1}{4}\lambda_{min} < \varepsilon(t) < \frac{1}{4}\lambda_{min}$

$$f(x_0 + tv) < f(x_0) + \frac{t^2}{2}\lambda_{min} + \frac{1}{4}\lambda_{min} t^2 = f(x_0) + \frac{\lambda_{min}}{4} t^2 < f(x_0)$$



$f(x_0 + tv) < f(x_0)$ $\quad \forall t \in (-\delta, \delta)$

$f(x_0) < f(x_0 + tw)$



$f(\underline{x}_0) < f(\underline{x}_0 + t\underline{w})$

Sia $\underline{w}$ autovettore relativo a λ_{MAX} $\quad \|\underline{w}\| = 1$

$$f(\underline{x}_0 + t\underline{w}) = f(\underline{x}_0) + \frac{1}{2}(t\underline{w})^t H_f(\underline{x}_0)(t\underline{w}) + o(\|t\underline{w}\|^2) =$$
$$= f(\underline{x}_0) + \frac{1}{2}t^2\, \underline{w}^t H_f(\underline{x}_0)\underline{w} + o(t^2)$$

$$\underline{w}^t H_f(\underline{x}_0)\underline{w} = \underline{w}^t \lambda_{MAX}\underline{w} = \lambda_{MAX}\, \underline{w}^t \cdot \underline{w} = \lambda_{MAX}\|\underline{w}\|^2 = \lambda_{MAX}$$

$$f(\underline{x}_0 + t\underline{w}) = f(\underline{x}_0) + \frac{1}{2}\lambda_{MAX} t^2 + t^2\varepsilon(t) \qquad \lim_{t\to 0}\varepsilon(t) = 0$$

$\forall \theta > 0 \;\; \exists \delta > 0$ t.c. $|t| < \delta \qquad |\varepsilon(t)| < \theta \qquad -\theta < \varepsilon(t) < \theta$

Scelgo $\theta = \frac{1}{4}\lambda_{MAX}$

$$f(\underline{x}_0 + t\underline{w}) \geq f(\underline{x}_0) + \frac{1}{2}\lambda_{MAX}t^2 + t^2\,\frac{-\lambda_{MAX}}{4} =$$
$$= f(\underline{x}_0) + \frac{1}{4}\lambda_{MAX}t^2 > f(\underline{x}_0) \quad \text{per } t \neq 0$$

OSS $\quad f(x,y) = x^2 + y^4 \qquad \begin{cases} f_x = 2x \\ f_y = 4y^3 \end{cases} \qquad \nabla f(x,y) = (0,0)$ sse $(x,y) = (0,0)$

$f_{xx} = 2 \qquad f_{xy} = f_{yx} = 0 \qquad f_{yy} = 12y^2$

$$H_f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \qquad \lambda_{min} = 0 \qquad \lambda_{MAX} = 2$$

$f(0,0) = 0 \qquad f(x,y) > 0$ se $(x,y) \neq (0,0)$

$\Rightarrow (0,0)$ è un pto di minimo locale (assoluto) stretto

$f(x,y) = x^2 - y^4 \qquad \begin{cases} f_x = 2x \\ f_y = -4y^3 \end{cases} \qquad \nabla f(x,y) = (0,0)$ sse $(x,y) = (0,0)$

$f_{xx} = 2 \qquad f_{xy} = f_{yx} = 0 \qquad f_{yy} = -12y^2$

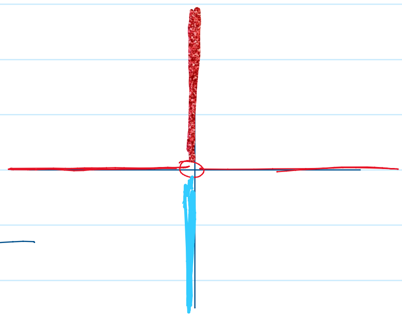
$$H_f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$f(0,0) = 0$

$x \neq 0 \quad f(x,0) = x^2 > 0$

$y \neq 0 \quad f(0,y) = -y^4 < 0$

SELLA



$y \neq 0 \quad f(0,y) = -y^4 < 0$

CASO $n=2$ $\quad q(\underline{h}) = \underline{h}^t M \underline{h}$ forma quadratica $\quad M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

