venerdì 9 ottobre 2020 05:54

$f: E \subseteq \mathbb{R}^n \to \mathbb{R}$, $\quad \underline{x_0} \in \operatorname{int}(E)$ $\qquad$ derivate parziali, differenziabilità di f in $\underline{x_0}$

PROPOSIZIONE Se f è differenziabile in $\underline{x_0} \in \operatorname{int}(E)$, $f: E \to \mathbb{R}$, allora f è continua in $\underline{x_0}$

DIM $\qquad f(\underline{x_0}+\underline{h}) = f(\underline{x_0}) + \nabla f(\underline{x_0}) \cdot \underline{h} + \varepsilon(\underline{h}) \qquad \lim_{\underline{h} \to 0} \frac{\varepsilon(\underline{h})}{|\underline{h}|} = 0$

$$\lim_{\underline{h}\to \underline{0}} f(\underline{x_0}+\underline{h}) = \lim_{\underline{h}\to \underline{0}} \Big(f(\underline{x_0}) + \underbrace{\nabla f(\underline{x_0}) \cdot \underline{h}}_{\downarrow\, 0 \ \ \underline{h}\to \underline{0}} + \underbrace{\varepsilon(\underline{h})}_{\to 0} \Big) = f(\underline{x_0})$$

DEF La funzione $\quad df(\underline{x_0}): \underline{h} \in \mathbb{R}^n \to \nabla f(\underline{x_0}) \cdot \underline{h} \in \mathbb{R}$ è una funzione lineare su $\mathbb{R}^n$, si chiama DIFFERENZIALE DI f IN $\underline{x_0}$

—— o ——

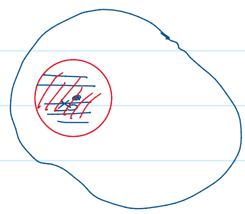
$$\lim_{\underline{h}\to 0} \frac{f(\underline{x_0}+\underline{h}) - f(\underline{x_0}) - \nabla f(\underline{x_0}) \cdot \underline{h}}{|\underline{h}|} \overset{?}{=} 0$$

TEOREMA (Condizione sufficiente per la differenziabilità)

Sia $A \subseteq \mathbb{R}^n$ aperto, sia $f: A \to \mathbb{R}$ e sia $\underline{x_0} \in A$

Supponiamo che le derivate parziali f esistano in almeno un intorno di x_0 e che siano continue nel pto $\underline{x_0}$

Allora f è differenziabile in $\underline{x_0}$



DIM $\quad n=2 \qquad \underline{h} = (h,k)$

$$f(x_0+h, y_0+k) - f(x_0,y_0) = \big(f(x_0+h, y_0+k) - f(x_0+h, y_0)\big) + \big(f(x_0+h, y_0) - f(x_0, y_0)\big)$$

$$f(x_0+h, y_0+k) - f(x_0+h, y_0) = \frac{\partial f}{\partial y}(x_0+h, y_0+\theta_1 k)\,(y_0+k-y_0) \qquad \theta_1 \in (0,1)$$

$y_0+\theta_1 k \quad \theta_1 \in (0,1)$

$y_0 \qquad y_0+k \qquad \to \mathbb{R}$

$$= k\, \frac{\partial f}{\partial y}(x_0+h, y_0+\theta_1 k)$$

$$f(x_0+h, y_0) - f(x_0,y_0) = \frac{\partial f}{\partial x}(x_0+\theta_2 h, y_0)\,(x_0+h-x_0) \qquad \theta_2 \in (0,1)$$

$x_0+\theta_2 h \quad \theta_2 \in (0,1)$

$x_0 \qquad x_0+h \qquad \to \mathbb{R}$

$$= h\, \frac{\partial f}{\partial x}(x_0+\theta_2 h, y_0)$$

$$f(x_0+h, y_0+k) - f(x_0,y_0) = h\, \frac{\partial f}{\partial x}(x_0+\theta_2 h, y_0) + k\, \frac{\partial f}{\partial y}(x_0+h, y_0+\theta_1 k) =$$

$$\frac{\partial f}{\partial x}(x_0+\theta_2 h, y_0) = \frac{\partial f}{\partial x}(x_0,y_0) + \varepsilon_2(h) \qquad \lim_{h\to 0} \varepsilon_2(h) = 0$$

