giovedì 21 novembre 2019 15:50

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix} \qquad P(t) = e^{tQ} \qquad t \geq 0$$

$$\alpha, \beta \geq 0 \qquad \alpha = \beta = 0 \Rightarrow P(t) = Id$$

$$\alpha, \beta \geq 0, \quad \alpha + \beta > 0$$

$$\lambda_1 = 0 \qquad \lambda_2 = -(\alpha+\beta) < 0$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -\alpha \\ \beta \end{pmatrix} \qquad M = (v_1, v_2) = \begin{pmatrix} 1 & -\alpha \\ 1 & \beta \end{pmatrix}$$

$$M^{-1} = \frac{1}{\alpha+\beta} \begin{pmatrix} \beta & \alpha \\ -1 & 1 \end{pmatrix}$$

$$Q = M \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} M^{-1}$$

$$P(t) = e^{tQ} = \sum_{k=0}^{\infty} \frac{t^k}{k!} Q^k$$

$$Q^2 = M \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cancel{M^{-1}} \cdot \cancel{M} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} M^{-1} = M \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} M^{-1}$$

e, per induzione

$$Q^k = M \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} M^{-1} \qquad \forall k \geq 0$$

$$P(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} Q^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} M \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} M^{-1} =$$

$$= M \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} \right) M^{-1} =$$

Posizione 1-1 $\qquad \displaystyle\sum_{k=0}^{\infty} \frac{(t\lambda_1)^k}{k!} = e^{t\lambda_1} = e^0 = 1$

Posizione 2-2 $\qquad \displaystyle\sum_{k=0}^{\infty} \frac{(t\lambda_2)^k}{k!} = e^{t\lambda_2} = e^{-t(\alpha+\beta)}$

$$P(t) = \begin{pmatrix} 1 & -\alpha \\ 1 & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-t(\alpha+\beta)} \end{pmatrix} \frac{1}{\alpha+\beta} \begin{pmatrix} \beta & \alpha \\ -1 & 1 \end{pmatrix} =$$

$$= \frac{1}{\ldots} \begin{pmatrix} \beta + \alpha e^{-t(\alpha+\beta)} & \alpha - \alpha e^{-t(\alpha+\beta)} \\ \ldots & \ldots \end{pmatrix}$$

$$= \frac{1}{\alpha+\beta} \begin{pmatrix} \beta + \alpha e^{-t(\alpha+\beta)} & \alpha - \alpha e^{-t(\alpha+\beta)} \\ \beta - \beta e^{-t(\alpha+\beta)} & \alpha + \beta e^{-t(\alpha+\beta)} \end{pmatrix}$$

$\pi \in \mathcal{J}$ $\quad \pi = (\pi_1, \pi_2)$ $\quad \pi_1, \pi_2 \geq 0$ $\quad \pi_1 + \pi_2 = 1$

$$\pi e^{tQ} = \pi P(t) = \frac{1}{\alpha+\beta} \cdot$$

$$\cdot \Big(\pi_1 \big(\beta + \alpha e^{-t(\alpha+\beta)}\big) + \pi_2 \big(\beta - \beta e^{-t(\alpha+\beta)}\big),$$
$$\pi_1 \big(\alpha - \alpha e^{-t(\alpha+\beta)}\big) + \pi_2 \big(\alpha + \beta e^{-t(\alpha+\beta)}\big) \Big)$$

$$= \frac{1}{\alpha+\beta} \Big(\beta + (\alpha\pi_1 - \beta\pi_2) e^{-t(\alpha+\beta)}, \ \alpha + (\beta\pi_2 - \alpha\pi_1) e^{-t(\alpha+\beta)} \Big)$$

