martedì 5 novembre 2019 14:16

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 1 & 0 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\|R^i - R^j\|_1$ $\quad R^1 - R^2 = \left(-1, \frac{1}{2}, \frac{1}{2}, 0\right)$ $\quad \|R^1 - R^2\|_1 = 1 + \frac{1}{2} + \frac{1}{2} + 0 = 2$

$C = 1$

$$P^2 = \begin{pmatrix} \frac{5}{8} & 0 & \frac{1}{8} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{16} & \frac{5}{8} & \frac{3}{16} & \frac{1}{8} \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$R^2 - R^4 = \left(-1, \frac{1}{2}, \frac{1}{2}, 0\right)$

$\|R^2 - R^4\|_1 = 1 + \frac{1}{2} + \frac{1}{2} + 0 = 2$

$\Rightarrow C(P^2) = 1$

$$P^3 = \begin{pmatrix} \frac{1}{32} & \frac{9}{16} & \frac{11}{32} & \frac{1}{16} \\ \frac{5}{8} & 0 & \frac{1}{8} & \frac{1}{4} \\ \frac{43}{64} & \frac{5}{32} & \frac{5}{64} & \frac{3}{32} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$R^2 - R^4 = \left(\frac{5}{8}, -\frac{1}{2}, -\frac{3}{8}, \frac{1}{4}\right)$

$\|R^2 - R^4\|_1 = \frac{5}{8} + \frac{1}{2} + \frac{3}{8} + \frac{1}{4} = \frac{7}{4}$

$\|R^i - R^j\|_1 \leq \frac{7}{4}$

$C(\underline{P}^3) = \frac{7}{8}$ $\quad \Rightarrow \quad \underline{P}^3 : x \in \mathcal{J} \longmapsto xP^3 \in \mathcal{J}$

$xP^3 = x$

$$\begin{cases} xP = x \\ \sum_{i=1}^{4} x_i = 1 \end{cases}$$

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$(x_1 \ x_2 \ x_3 \ x_4) P = (x_1 \ x_2 \ x_3 \ x_4)$$

$$\begin{cases} x_2 + \frac{x_3}{4} = x_1 \\ \frac{x_1}{2} + x_4 = x_2 \\ \frac{x_1}{2} + \frac{1}{4}x_3 = x_3 \\ \frac{x_3}{2} = x_4 \\ x_1 + x_2 + x_3 + x_4 = 1 \end{cases}$$

$x_2 = x_1 - \frac{x_3}{4}$

$\frac{x_1}{2} = \frac{3}{4}x_3$

$x_3 = 2x_4$

$x_1 = \frac{3}{2}x_3 = 3x_4$

$x_2 = 3x_4 - \frac{1}{4}2x_4 = \frac{5}{2}x_4$

$3x_4 + \frac{5}{2}x_4 + 2x_4 + x_4 = 1$

$x_4\left(3 + \frac{5}{2} + 2 + 1\right) = 1$

$x_4 \cdot \frac{17}{2} = 1$ $\quad x_4 = \frac{2}{17}$ $\quad x_3 = 2 \cdot \frac{2}{17} = \frac{4}{17}$ $\quad x_2 = \frac{5}{2} \cdot \frac{2}{17} = \frac{5}{17}$

$x_1 = 3 \cdot \frac{2}{17} = \frac{6}{17}$

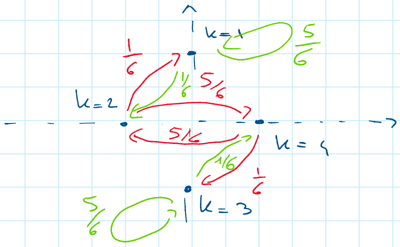
Il ~~petto~~ è $w = \left(\frac{6}{17}, \frac{5}{17}, \frac{4}{17}, \frac{2}{17}\right)$

$e^{ik\frac{\pi}{2}}$ $\quad k = 1, 2, 3, 4$

2 dadi non Truccati $\quad \mathbb{P}(D_1 = D_2) = \frac{1}{6}$ $\quad \mathbb{P}(D_1 \neq D_2) = \frac{5}{6}$

$k = 1,2,3,4$

2 dadi non Truccati $\quad \mathbb{P}(D_1 = D_2) = \frac{1}{6} \quad \mathbb{P}(D_1 \neq D_2) = \frac{5}{6}$



$$e^{ix} = \cos(x) + i\sin(x)$$

$$\begin{pmatrix} \frac{5}{6} & \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & 0 & 0 & \frac{5}{6} \\ 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & \frac{5}{6} & \frac{1}{6} & 0 \end{pmatrix}$$