$$\frac{\partial f}{\partial y}(x_0+h, y_0+\theta_1 k) = \frac{\partial f}{\partial y}(x_0,y_0) + \varepsilon_1(h,k) \qquad \lim \varepsilon_1(h,k) = 0$$

$$\frac{\partial f}{\partial x}(x_0+\theta_2 h, y_0) = \frac{\partial f}{\partial x}(x_0,y_0) + \varepsilon_2(h) \qquad \lim_{h\to 0} \varepsilon_2(h) = 0$$

$$\frac{\partial f}{\partial y}(x_0+h, y_0+\theta_1 k) = \frac{\partial f}{\partial y}(x_0,y_0) + \varepsilon_1(h,k) \qquad \lim_{(h,k)\to(0,0)} \varepsilon_1(h,k) = 0$$

$$f(x_0+h, y_0+k) - f(x_0,y_0) = \underbrace{h\frac{\partial f}{\partial x}(x_0,y_0)} + h\,\varepsilon_2(h) + \underbrace{k\frac{\partial f}{\partial y}(x_0,y_0)} + k\,\varepsilon_1(h,k)$$

$$\left| \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - \nabla f(x_0,y_0)\cdot(h,k)}{\sqrt{h^2+k^2}} \right| = \left| \frac{h\varepsilon_2(h) + k\varepsilon_1(h,k)}{\sqrt{h^2+k^2}} \right| \leq$$

$$\leq \underbrace{\frac{|h|}{\sqrt{h^2+k^2}}}_{\leq 1} |\varepsilon_2(h)| + \underbrace{\frac{|k|}{\sqrt{h^2+k^2}}}_{\leq 1} |\varepsilon_1(h,k)| \leq \underbrace{|\varepsilon_2(h)| + |\varepsilon_1(h,k)|}_{\to 0}$$

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - \nabla f(x_0,y_0)\cdot(h,k)}{\|(h,k)\|} = 0$$

DERIVATE DIREZIONALI

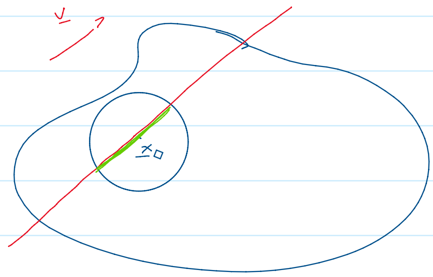
Le derivata parziale di f in un pto $\underline{x}_0$ rispetto alla i-esima variabile

$$\frac{d}{dt} f(\underline{x}_0 + t\underline{e}_i)\Big|_{t=0} \qquad \underline{e}_i = (0, \_\_, 0, 1, 0, \_\_\_ 0)$$

$\to$ i-esima componente

$f : E \subseteq \mathbb{R}^n \to \mathbb{R}$ $\qquad \underline{x}_0 \in \text{int}(E)$

Sia $\underline{v} \in \mathbb{R}^n : \|\underline{v}\| = 1$



$g(t) = f(\underline{x}_0 + t\underline{v}) \qquad t \in (-r, r)$

Se $\exists\, g'(0)$, dico che il suo valore è la derivata direzionale di f nel pto $\underline{x}_0$ e nella direzione $\underline{v}$ e la indico $D_{\underline{v}} f(\underline{x}_0)$

cioè

$$D_{\underline{v}} f(\underline{x}_0) = \lim_{t\to 0} \frac{f(\underline{x}_0 + t\underline{v}) - f(\underline{x}_0)}{t}$$ se questo limite esiste finito

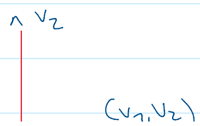
N.B. $\frac{\partial f}{\partial x_i}(\underline{x}_0) = D_{\underline{e}_i} f(\underline{x}_0)$

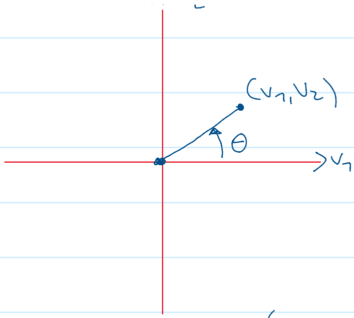
ESEMPIO $\qquad f(x,y) = x^2\cos(y-x) \qquad D_{\underline{v}} f(x_0,y_0)$