$\underline{t \to +\infty}$ $\quad$ Poiché $\alpha + \beta > 0$ $\quad e^{-t(\alpha+\beta)} \to 0$

$$\pi P(t) \to \frac{1}{\alpha+\beta} (\beta, \alpha) = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta} \right)$$

$$wQ = 0 \qquad (w_1 \ w_2) \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix} = (0, 0)$$

$$\begin{cases} -\alpha w_1 + \beta w_2 = 0 \\ \alpha w_1 - \beta w_2 = 0 \\ w_1 + w_2 = 1 \end{cases} \qquad \begin{cases} \beta w_2 = \alpha w_1 \\ \beta w_1 + \beta w_2 = \beta \end{cases}$$

$$\beta w_1 + \alpha w_1 = \beta$$

$$w_1 = \frac{\beta}{\alpha+\beta} \Rightarrow w_2 = \frac{\alpha}{\alpha+\beta}$$

$(X_t)_{t \geq 0}$ processo stocastico continuo da destra su uno spazio probabilizzato completo $(\Omega, \mathcal{E}, \mathbb{P})$

$$SJ_i(\omega) := \inf \{ t > 0 : X_t(\omega) \neq i \}$$

N.B. Se $X_0(\omega) \neq i \Rightarrow SJ_i(\omega) = 0$

Vogliamo vedere se SJ_i è una v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

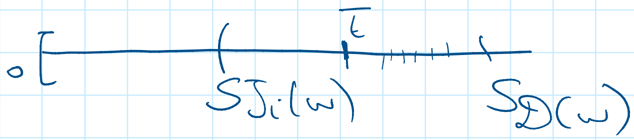
Sia $\mathcal{D} \subset [0,+\infty)$ denso e numerabile.

$$S_{\mathcal{D}}(\omega) := \inf\{t>0,\ t\in\mathcal{D} : X_t(\omega)\neq i\}$$

$$SJ_i(\omega) \le S_{\mathcal{D}}(\omega)$$

Fisso $\omega\in\Omega$ t.c. $t\mapsto X_t(\omega)$ è continua da destra

Supponiamo per assurdo $SJ_i(\omega) < S_{\mathcal{D}}(\omega)$



$X_{\bar t}(\omega)\neq i \qquad \bar t\notin\mathcal{D}$

$\exists (t_n)_{n\in\mathbb{N}}$ $t_n\in\mathcal{D}$ $\forall n$ $\quad t_n \searrow \bar t$: $\exists \bar n$ t.c. $t_n < S_{\mathcal{D}}(\omega)$ $\forall n>\bar n$

$\Rightarrow X_{t_n}(\omega) = i \quad \forall n\ge\bar n$

$$\lim_{n\to\infty} X_{t_n}(\omega) = X_{\bar t}(\omega) \neq i$$

$$\boxed{\lim_{n\to\infty} i \neq i}$$ ASSURDO.

$\forall\omega\in\Omega$ t.c. $t\mapsto X_t(\omega)$ è continua da destra si ha $SJ_i(\omega)=S_{\mathcal{D}}(\omega)$

So che $\exists\Omega_0\subset\Omega$ t.c. $\mathbb{P}(\Omega_0)=1$ e la mappa $t\mapsto X_t(\omega)$ è continua (in t) per ogni $\omega\in\Omega_0$.

Sia $t\in\mathbb{R}, t\ge 0$ $\quad \boxed{\{SJ_i>t\}} \subset (\Omega\setminus\Omega_0)\cup \{\omega\in\Omega_0 : S_{\mathcal{D}}(\omega)>t\} =$

$= (\Omega\setminus\Omega_0)\cup\{\omega\in\Omega_0 : X_s(\omega)=i \ \underline{\forall s\in\mathcal{D},\ s\le t}\}$

$= (\Omega\setminus\Omega_0)\cup\{\omega\in\Omega_0: X_s(\omega)=i\ \ \forall s\in\mathcal{D}\cap[0,t]\}$

$= (\Omega\setminus\Omega_0)\cup \bigcap_{s\in\mathcal{D}\cap[0,t]} \{\omega\in\Omega_0 : X_s(\omega)=i\} \in \mathcal{E}$

$$= \underbrace{(\Omega \setminus \Omega_0)}_{P(\Omega\setminus\Omega_0)=0} \cup \underbrace{\bigcap_{\underbrace{s\in\mathcal{D}\cap[0,t]}_{\text{numerabile}}} \underbrace{\{\omega\in\Omega_0 : X_s(\omega)=i\}}_{\in \mathcal{E}\ \forall s}}_{\in\mathcal{E}} \in \mathcal{E}$$