1) definita positiva SSE $\det M > 0,\ a > 0$

$$\det H_f(x_0,y_0) > 0 \qquad f_{xx}(x_0,y_0) > 0$$

e ANALOGHI

$$\det\left(H_f(x_0,y_0) - \lambda Id_2\right) = \det\begin{pmatrix} f_{xx}(x_0,y_0)-\lambda & f_{xy}(x_0,y_0) \\ f_{xy}(x_0,y_0) & f_{yy}(x_0,y_0)-\lambda \end{pmatrix} =$$

$$= (\lambda - f_{xx})(\lambda - f_{yy}) - f_{xy}^2 = \lambda^2 - \lambda\underbrace{(f_{xx}+f_{yy})}_{tr\, H_f(x_0,y_0)} + \underbrace{f_{xx}f_{yy} - f_{xy}^2}_{\det H_f(x_0,y_0)}$$

$$= \lambda^2 - \lambda\, tr\, H_f(x_0,y_0) + \det H_f(x_0,y_0)$$

$$\Rightarrow (\lambda - \lambda_1)(\lambda - \lambda_2) = (\lambda - \lambda_{min})(\lambda - \lambda_{MAX}) =$$

$$= \lambda^2 - \lambda(\lambda_{min} + \lambda_{MAX}) + \lambda_{min}\lambda_{MAX}$$

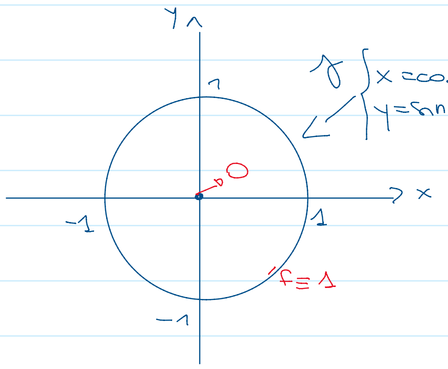
$$\begin{cases} \lambda_{min} + \lambda_{MAX} = tr\, H_f(x_0,y_0) \\ \lambda_{min}\lambda_{MAX} = \det H_f(x_0,y_0) \end{cases} \qquad \begin{matrix} 0 < \lambda_{min} < \lambda_{MAX} \\ \lambda_{min} < \lambda_{MAX} < 0 \end{matrix}$$

TEOREMA DI WEIERSTRASS

$E \subset \mathbb{R}^n$ chiuso e limitato, $f \in C^0(E) \Rightarrow f$ ammette MAX e min assoluti

1) cerco i pti critici di f in $int(E)$
2) studio il valore di f nei pti $\underline{x} \in int(E)$ e in cui f non è derivabile
3) Studio f su ∂E

ESEMPIO $\quad D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\} \qquad f(x,y) = \sqrt{x^2+y^2} = (x^2+y^2)^{1/2}$



$\gamma \begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases} \quad t \in [0, 2\pi)$

$(x,y) \in int(D)$ t.c. f non è derivabile in (x,y)

$$f_x(x,y) = \frac{1}{2}(x^2+y^2)^{-1/2}\, 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

$(x,y) \in B_1(0,0) \setminus \{(0,0)\} \Rightarrow f$ è derivabile in (x,y) e

$(x,y)\in B_1(0,0)\setminus\{(0,0)\} \Rightarrow f$ è derivabile in (x,y) e
$\nabla f(x,y)\neq(0,0)$

$\Rightarrow \nexists$ pti critici interni al dominio D

Calcolo $f(0,0)=0$

$$\gamma\begin{cases} x=\cos(t) \\ y=\sin(t)\end{cases} \qquad t\in[0,2\pi]$$

$$f(\gamma(t)) = f(\cos(t),\sin(t)) = \sqrt{\cos^2(t)+\sin^2(t)} = 1$$

$\Rightarrow$ 0 è il minimo di f e $(0,0)$ è l'unico pto di minimo

1 è il massimo di f e la circonferenza $x^2+y^2=1$ è l'insieme dei pti di massimo