$\underline{v} = (v_1, v_2)$

$v_1^2 + v_2^2 = 1$

$$g(t) = f\big((x_0,y_0) + t(\cos\theta, \sin\theta)\big) = f(x_0 + t\cos\theta,\ y_0 + t\sin\theta)$$





$f(x_0+t\cos\theta,\ y_0+t\sin\theta)$

$(v_1,v_2)=(\cos\theta,\sin\theta)$

$g(t)=(x_0+t\cos\theta)^2\cos(y_0+t\sin\theta-x_0-t\cos\theta)$

$g(t)=(x_0+t\cos\theta)^2\cos\left(y_0-x_0+t(\sin\theta-\cos\theta)\right)$

$$g'(t)=2(x_0+t\cos\theta)\cos(\theta)\cos(y_0-x_0+t(\sin\theta-\cos\theta))+$$
$$-(x_0+t\cos\theta)^2\cdot\sin(y_0-x_0+t(\sin\theta-\cos\theta))\cdot(\sin\theta-\cos\theta)$$

$$D_{\underline{v}}f(x_0,y_0)=g'(0)=2\cos\theta\cos(y_0-x_0)-x_0^2\sin(y_0-x_0)(\sin\theta-\cos\theta)$$

TEO (FORMULA DEL GRADIENTE)

Sia $f:E\subseteq\mathbb{R}^n\to\mathbb{R}$, $\underline{x}_0\in\operatorname{int}(E)$, Supponiamo che f sia differenziabile in x_0.

Allora $\forall\underline{v}$ direzione, $D_{\underline{v}}f(\underline{x}_0)=\nabla f(\underline{x}_0)\cdot\underline{v}=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\underline{x}_0)v_i$

$\underline{v}=(v_1,v_2,\ldots v_n)$

DIM $f(\underline{x}_0+\underline{h})=f(\underline{x}_0)+\nabla f(\underline{x}_0)\cdot\underline{h}+o(\|\underline{h}\|)$ $\qquad\lim_{\underline{h}\to0}\frac{o(\|\underline{h}\|)}{\|\underline{h}\|}=0$

Scelgo $\underline{h}=t\underline{v}\Rightarrow\|\underline{h}\|=|t|$

$f(\underline{x}_0+t\underline{v})-f(\underline{x}_0)=t\nabla f(\underline{x}_0)\cdot\underline{v}+o(|t|)$

$$\frac{f(\underline{x}_0+t\underline{v})-f(\underline{x}_0)}{t}=\nabla f(\underline{x}_0)\cdot\underline{v}+\underbrace{\boxed{\frac{o(|t|)}{t}}}_{\to0}\xrightarrow{t\to0}\nabla f(\underline{x}_0)\cdot\underline{v}$$

$$\exists\lim_{t\to0}\frac{f(\underline{x}_0+t\underline{v})-f(\underline{x}_0)}{t}=\nabla f(\underline{x}_0)\cdot\underline{v}$$

cioè $D_{\underline{v}}f(\underline{x}_0)=\nabla f(\underline{x}_0)\cdot\underline{v}$

CALCOLO DELLE DERIVATE

f e g derivabili in un pto x_0 $\quad\alpha,\beta\in\mathbb{R}\Rightarrow\alpha f+\beta g$ è derivabile in x_0

e $\frac{\partial}{\partial x_i}(\alpha f+\beta g)(\underline{x}_0)=\alpha\frac{\partial f}{\partial x_i}(\underline{x}_0)+\beta\frac{\partial g}{\partial x_i}(\underline{x}_0)\quad\forall i=1\ldots n$

$$\nabla(\alpha f+\beta g)(\underline{x}_0)=\alpha\nabla f(\underline{x}_0)+\beta\nabla g(\underline{x}_0)$$

$$\frac{\partial}{\partial x_i}(fg)(\underline{x}_0)=f(\underline{x}_0)\frac{\partial g}{\partial x_i}(\underline{x}_0)+g(\underline{x}_0)\frac{\partial f}{\partial x_i}(\underline{x}_0)\quad\forall i=1\ldots n$$