$k=2 \qquad k=1$

$k=4 \qquad k=3$

$$B = Id + P + P^2 + P^3$$

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.2 & 0.4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad 8 \times 8$$

$$B = \sum_{k=0}^{7} P^k$$

	1	2	3	4	5	6	7	8
1	x	x	x	x	0	0	0	0
2	0	x	x	x	0	0	0	0
3	0	x	x	x	0	0	0	0
4	0	x	x	x	0	0	0	0
5	x	x	x	x	x	0	0	0
6	0	x	x	x	0	x	0	0
7	0	0	0	0	0	0	x	0
8	0	x	x	x	0	0	0	x

$1 \to 1,2,3,4 \qquad 2 \not\to 1 \qquad 1 \in T$

$2 \to 2,3,4 \qquad C_1 = \{2,3,4\}$

$5 \to 1,2,3,4,5 \qquad 1 \not\to 5 \qquad 5 \in T$

$6 \to 2,3,4,6 \qquad 2 \not\to 6 \qquad 6 \in T$

$7 \to 7 \qquad C_2 = \{7\}$

$8 \to 2,3,4,8 \qquad 2 \not\to 8 \qquad 8 \in T$

$C_1 = \{2,3,4\} \quad C_2 = \{7\} \qquad C = C_1 \cup C_2$ – insieme degli stati ricorrenti

$T = \{1,5,6,8\} \qquad T =$ insieme degli stati Transienti.

— ∘ —

$1 \qquad\qquad 2 \qquad\qquad 3 \qquad\qquad 4$

4 monete non Truccate

— ∘ —

n monete, la prob. di ottenere T in ogni singola moneta è p

$k = 0, 1, 2, \ldots n \qquad \mathbb{P}(T = k) = \mathcal{B}(n, p)(\{k\}) = \binom{n}{k} p^k (1-p)^{n-k}$

$p = \frac{1}{2} \quad n = 4 \qquad k = 0, 1, 2, 3, 4$

$$\mathbb{P}(T = k) = \binom{4}{k} \cancel{\left(\frac{1}{2}\right)^k} \left(1 - \frac{1}{2}\right)^{4-\cancel{k}} = \frac{1}{16} \binom{4}{k}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

$\mathbb{P}(T = 0) = \mathbb{P}(T = 4) = \frac{1}{16}$

$\mathbb{P}(T = 1) = \mathbb{P}(T = 3) = \frac{1}{16} \cdot 4 = \frac{1}{4}$

$$\binom{4}{2} = \frac{4 \cdot 3 \cdot 2}{2 \cdot 2} = 6$$

$\mathbb{P}(T = 2) = \frac{6}{16} = \frac{3}{8}$

$i = 1 \quad o \quad i = 3 \qquad \begin{cases} i + t & i + t \leq 4 \\ t & \text{altrimenti} \end{cases}$

$i = 1$ se $t = 0$ rimango in 1 con $\mathbb{P}(T = 0) = \frac{1}{16}$

se $t = 1$ vado in 2 con $\mathbb{P}(T = 1) = \frac{1}{4}$

Se $t = 2$ vado in 3 con $\mathbb{P}(T = 2) = \frac{3}{8}$

Se $t = 3$ vado in 4 con $\mathbb{P}(T = 3) = \frac{1}{4}$

Se $t = 4$ vado in 4 con $\mathbb{P}(T = 4) = \frac{1}{16}$

$i = 3$ $t = 0$ rimango in 3 con $\mathbb{P}(T = 0) = \frac{1}{16}$

$t = 1$ vado in 4 con $\mathbb{P}(T = 1) = \frac{1}{4}$

$t = 2$ vado in 2 con $\mathbb{P}(T = 2) = \frac{3}{8}$

$t = 3$ vado in 3 con $\mathbb{P}(T = 3) = \frac{1}{4}$

$t = 4$ vado in 4 con $\mathbb{P}(T = 4) = \frac{1}{16}$

$i = 2, \ i = 4 \qquad \begin{cases} i - t & \text{se } i - t \geq 1 \\ i & \text{altrimenti} \end{cases}$

$i = 2$ Se $t = 0$ rimango in 2 con $\mathbb{P}(T = 0) = \frac{1}{16}$

Se $t = 1$ vado in 1 con $\mathbb{P}(T = 1) = \frac{1}{4}$

Se $t = 2, 3, 4 \Rightarrow i - t \leq 0 \Rightarrow$ rimango in 2

Se $t=1$ vado in 1 con $\mathbb{P}(T=1)=\frac{1}{4}$

Se $t=2,3,4 \Rightarrow i-t\leq 0 \Rightarrow$ rimango in 2

con $\mathbb{P}(T=2)+\mathbb{P}(T=3)+\mathbb{P}(T=4)=\frac{3}{8}+\frac{1}{4}+\frac{1}{16}=\frac{11}{16}$