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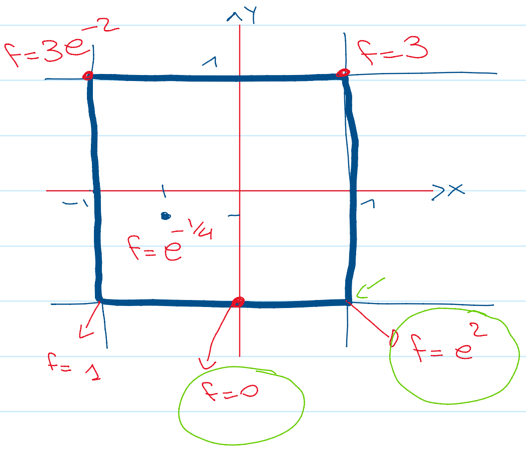
Esercizio Determinare estremi assoluti e natura degli eventuali pti critici

$f(x,y)=(1+x^2+y)\exp(x-y)$ in $E=\{(x,y)\in\mathbb{R}^2, : |x|\le 1, |y|\le 1\}$



$-1\le x\le 1 \;\wedge\; -1\le y\le 1$

$$f_x = 2x\exp(x-y)+(1+x^2+y)\exp(x-y)\cdot 1 = $$
$$= (\underbrace{1+2x+x^2}_{(1+x)^2}+y)\exp(x-y)$$

$$f_y = 1\exp(x-y)+(1+x^2+y)\exp(x-y)(-1)$$
$$=(-x^2-y)\exp(x-y)$$
$$=-(x^2+y)\exp(x-y)$$

Il minimo assoluto è 0 e l'unico pto di minimo assoluto è $(0,-1)$

Il massimo assoluto è e^2 e l'unico pto di max assoluto è $(1,-1)$

$$\begin{cases} -1<x<1 \\ -1<y<1 \\ ((1+x)^2+y)\exp(x-y)=0 \\ -(x^2+y)\exp(x-y)=0\end{cases}$$

$$\begin{cases} -1<x<1 \\ -1<y<1 \\ (1+x)^2+y=0 \\ x^2+y=0\end{cases} \qquad \begin{cases} -1<x<1 \\ -1<y<1 \\ x^2+y=0 \\ (1+x)^2=x^2 \longrightarrow \end{cases}$$

IMPOSSIBILE

$\cancel{1+x=x}$ $\vee$

$1+x=-x$

$2x=-1$ $\quad x=-\frac{1}{2}$

$$\begin{cases} x=-\frac{1}{2} \\ -1<y<1 \end{cases} \qquad \begin{cases} x=-\frac{1}{2} \\ -1<y<1 \leftarrow V\end{cases} \qquad P\left(-\frac{1}{2}, -\frac{1}{\;}\right) \text{ unico}$$

$$\begin{cases} x = -\frac{1}{2} \\ -1 < y < 1 \\ y = -x^2 \end{cases} \qquad \begin{cases} x = -\frac{1}{2} \\ -1 < y < 1 \ \checkmark \\ y = -\frac{1}{4} \end{cases} \qquad P\left(-\frac{1}{2}, -\frac{1}{4}\right)$$

unico pto critico

$$f(x,y) = (1+x^2+y)\exp(x-y) \qquad f\left(-\frac{1}{2}, -\frac{1}{4}\right) = \left(1+\frac{1}{4}-\frac{1}{4}\right)\exp\left(-\frac{1}{2}+\frac{1}{4}\right)$$

$$= \exp\left(\frac{-1}{4}\right) = \frac{1}{\sqrt[4]{e}}$$

$$\gamma_1 \begin{cases} x = t \\ y = -1 \end{cases} \qquad t \in [-1,1]$$

$$g_1(t) = f(\gamma_1(t)) = f(t,-1) = (1+t^2-1)\exp(t+1) = t^2 e^{t+1} \qquad t \in [-1,1]$$

$$g_1(-1) = 1 \qquad g_1(1) = 1 \cdot e^2 = e^2$$

$$g_1'(t) = 2t e^{t+1} + t^2 e^{t+1} = e^{t+1}(t^2+2t) = \underbrace{e^{t+1}}_{>0}\underbrace{(t+2)}_{>0}t \qquad t \in (-1,1)$$