$$\nabla(fg)(x_0) = f(x_0)\nabla g(x_0) + g(x_0)\nabla f(x_0)$$

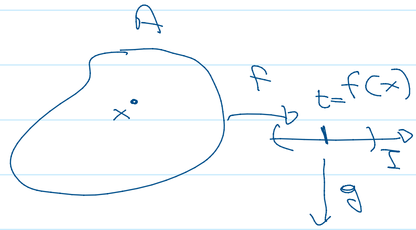
Se $g(x_0) \neq 0$ $\quad \dfrac{\partial}{\partial x_i}\left(\dfrac{f}{g}\right)(x_0) = \dfrac{g(x_0)\frac{\partial f}{\partial x_i}(x_0) - f(x_0)\frac{\partial g}{\partial x_i}(x_0)}{g^2(x_0)}$

$$\nabla\left(\frac{f}{g}\right)(x_0) = \frac{1}{g^2(x_0)}\Big(g(x_0)\nabla f(x_0) - f(x_0)\nabla g(x_0)\Big)$$

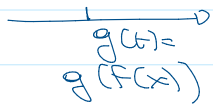
DERIVABILITÀ DI FUNZIONI COMPOSTE

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A aperto,

$g : I \subseteq \mathbb{R} \to \mathbb{R}$ $\quad$ T.c. $\quad f(A) \subseteq I$



$h(x) := g(f(x))$ $\qquad h : x \in A \mapsto g(f(x)) \in \mathbb{R}$



PROPRIETÀ Se f è DIFFERENZIABILE in un pto $x_0 \in A$ e g è derivabile in $f(x_0)$, allora h è DIFFERENZIABILE in x_0 e $\nabla h(x_0) = g'(f(x_0))\nabla f(x_0)$

DIM $\quad h(\underline{x}) = g(f(\underline{x}))$

$\qquad h(x_1 \ldots x_n) = g(f(x_1 \ldots x_n))$

$$\frac{\partial h}{\partial x_i}(x_1 \ldots x_n) = g'(f(x_1 \ldots x_n))\frac{\partial f}{\partial x_i}(x_1 \ldots x_n) \qquad \forall i = 1 \ldots n$$

$$\nabla h(\underline{x}_0) = g'(f(\underline{x}_0))\nabla f(\underline{x}_0)$$

$k \in \mathbb{R}$ $\quad g(\underline{f(x_0)} + k) = g(f(x_0)) + k\,g'(f(x_0)) + o(k) =$

$\qquad = g(f(x_0)) + k\,g'(f(x_0)) + k\,\theta(k) \qquad \lim_{k \to 0}\theta(k) = 0$

Scelgo $\quad k = f(\underline{x}_0 + \underline{w}) - f(\underline{x}_0) \qquad \underline{w} \in \mathbb{R}^n$

$\qquad f(\underline{x}_0) + k = f(\underline{x}_0 + \underline{w})$

$$g(f(x_0 + \underline{w})) = g(f(x_0)) + \big(\underline{f(x_0 + \underline{w}) - f(x_0)}\big)\,g'(f(x_0)) +$$

$$+ \big(\underline{f(x_0 + \underline{w}) - f(x_0)}\big)\,\theta\big(f(x_0 + \underline{w}) - f(x_0)\big)$$

f differenziabile in $\underline{x}_0$ $\Rightarrow$ $\quad f(\underline{x}_0 + \underline{w}) - f(\underline{x}_0) = \nabla f(\underline{x}_0)\cdot\underline{w} + o(\|\underline{w}\|)$

$$\underbrace{g(f(x_0 + \underline{w})) - g(f(\underline{x}_0))}_{h(x_0 + \underline{w}) - h(x_0)} = g'(f(x_0))\,\nabla f(\underline{x}_0)\cdot\underline{w} + g'(f(x_0))\,o(\|\underline{w}\|)$$

$$+\big(\underline{f(x_0 + \underline{w}) - f(x_0)}\big)\,\underline{\theta\big(f(\underline{x}_0 + \underline{w}) - f(\underline{x}_0)\big)}$$

