APPLICAZIONI DELLA LEGGE DEI GRANDI NUMERI

Note Title 03/10/2019

$Q = [0,1]^N$ $\qquad f \in C^0(Q)$ $\qquad \int_Q f(x)\,dx = \int_{\mathbb{R}^N} f(x)\, \mathcal{L}^N\big|_Q (dx)$

Divido ciascun lato in K parti uguali $\Rightarrow$ ho suddiviso Q in K^N parti uguali, chiamandole Q_i $\quad i = 1 \ldots K^N$

Per ogni $i = 1 \ldots K^N$ considero un p.to $x_i \in Q_i$ e calcolo $\sum_{i=1}^{K^N} \frac{1}{K^N} f(x_i)$

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$(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N), \mu)$ $\qquad f : \mathbb{R}^N \to \mathbb{R}$ limitata e Borel misurabile

voglio calcolare $\int_{\mathbb{R}^N} f(t_1 \ldots t_N)\, \mu(dt_1 \ldots dt_N) =$

$$= \int_{\mathbb{R}^N} f(t)\, \mu(dt)$$

$(\Omega, \mathcal{E}, \mathbb{P})$, $\{X_i\}_{i\in\mathbb{N}}$ successione di v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ t.c. le $\{X_i\}_{i\in\mathbb{N}}$ sono i.i.d. $\mathbb{P}_{X_i} = \mu$ $\qquad$ N.B. $X_i : \Omega \to \mathbb{R}^N$

Pongo $Y_i = f \circ X_i$. Le Y_i sono i.i.d. $\quad Y_i : \Omega \to \mathbb{R}$

$$\mathbb{E}[|Y_i|] = \int_\Omega |f(X_i(\omega))|\, \mathbb{P}(d\omega) = \int_{\mathbb{R}^N} |f(t)|\, \underbrace{\mathbb{P}_{X_i}(dt)}_{\mu(dt)\ \forall i}$$

$$= \int_{\mathbb{R}^N} |f(t)|\, \mu(dt)$$

f è limitata $\Rightarrow \exists M$ t.c. $|f(t)| \le M \quad \forall t \in \mathbb{R}^N$

$$\le \int_{\mathbb{R}^N} M\, \mu(dt) = M \int_{\mathbb{R}^N} \mu(dt)$$

$$= M\mu(\mathbb{R}^N) = M \cdot 1 = M.$$

$$\mathbb{E}[Y_i] = \int_\Omega f(X_i(\omega))\,\mathbb{P}(d\omega) = \int_{\mathbb{R}^N} f(t)\,\overbrace{\mathbb{P}_{X_i}(dt)}^{\mu(dt)} =$$

$$= \int_{\mathbb{R}^N} f(t)\,\mu(dt) = \text{integrale che voglio calcolare, lo indico } I$$

$$\mathrm{Var}[Y_i] = \int_\Omega |Y_i(\omega) - I|^2\,\mathbb{P}(d\omega) = \int_\Omega (f(X_i(\omega)) - I)^2\,\mathbb{P}(d\omega)$$

$$= \int_{\mathbb{R}^N} (f(t) - I)^2\,\underbrace{\mathbb{P}_{X_i}(dt)}_{\mu(dt)} = \int_{\mathbb{R}^N} (f(t) - I)^2\,\mu(dt)$$

$$(f(t) - I)^2 = f^2(t) - 2I f(t) + I^2 \le \underbrace{M^2 + 2|I|M + I^2}_{C \text{ costante}}$$

$$\mathrm{Var}[Y_i] \le \int_{\mathbb{R}^N} C\,\mu(dt) = C\mu(\mathbb{R}^N) = C \cdot 1 = C$$

$$|I| = \left|\int_{\mathbb{R}^N} f(t)\,\mu(dt)\right| \le \int_{\mathbb{R}^N} |f(t)|\,\mu(dt) \le \int_{\mathbb{R}^N} M\,\mu(dt) = M$$

$$\mathrm{Var}[Y_i] \le M^2 + 2|I|M + I^2 \le M^2 + 2M^2 + M^2 = 4M^2$$

$$S_n = \sum_{i=1}^n Y_i \qquad \bar{Y}_n = \frac{1}{n} S_n = \frac{1}{n}\sum_{i=1}^n Y_i$$

converge $\mathbb{P}$-qc, in media quadratica e in probabilità a $\mathbb{E}[Y_i] = I$

$$\mathbb{P}(|\bar{Y}_n - I| \ge \delta) \le \frac{4M^2}{n\delta^2} \qquad \forall \delta > 0$$

$$\mathbb{P}(|\bar{Y}_n - I| < \delta) \ge 1 - \frac{4M^2}{n\delta^2} \qquad \forall \delta > 0$$

$$\delta = 10^{-2} \qquad 1 - \frac{4M^2}{n\delta^2} \ge 95\%$$

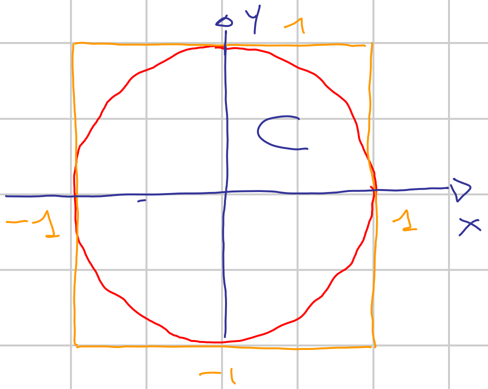
$$\frac{4M^2}{n\delta^2} \le 5\% = \frac{5}{100} \qquad n \ge \frac{4M^2}{\delta^2}\,\frac{1}{5/100}$$