$i=4$ Se $t=0$ rimango in 4 con $\mathbb{P}(T=0)=\frac{1}{16}$

Se $t=1$ vado in 3 con $\mathbb{P}(T=1)=\frac{1}{4}$

Se $t=2$ vado in 2 con $\mathbb{P}(T=2)=\frac{3}{8}$

Se $t=3$ vado in 1 con $\mathbb{P}(T=3)=\frac{1}{4}$

Se $t=4$ rimango in 4 con $\mathbb{P}(T=4)=\frac{1}{16}$

— - —

$(X_n)_{n\in\mathbb{N}}$ catena di Markov omogenea e matrice di transizione P indicizzata sull'insieme degli stati S

Per $C\subset S$ $t_C(\omega)$ t_C è una v.a. a valori in

$\rightarrow \mathbb{N}_{\geq 1}\cup\{+\infty\}$

$\mathbb{P}(\cdot \mid X_0=i)=\mathbb{P}_i(\cdot)$

$$E_i[t_C]=\int_\Omega t_C(\omega)\,\mathbb{P}_i(d\omega)$$

Se $C=\{j\}$ e se $i=j$ $\quad E_j[t_j]=\overline{T}_{jj}=\int_\Omega t_j(\omega)\,\mathbb{P}_j(d\omega)$

$$E_i[t_C]=\int_\Omega t_C(\omega)\,\mathbb{P}_i(d\omega)=(+\infty)\,\mathbb{P}_i(t_C=+\infty)+\sum_{k=1}^{\infty}k\,\mathbb{P}_i(t_C=k)$$

$$=\begin{cases}+\infty & \mathbb{P}_i(t_C=+\infty)>0\\ \sum_{k=1}^{\infty}k\,\mathbb{P}_i(t_C=k) & \text{se } \mathbb{P}_i(t_C=+\infty)=0\end{cases}$$

$$=\begin{cases}+\infty & \mathbb{P}_i(t_C<+\infty)<1 \quad f_{iC}<1\\ \sum_{k=1}^{\infty}k\,f_{iC}^{(k)} & \mathbb{P}_i(t_C<+\infty)=1 \quad f_{iC}=1\end{cases}$$

$C=\{j\}$ $i=j$

$$\overline{T}_{jj} = \begin{cases} +\infty & \text{se } f_{jj} < 1 \iff j \text{ transiente} \\ \sum_{k=1}^{\infty} k f_{jj}^{(k)} & \text{se } f_{jj} = 1 \iff j \text{ ricorrente} \end{cases}$$

Se j è ricorrente e $\overline{T}_{jj} < +\infty$, dico che lo stato j è positivamente ricorrente.

— ∘ —

Sia $(X_n)_{n\in\mathbb{N}}$ catena di Markov omogenea con spazio degli stati S (discreto) e matrice di transizione $P = (p_{ij})_{i,j\in S}$. Allora $\forall i,j \in S$ e $\forall n \in \mathbb{N}$

$$p_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} \, p_{jj}^{(n-k)}$$

DIM $E := \{X_n = j\}$ $\quad G := \{X_0 = i\}$

$$p_{ij}^{(n)} = \mathbb{P}(E \mid G)$$

$k = 1 \ldots n$ $\quad F_k = \{X_k = j, X_{k-1} \neq j, \ldots, X_1 \neq j\}$

$$F_k \cap F_\ell = \emptyset \quad \text{se } k \neq \ell$$

$$E = \bigcup_{k=1}^{n} (E \cap F_k)$$

$$p_{ij}^{(n)} = \mathbb{P}(E \mid G) = \mathbb{P}\left(\bigcup_{k=1}^{n} (E\cap F_k) \,\middle|\, G\right) =$$

$$= \sum_{k=1}^{n} \mathbb{P}(E \cap F_k \mid G) = \sum_{k=1}^{n} \frac{\mathbb{P}(E \cap F_k \cap G)}{\mathbb{P}(F_k \cap G)} \frac{\mathbb{P}(F_k \cap G)}{\mathbb{P}(G)}$$

$$= \sum_{k=1}^{n} \underbrace{\mathbb{P}(E \mid F_k \cap G)}_{\mathbb{P}(X_n = j \mid X_k = j)} \cdot \underbrace{\mathbb{P}(F_k \mid G)}_{f_{ij}^{(k)}} = \sum_{k=1}^{n} f_{ij}^{(k)} \, p_{jj}^{(n-k)} \qquad \boxtimes$$

$$\mathbb{P}(X_n = j \mid X_k = j) = \mathbb{P}(X_{n-k} = j \mid X_0 = j) = p_{jj}^{(n-k)}$$