$$h(\underline{x}_0+\underline{w}) - h(\underline{x}_0) = g'(f(\underline{x}_0))\, \nabla f(\underline{x}_0)\cdot \underline{w} + o(\|\underline{w}\|) + \underbrace{(\nabla f(\underline{x}_0)\cdot \underline{w} + o(\|\underline{w}\|))\, \vartheta(\nabla f(\underline{x}_0)\cdot\underline{w} + o(\|\underline{w}\|))}_{o(\|\underline{w}\|)}$$

$$h(\underline{x}_0+\underline{w}) - h(\underline{x}_0) = g'(f(\underline{x}_0))\, \nabla f(\underline{x}_0)\cdot \underline{w} + o(\|\underline{w}\|)$$

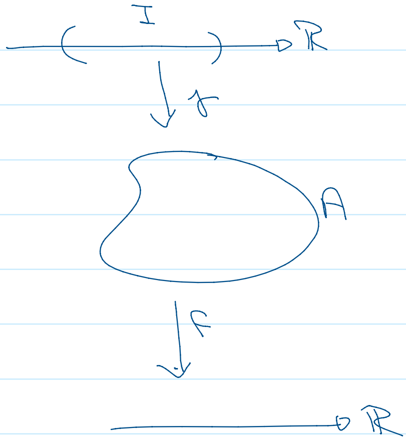
I intervallo aperto $\qquad \gamma : I \to \mathbb{R}^n$

$A \subseteq \mathbb{R}^n$ aperto $\qquad f : A \to \mathbb{R}$

Se $\gamma(I) \subseteq A$ posso considerare

$$f(\gamma(t)) \quad \forall t \in I$$

$$g := f\circ\gamma : t \in I \longmapsto f(\gamma(t)) \in \mathbb{R}$$



PROPRIETÀ Se γ è derivabile in t_0 e se f è differenziabile in $\gamma(t_0)$, allora g è derivabile in t_0 e

$$g'(t_0) = \nabla f(\gamma(t_0))\cdot\gamma'(t_0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\gamma(t_0))\,\gamma_i'(t_0)$$

DIM

So che f è differenziabile in $\gamma(t_0)$

$$f(\gamma(t_0)+\underline{k}) - f(\gamma(t_0)) = \nabla f(\gamma(t_0))\cdot\underline{k} + \|\underline{k}\|\,\varepsilon(\underline{k}) \qquad \lim_{\underline{k}\to 0}\varepsilon(\underline{k}) = 0$$

Scelgo $\underline{k} = \gamma(t_0+h) - \gamma(t_0) \qquad h \in \mathbb{R}$

$$\gamma(t_0)+k = \gamma(t_0+h)$$

$$f(\gamma(t_0+h)) - f(\gamma(t_0)) = \nabla f(\gamma(t_0))\cdot(\gamma(t_0+h)-\gamma(t_0)) + \|\gamma(t_0+h)-\gamma(t_0)\|\,\varepsilon(\underbrace{\gamma(t_0+h)-\gamma(t_0)}_{\to 0 \text{ per } h\to 0})$$

$\gamma(t_0+h) = \gamma(t_0) + h\gamma'(t_0) + o(h)$ perché γ è derivabile in t_0

$$f(\gamma(t_0+h)) - f(\gamma(t_0)) = \nabla f(\gamma(t_0))\cdot(h\gamma'(t_0) + o(h)) + o(h)$$

$$g(t_0+h) - g(t_0) = h\,\nabla f(\gamma(t_0))\cdot\gamma'(t_0) + o(h)$$

$$\exists\, g'(t_0) = \nabla f(\gamma(t_0))\cdot\gamma'(t_0)$$

$f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ differenziabile in $\underline{x}_0$

$\underline{x}_0 = (x_1^0, x_2^0, \_\_\_ \, x_n^0)$

$Gr(f) \subset \mathbb{R}^{n+1}$ $\qquad (\underline{x}, x_{n+1}) = (x_1 \_\_\_ \, x_n, x_{n+1})$

$$x_{n+1} = f(\underline{x}_0) + Df(\underline{x}_0) \cdot (\underline{x} - \underline{x}_0)$$

$$x_{n+1} = f(x_1^0 \_\_\_ \, x_n^0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_1^0 \_\_ \, x_n^0)(x_i - x_i^0)$$