— o —

$(X_t)_{t\geq 0}$ sia catena di Markov a tempo continuo

Fisso $t>0$ $\quad P(SJ_i > t \mid X_0 = i) \quad i \in S$

$$\mathcal{D} = \left\{ \frac{jt}{2^n} : j\in\mathbb{N},\ n\in\mathbb{N} \right\}$$

$$P_i(SJ_i > t) = P_i\left(\bigcap_{s\in\mathcal{D}\cap[0,t]} \{\omega\in\Omega_0 : X_s(\omega)=i\} \right) = \circledast$$

$$\mathcal{D}\cap[0,t] = \left\{ \frac{jt}{2^n} \quad 0\leq j\leq 2^n,\ n\in\mathbb{N} \right\}$$

$$\mathcal{D}_n = \left\{ \frac{jt}{2^n} : 0\leq j\leq 2^n \right\}$$

$$\mathcal{D}_{n+1} \supseteq \mathcal{D}_n$$
$$\bigcup_{n=0}^{\infty} \mathcal{D}_n = \mathcal{D}\cap[0,t]$$

$$\circledast\ P_i\left(\bigcap_{n\in\mathbb{N}} \left\{ X_{\frac{jt}{2^n}} = i \quad \forall j = 1,\dots,2^n \right\} \right) =$$

$$P_i(\cdot) = P(\cdot \mid X_0 = i)$$

$$= \lim_{n\to\infty} P_i\left(X_{\frac{jt}{2^n}} = i \quad \forall j = 1 \dots 2^n \right)$$

$$= \lim_{n\to\infty} \prod_{j=0}^{2^n-1} P\left(X_{\frac{(j+1)t}{2^n}} = i \,\middle|\, X_{\frac{jt}{2^n}} = i \right)$$

Se la catena è omogenea

$$= \lim_{n\to\infty} \prod_{j=0}^{2^n-1} P\left(X_{\frac{t}{2^n}} = i \,\middle|\, X_0 = i \right)$$

$$= \lim_{n\to\infty} \left(P\left(\frac{t}{2^n}\right)_{i\,i} \right)^{2^n}$$

$$P_i(X_t = i,\ X_s = i) \quad t>s>0$$

$$\frac{P(X_t = i,\ X_s = i,\ X_0 = i)}{P(X_0 = i)} =$$

$$= \frac{P(X_t = i \mid X_s = i, \cancel{X_0 = i})\, P(X_s = i, X_0 = i)}{P(X_0 = i)}$$

$$= P(X_t = i \mid X_s = i)\, P(X_s = i \mid X_0 = i)$$

$$= \lim_{n \to \infty} \left(P\left(\frac{t}{2^n}\right)_{ii} \right)^{2^n}$$

$n \to \infty \quad \frac{t}{2^n} \to 0$ usando $P(s) = Id + sQ + s^2 A(s)$

$$\left(P\left(\frac{t}{2^n}\right)_{ii} \right)^{2^n} = \left(1 + s q_{ii} + O(s^2)\right)^{2^n} \qquad \left(s = \frac{t}{2^n}\right)$$

$$= \left(\underbrace{\left(1 + \frac{q_{ii} t}{2^n} + O\left(\frac{t^2}{4^n}\right)\right)^{\frac{2^n}{t q_{ii}}}} \right)^{t q_{ii}} \longrightarrow e^{t q_{ii}}$$

$$\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$$

$$\mathbb{P}_i(SJ_i > t) = \mathbb{P}(SJ_i > t \mid X_0 = i) = e^{t q_{ii}} =$$

$$= e^{-(-q_{ii})t}$$

$$\forall t \geq 0 \quad \mathbb{P}(SJ_i \leq t \mid X_0 = i) = 1 - e^{-(-q_{ii})t}$$

È la legge della distribuzione esponenziale di parametro $\lambda = -q_{ii} > 0$