$$g_1'(t) \gtrless 0 \quad \text{SSE} \quad t \gtrless 0$$

$g_1'<0$ $\quad$ $g_1'>0$

$-1 \quad 0 \quad 1$

$$g_1(0) = 0$$

$$\gamma_2 \begin{cases} x = 1 \\ y = t \end{cases} \qquad t \in [-1,1] \qquad f = (1+x^2+y)\exp(x-y)$$

$$g_2(t) = f(\gamma_2(t)) = f(1,t) = (2+t)\exp(1-t)$$

$$g_2(-1) = e^2 \qquad g_2(1) = 3e^0 = 3$$

$$g_2'(t) = 1\exp(1-t) + (2+t)\exp(1-t)\cdot(-1) = \exp(1-t)(1-2-t) =$$

$$= -\underbrace{(t+1)}_{>0}\underbrace{\exp(1-t)}_{>0} < 0 \qquad t \in (-1,1)$$

$-1 \quad 1$

$$\gamma_3 \begin{cases} x = t \\ y = 1 \end{cases} \qquad t \in [-1,1] \qquad f = (1+x^2+y)\exp(x-y)$$

$$g_3(t) = f(\gamma_3(t)) = f(t,1) = (2+t^2)\exp(t-1)$$

$$g_3(-1) = 3e^{-2} \qquad g_3(1) = 3$$

$g_3(t) = f(\gamma_3(t)) = f(t,1) = (2+t^2)\exp(t-1)$

$g_3(-1) = 3e^{-2} \qquad g_3(1) = 3$

$$g_3'(t) = 2t\exp(t-1) + (2+t^2)\exp(t-1)(1) =$$
$$= \exp(t-1)\cdot(2+2t+t^2) = \qquad \underbrace{1+1+2t+t^2}_{(t+1)^2}$$
$$= \underbrace{\exp(t-1)}_{>0}\cdot\underbrace{(1+(1+t)^2)}_{>0} > 0$$

$$\gamma_4 \begin{cases} x=-1 \\ y=t \end{cases} \quad t\in[-1,1] \qquad f=(1+x^2+y)\exp(x-y)$$

$$g_4(t) = f(\gamma_4(t)) = f(-1,t) = (1+1+t)\exp(-1-t) = (t+2)\exp(-t-1)$$

$$g_4'(t) = \exp(-t-1) + (t+2)\exp(-t-1)(-1) = \exp(-t-1)(1-t-2) =$$
$$= \underbrace{\exp(-t-1)}_{>0}\underbrace{(-t-1)}_{<0} < 0 \qquad t\in(-1,1)$$

$$f_x = e^{x-y}(1+2x+x^2+y)$$
$$f_y = e^{x-y}(-x^2-y) \qquad P\left(-\frac{1}{2}, -\frac{1}{4}\right)$$

$$f_{xx} = e^{x-y}(1+2x+x^2+y) + e^{x-y}(2+2x) =$$
$$= e^{x-y}(1+2x+x^2+y+2+2x) = e^{x-y}(3+4x+x^2+y)$$

$$f_{xy} = f_{yx} = e^{x-y}(-x^2-y) + e^{x-y}(-2x) = e^{x-y}(-x^2-2x-y)$$

$$f_{yy} = e^{x-y}(-1)(-x^2-y) + e^{x-y}(-1) = e^{x-y}(x^2+y-1)$$

$$f_{xx}\left(-\frac{1}{2}, -\frac{1}{4}\right) = e^{-1/4}\left(3-2+\cancel{\frac{1}{4}}-\cancel{\frac{1}{4}}\right) = e^{-1/4}$$

$$f_{xy}\left(-\frac{1}{2}, -\frac{1}{4}\right) = f_{yx}\left(-\frac{1}{2}, -\frac{1}{4}\right) = e^{-1/4}\left(-\cancel{\frac{1}{4}}+1+\cancel{\frac{1}{4}}\right) = e^{-1/4}$$

$$f_{yy}\left(-\frac{1}{2}, -\frac{1}{4}\right) = e^{-1/4}\left(\frac{1}{4}-\frac{1}{4}-1\right) = -e^{-1/4}$$

$$H_f\left(-\frac{1}{2}, -\frac{1}{4}\right) = \begin{pmatrix} e^{-1/4} & e^{-1/4} \\ e^{-1/4} & -e^{-1/4} \end{pmatrix}$$

$$H_f\left(-\frac{1}{2}, -\frac{1}{4}\right) = \begin{vmatrix} e^{-1/4} & -e^{-1/4} \end{vmatrix}$$

$$\det H_f\left(-\frac{1}{2}, -\frac{1}{4}\right) = e^{-1/4}\left(-e^{-1/4}\right) - \left(e^{-1/4}\right)^2 = -e^{-1/2} - e^{-1/2} = -2e^{-1/2} < 0$$