$n = 2$ $\quad Gr(f) \subset \mathbb{R}^3$ $\quad (x, y, z)$ $\quad (x_0, y_0)$

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$\gamma: t \in I \mapsto \gamma(t) \in A$ $\qquad \gamma(t_0) = (x_0, y_0)$ $\qquad f(\gamma(t))$

$\gamma(t) = (\gamma_1(t), \gamma_2(t))$

$\eta: t \in I \mapsto \eta(t) \in \mathbb{R}^3$

$$\eta(t) = \begin{cases} x = \gamma_1(t) \\ y = \gamma_2(t) \\ z = \underbrace{f(\gamma_1(t), \gamma_2(t))}_{(f \circ \gamma)(t)} \end{cases} \qquad t \in I$$

$$\eta'(t_0) = \left(\gamma_1'(t_0), \gamma_2'(t_0), Df(\underbrace{\gamma(t_0)}_{(x_0, y_0)}) \cdot \gamma'(t_0)\right) =$$

$$= \left(\gamma_1'(t_0), \gamma_2'(t_0), f_x(x_0, y_0)\gamma_1'(t_0) + f_y(x_0, y_0)\gamma_2'(t_0)\right)$$

$$\eta(t_0) = (x_0, y_0, f(x_0, y_0))$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \eta(t_0) = \lambda \eta'(t_0) \qquad \lambda \in \mathbb{R}$$

$$\to \begin{cases} x - x_0 = \lambda \gamma_1'(t_0) \\ y - y_0 = \lambda \gamma_2'(t_0) \\ z - f(x_0, y_0) = \lambda \left(f_x(x_0, y_0)\gamma_1'(t_0) + f_y(x_0, y_0)\gamma_2'(t_0)\right) \end{cases} \qquad \lambda \in \mathbb{R}$$

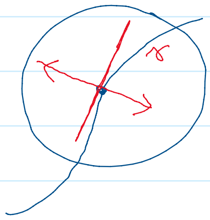
$$\to z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$\frac{y - y_0}{x - x_0} = \frac{\gamma_2'(t_0)}{\gamma_1'(t_0)}$$

$f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ $\quad A$ aperto $\quad \underline{x}_0 \in A$ $\quad$ t.c. f è differenziabile in A

$c := f(\underline{x}_0)$

$$L_c = \{\underline{x} \in A : f(\underline{x}) = c\}$$



Suppongo che $\exists r>0$ T.c. $L_c \cap B_r(x_0)$ sia una curva derivabile
$\gamma: t \in (-r,r) \mapsto \gamma(t) \in \mathbb{R}^n$ derivabile
$\gamma(0) = x_0$

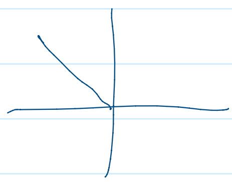
$g(t) = f(\gamma(t)) = c \qquad g'(t) = 0$

$0 = g'(0) = \nabla f(\gamma(0)) \cdot \gamma'(0) = \nabla f(x_0) \cdot \gamma'(0)$

ESERCIZIO $\quad f(x,y) = \ln(xy) + \cos(x+y) \qquad P\left(\frac{\pi}{2}, \frac{\pi}{4}, f\left(\frac{\pi}{2}, \frac{\pi}{4}\right)\right)$

$f\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = \ln\left(\frac{\pi^2}{8}\right) + \cos\left(\frac{3}{4}\pi\right) = \underbrace{\ln\left(\frac{\pi^2}{8}\right) - \frac{\sqrt{2}}{2}}$

$f_x(x,y) = \frac{1}{xy} y - \sin(x+y) = \frac{1}{x} - \sin(x+y)$



$f_x\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = \frac{2}{\pi} - \sin\left(\frac{3}{4}\pi\right) = \underbrace{\frac{2}{\pi} - \frac{\sqrt{2}}{2}}$

$f_y(x,y) = \frac{1}{xy} x - \sin(x+y) = \frac{1}{y} - \sin(x+y)$

$f_y\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = \frac{4}{\pi} - \sin\left(\frac{3}{4}\pi\right) = \underbrace{\frac{4}{\pi} - \frac{\sqrt{2}}{2}}$