— ∘ —

$$p_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} \, p_{jj}^{(n-k)}$$

In particolare $p_{jj}^{(n)} = \sum_{k=1}^{n} f_{jj}^{(k)} p_{jj}^{(n-k)}$

— o —

PROP (no dim) Siano $(f_n)_{n\in\mathbb{N}}$ e $(p_n)_{n\in\mathbb{N}}$ due successioni reali non negative che hanno le seguenti proprietà

→ $\sum_{n=1}^{\infty} f_n = 1$

→ $p_0 = 1$ → $p_n = \sum_{k=1}^{n} f_k p_{n-k}$ $\forall n \geq 1$

Se $\exists \lim_{n\to\infty} p_n = \lambda$, allora $\lambda = \dfrac{1}{\sum_{k\geq 1}^{\infty} k f_k}$ —

— o —

Fisso $j \in S$ pongo $f_n = f_{jj}^{(n)}$ e $p_n = p_{jj}^{(n)}$

Se j è stato ricorrente allora valgono tutte le ipotesi della proposizione. Quindi,

Se $\exists \lim_{n\to\infty} p_{jj}^{(n)} =: w_j$, allora $w_j = \dfrac{1}{l_{jj}}$

— o —

Se $\exists \lim_{n\to\infty} p_{jj}^{(n)} = w_j$, allora $\exists \lim_{n\to\infty} p_{ij}^{(n)} = w_j f_{ij}$ ←

Se $\exists \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} p_{jj}^{(k)} = w_j$, allora $\exists \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} p_{ij}^{(k)} = w_j f_{ij}$

DIM $\mathbb{N}_{\geq 1}$ $\quad \mathcal{E} = \mathcal{P}(\mathbb{N}_{\geq 1})$ $\quad \mu(\{n\}) = 1$

$E \subset \mathbb{N}_{\geq 1}$ $\quad \mu(E) = \#$ degli elementi di E

$k \in \mathbb{N}_{\geq 1}$ $\quad f_n(k) = \begin{cases} f_{ij}^{(k)} p_{jj}^{(n-k)} & k = 1 - n \\ 0 & k \geq n+1 \end{cases}$

$$p_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)} = \sum_{k=1}^{\infty} f_n(k) = \int_{\mathbb{N}_{\geq 1}} f_n(k)\,\mu(dk)$$

$$\lim_{n\to\infty} f_n(k) = \lim_{n\to\infty} f_{ij}^{(k)} p_{jj}^{(n-k)} = f_{ij}^{(k)} w_j$$

$$|\varphi_n(k)| \begin{cases} 0 & \text{se } k \geq n+1 \\ = f_{ij}^{(k)} \, p_{jj}^{(n-k)} & \text{se } k \leq n \end{cases}$$

$$|\varphi_n(k)| \leq f_{ij}^{(k)} \qquad \forall k \in \mathbb{N}_{\geq 1}$$

$$\psi(k) = f_{ij}^{(k)} \qquad \forall k \in \mathbb{N}_{\geq 1}$$

$$\int_{\mathbb{N}_{\geq 1}} \psi(k)\mu(dk) = \sum_{k=1}^{\infty} \psi(k) = \sum_{k=1}^{\infty} f_{ij}^{(k)} = f_{ij} \leq 1 < +\infty$$

Posso applicare il Teorema di convergenza dominata di Lebesgue

$$p_{ij}^{(n)} = \int_{\mathbb{N}_{\geq 1}} \varphi_n(k)\mu(dk) \xrightarrow[n\to\infty]{} \int_{\mathbb{N}_{\geq 1}} \left(\lim_{n\to\infty} \varphi_n(k)\right)\mu(dk)$$

$$= \int_{\mathbb{N}_{\geq 1}} f_{ij}^{(k)} w_j \, \mu(dk)$$

$$= w_j \sum_{k=1}^{\infty} f_{ij}^{(k)} = w_j f_{ij}$$

TEOREMA Sia $(X_n)_{n\in\mathbb{N}}$ catena di Markov omogenea con spazio degli stati discreto S e matrice di transizione $P=(p_{ij})_{i,j\in S}$

Siano $i,j \in S$.

Se $\exists \lim_{n\to\infty} p_{jj}^{(n)} =: w_j$, allora $\begin{cases} w_j = \dfrac{1}{\overline{T_{jj}}} \\ p_{ij}^{(n)} \to f_{ij} w_j \end{cases}$ ←

DIM Rimane solo da dimostrare $w_j = \dfrac{1}{\overline{T_{jj}}}$ nel caso j transiente

Se j è transiente, so che $\sum_{n=1}^{\infty} p_{jj}^{(n)} < +\infty$ e quindi sicuramente

$$\exists \lim_{n\to\infty} p_{jj}^{(n)} = 0$$

D'altra parte, se j è transiente $\Rightarrow \overline{T_{jj}} = +\infty$ "=" $\dfrac{1}{0}$