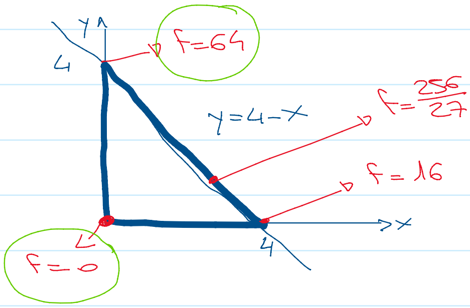
$\Rightarrow$ $\lambda_{min}\lambda_{MAX} < 0$

$\lambda_{min} < 0 < \lambda_{MAX}$ $\Rightarrow$ forma indefinita

$\Downarrow$

$\left(-\frac{1}{2}, -\frac{1}{4}\right)$ è PTO DI SELLA

Esercizio Determinare gli estremi assoluti di $f(x,y) = x^2 + y^3$ in $D = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x+y \leq 4\}$



$x+y \leq 4$

$x+y=4$ $y=4-x$ $y \leq 4-x$

$\Rightarrow$ Il min è 0 e l'unico pto di min è $(0,0)$

Il MAX è 64 e l'unico pto di MAX è $(0,4)$

$f(x,y) = x^2 + y^3$

$f_x = 2x$ $f_y = 3y^2$

$\nabla f(x,y) = (0,0)$ SSE $(x,y) = (0,0)$

$$\begin{cases} (x,y) \in int(D) \\ 2x = 0 \\ 3y^2 = 0 \end{cases} \Rightarrow (x,y) = (0,0) \notin int(D)$$

$\gamma_1 \begin{cases} x = t \\ y = 0 \end{cases}$ $t \in [0,4]$ $f = x^2 + y^3$

$g_1(t) = f(\gamma_1(t)) = f(t,0) = t^2$

0 4

$g_1(0) = 0$ $g_1(4) = 16$

$\gamma_2 \begin{cases} x = t \\ y = 4-t \end{cases}$ $t \in [0,4]$ $f = x^2 + y^3$

$g_2(t) = f(\gamma_2(t)) = f(t, 4-t) = t^2 + (4-t)^3$

$g_2(0) = 0 + 4^3 = 64$

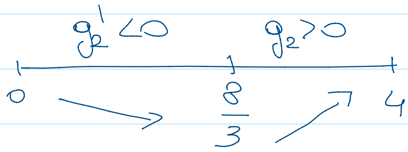
$$g_2'(t) = 2t + 3(4-t)^2(-1) = 2t - 3(t-4)^2 = 2t - 3(t^2 - 8t + 16)$$

$$= -(3t^2 - 24t + 48 - 2t) = -(3t^2 - 26t + 48) = \underbrace{-3}_{<0}\underbrace{(t-6)}_{<0}\left(t - \frac{8}{3}\right)$$

$t \in (0,4)$

$$\frac{\Delta}{4} = (13)^2 - 3 \cdot 48 = 169 - 144 = 25$$

$$t_{1,2} = \frac{13 \pm 5}{3} \left\langle \begin{array}{l} 6 \\ \frac{8}{3} \end{array} \right.$$



$$g_2\left(\frac{8}{3}\right) = t^2 + (4-t)^3 \Big|_{t=\frac{8}{3}} = \frac{64}{9} + \left(\frac{4}{3}\right)^3 = \frac{64}{9} + \frac{64}{27} = \frac{256}{27}$$

$$\gamma_3 \begin{cases} x = 0 \\ y = t \end{cases} \qquad t \in [0,4] \qquad f = x^2 + y^3$$

$$g_3(t) = f(\gamma_3(t)) = f(0,t) = t^3$$