$z = \ln\left(\frac{\pi^2}{8}\right) - \frac{\sqrt{2}}{2} + \left(\frac{2}{\pi} - \frac{\sqrt{2}}{2}\right)\left(x - \frac{\pi}{2}\right) + \left(\frac{4}{\pi} - \frac{\sqrt{2}}{2}\right)\left(y - \frac{\pi}{4}\right)$

ESERCIZIO $\quad \alpha: t \in \mathbb{R} \mapsto (t-1, t+1) \in \mathbb{R} \leftarrow$
$\beta: t \in \mathbb{R} \mapsto (t^2-1, 2) \in \mathbb{R} \leftarrow$

$f: \mathbb{R}^2 \to \mathbb{R}$ differenziabile $\leftarrow$

$\frac{d}{dt}(f \circ \alpha)(t)\Big|_{\underline{\underline{t=1}}} = 2 \qquad \frac{d}{dt}(f \circ \beta)(t)\Big|_{\underline{\underline{t=1}}} = 3$

$\nabla f(0,2) = ?$

$\alpha(1) = (0,2) \qquad \alpha'(t) = (1,1)$
$\beta(1) = (0,2) \qquad \beta'(t) = (2t, 0)$
$\beta'(1) = (2,0)$

$2 = \frac{d}{dt}(f \circ \alpha)(t)\Big|_{t=1} = \nabla f(\alpha(1)) \cdot \alpha'(1) = (f_x(0,2), f_y(0,2)) \cdot (1,1) = f_x(0,2) + f_y(0,2)$

$$\boxed{f_x(0,2) + f_y(0,2) = 2}$$

$$\boxed{f_x(0,2)+f_y(0,2)=2}$$

$$3=\frac{d}{dt}(f\circ\beta)(t)\Big|_{t=1}=\nabla f(\beta(1))\cdot\beta'(1)=\left(f_x(0,2),f_y(0,2)\right)\cdot(2,0)=$$
$$2f_x(0,2)$$

$$\boxed{2f_x(0,2)=3}$$

$$\begin{cases}2f_x(0,2)=3\\ f_x(0,2)+f_y(0,2)=2\end{cases}$$

$$f_x(0,2)=3/2$$
$$f_y(0,2)=2-f_x(0,2)=2-\frac{3}{2}=\frac{1}{2}$$

$$\nabla f(0,2)=\left(\frac{3}{2},\frac{1}{2}\right)$$

ESERCIZIO $\longrightarrow f(x,y)=x^4e^{xy^2}$ $\quad P(2,1)$ $\quad \underline{v}$ direzione individuata da $\underline{w}=(1,3)$

$$\underline{v}=\frac{\underline{w}}{\|\underline{w}\|}\qquad \|\underline{w}\|=\sqrt{1^2+3^2}=\sqrt{10}$$

$$\underline{v}=\left(\frac{1}{\sqrt{10}},\frac{3}{\sqrt{10}}\right)$$

$F(x,y)=x^4e^{xy^2}$
$P(2,1)$

$$f_x=4x^3\cdot e^{xy^2}+x^4\cdot e^{xy^2}y^2=e^{xy^2}\cdot x^3(4+xy^2)$$
$$f_y=x^4e^{xy^2}2xy=2x^5ye^{xy^2}$$

$$f_x(2,1)=e^2\cdot 8(4+2)=48e^2\qquad \nabla f(2,1)=(48e^2,64e^2)$$
$$f_y(2,1)=64e^2$$

$$D_{\underline{v}}f(2,1)=\nabla f(2,1)\cdot\underline{v}=(48e^2,64e^2)\cdot\left(\frac{1}{\sqrt{10}},\frac{3}{\sqrt{10}}\right)$$
$$=\frac{1}{\sqrt{10}}\left(48e^2+192e^2\right)=\frac{e^2}{\sqrt{10}}240=\frac{e^2}{\cancel{\sqrt{10}}}24\cdot\cancel{10}^{\sqrt{10}}$$
$$=24e^2\sqrt{10}$$

— o —

$f: A\subseteq\mathbb{R}^n\to\mathbb{R}$, A aperto - Supponiamo che f è derivabile rispetto ad x_i in ogni pto di A $\quad\exists f_{x_i}(\underline{x})\quad\forall\underline{x}\in A$

$$f_{x_i}:\underline{x}\in A\longmapsto f_{x_i}(\underline{x})\in\mathbb{R}$$

— o —

$n=2$ $\quad f: A \subseteq \mathbb{R}^2 \to \mathbb{R}$, A aperto $\quad (x,y)$

$\forall (x,y)\in A \quad f_x(x,y) \qquad f_x : (x,y)\in A \mapsto f_x(x,y) \in \mathbb{R}$

$$\frac{\partial}{\partial x}\left(f_x\right) \,? \qquad \frac{\partial}{\partial y}\left(f_x\right)\,?$$

Se $(x_0,y_0)\in A$ e f_x è derivabile in (x_0,y_0) rispetto a x
cioè se

$$\exists \frac{\partial}{\partial x}\left(f_x\right)(x_0,y_0) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f\right)(x_0,y_0),$$ la indico

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) \quad \text{o anche} \quad f_{xx}(x_0,y_0)$$

e dico che f è derivabile due volte rispetto a x nel pto (x_0,y_0)

Se esiste $\frac{\partial}{\partial y}\left(f_x\right)(x_0,y_0)$ $\quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x_0,y_0)$, la indico

$$\frac{\partial^2}{\partial y\, \partial x} f(x_0,y_0)$$ derivata mista di f rispetto a x e poi rispetto a y

Se $f_y(x,y)$ è definita $\forall (x,y)\in A$, mi posso chiedere se

$$\exists \frac{\partial}{\partial x}\left(f_y(x,y)\right) = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x,y)$$

$$\exists \frac{\partial}{\partial y}\left(f_y(x,y)\right) = \frac{\partial}{\partial y}\frac{\partial f}{\partial y}(x,y)$$

Se $\frac{\partial}{\partial x} f_y(x_0,y_0)$ esiste, dico che f è derivabile prima rispetto a y e poi rispetto a x nel pto (x_0,y_0)

$$\frac{\partial^2}{\partial x\, \partial y} f(x_0,y_0)$$

Se $\frac{\partial}{\partial y} f_y(x_0,y_0)$ esiste, dico che f è derivabile due volte rispetto a y nel pto (x_0,y_0) e la indico

$$\frac{\partial^2}{\partial y^2} f(x_0,y_0) \qquad \frac{\partial^2 f}{\partial y^2}(x_0,y_0)$$

$f: A\subseteq \mathbb{R}^n \to \mathbb{R}$ derivabile in ogni pto di A (A aperto)

$\forall \underline{x} = (x_1 \dots x_n) \in A$ $\qquad \frac{\partial f}{\partial x_i}(\underline{x}) \quad \forall i = 1 \dots n$

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(\underline{x}) \qquad j = 1 \dots n$$

Se $\forall i, j = 1 \dots n$ esiste la derivata $\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(\underline{x}_0)$, $\underline{x}_0 \in A$, dico che f è derivabile due volte nel pto $\underline{x}_0$

ESEMPIO $\quad f(x,y,z) = x^2 z\, e^{y-z}$

$$f_x(x,y,z) = 2x\, z e^{y-z}$$
$$f_y(x,y,z) = x^2 z e^{y-z}$$
$$f_z(x,y,z) = x^2 \cdot e^{y-z} + x^2 z\, e^{y-z} \cdot (-1) = e^{y-z}\left(x^2 - x^2 z\right) = x^2(1-z)e^{y-z}$$

$$\frac{\partial^2 f}{\partial x^2} = 2z e^{y-z}$$

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = 2xz e^{y-z}$$

$\frac{\partial^2 f}{\partial x^2}$ $\quad f_{xx}$

$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$$\frac{\partial}{\partial z}\frac{\partial f}{\partial x} = 2x e^{y-z} + 2xz\, e^{y-z}(-1) = 2x e^{y-z}(1-z)$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial x}\left(x^2 z e^{y-z}\right) = 2xz e^{y-z}$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial z} = \frac{\partial}{\partial x}\left(x^2(1-z)e^{y-z}\right) = 2x(1-z)e^{y-z}$$

CALCOLARE LE DERIVATE SECONDE MANCANTI