Fisso $\alpha \in (0,1)$ $\qquad \mathbb{P}\left(|\bar{Y}_n - I| < \delta\right) \ge 1-\alpha$

$$1 - \frac{4M^2}{n\delta^2} \ge 1-\alpha \qquad \frac{4M^2}{n\delta^2} \le \alpha$$

$$n \ge \frac{4M^2}{\alpha\delta^2}$$

π = Area del cerchio di raggio 1 $\qquad C = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\}$



π = Area di C =

$$= \int_{\mathbb{R}^N} \mathbb{1}_C(x,y)\,dx\,dy$$

$$Q = [-1,1]^2, \quad C \subset Q$$

$$= \int_Q \mathbb{1}_C(x,y)\,dx\,dy = \int_{\mathbb{R}^N} 4\,\mathbb{1}_C(x,y)\,\frac{1}{4}\mathcal{L}^2|_Q(dx\,dy)$$

$$\mathcal{L}^2(Q) = 2\cdot 2 = 4 \qquad \mu = \frac{1}{4}\mathcal{L}^2|_Q = U([-1,1]) \times U([-1,1])$$

$$= \int_{\mathbb{R}^N} 4\,\mathbb{1}_C(x,y)\,\mu(dx\,dy) \qquad (X_i, Y_i)$$

Genero $\{(X_i,Y_i)\}_{i=1}^{\infty}$ i.i.d $\quad \mathbb{P}_{X_i} = \mathbb{P}_{Y_i} = U([-1,1])$

$$\frac{1}{n}\sum_{i=1}^{n} 4\,\mathbb{1}_C\big(X_i(\omega), Y_i(\omega)\big) = \frac{4}{n}\sum_{i=1}^{n} \mathbb{1}_C\big(X_i(\omega), Y_i(\omega)\big)$$

$$\frac{\pi}{6} = \arcsin\left(\frac{1}{2}\right) - \arcsin(0) = \int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = \int_0^{1/2} \frac{1}{2}\frac{1}{\sqrt{1-x^2}}\,(2\,dx)$$

$$= \int_{\mathbb{R}} \frac{1}{2}\frac{1}{\sqrt{1-x^2}}\, \underbrace{2\,\mathbb{1}_{[0,\frac{1}{2}]}(x)}_{}\,dx$$

densità di $U([0,\tfrac{1}{2}]) =: \mu$

$$= \int_{\mathbb{R}} \frac{1}{2\sqrt{1-x^2}}\,\mu(dx)$$

$$\pi = \int_{\mathbb{R}} \frac{3}{\sqrt{1-x^2}}\,\mu(dx) \qquad f(x) = \frac{3}{\sqrt{1-x^2}}$$

$\{X_i\}_{i=1}^{\infty}$ v.a. i.i.d. $\quad \mathbb{P}_{X_i} = U([0,\tfrac{1}{2}])$

$$\frac{1}{n}\sum_{i=1}^{n} \frac{3}{\sqrt{1-X_i^2(\omega)}}$$

converge $\mathbb{P}$-qc, in media quadratica e in probabilità a π.

FUNZIONE DI RIPARTIZIONE EMPIRICA

Sia X v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$. Considero una successione di v.a. $\{X_i\}_{i=1}^{\infty}$ su $(\Omega, \mathcal{E}, \mathbb{P})$ sono i.i.d. e $\mathbb{P}_{X_i} = \mathbb{P}_X$ $\forall i$

Fisso $t \in \mathbb{R}$

$$Y_i(\omega) := \begin{cases} 1 & \text{se } X_i(\omega) \le t \\ 0 & \text{se } X_i(\omega) > t \end{cases} = \mathbb{1}_{\{X_i \le t\}}(\omega)$$

Le Y_i sono sicuramente indipendenti. Sono anche identicamente distribuite.

$\mathbb{P}_{Y_i} = B(p)$ per un certo $p \in [0,1]$

$p = \mathbb{P}(Y_i = 1) = \mathbb{P}(X_i \le t) = F_{X_i}(t) = F_X(t) \quad \forall i$

$\mathbb{E}[Y_i] = p = F_X(t)$

$\mathrm{Var}[Y_i] = p(1-p) = F_X(t)\,(1 - F_X(t)) \le \frac{1}{4}$

Applico la legge dei grandi numeri a $\{Y_i\}_{i=1}^{\infty}$

$$T_{n,t}(\omega) = S_n^{(\omega)} = \sum_{i=1}^{n} Y_i(\omega) = \#\{j \in \{1 \dots n\} : X_j(\omega) \le t\}$$

$\overline{Y}_n^{(\omega)} = \frac{1}{n} S_n(\omega)$ converge $\mathbb{P}$-qc, in media quadratica e in probabilità a $F_X(t)$

$= \frac{1}{n} T_{n,t}(\omega)$

$$\mathbb{P}\left(\left| \frac{1}{n} T_{n,t} - F_X(t) \right| > \delta \right) \le \frac{F_X(t)(1-F_X(t))}{n\delta^2} \quad \forall t \in \mathbb{R} \ \forall n \in \mathbb{N} \ \forall \delta > 0$$

$$\mathbb{P}\left(\left| \frac{1}{n} T_{n,t} - F_X(t) \right| > \delta \right) \le \frac{1}{4n\delta^2} \quad \forall t \in \mathbb{R} \ \forall n \in \mathbb{N} \ \forall \delta > 0$$

$$\sup_{t \in \mathbb{R}} \mathbb{P}\left(\left| \frac{1}{n} T_{n,t} - F_X(t) \right| > \delta \right) \le \frac{1}{4n\delta^2} \quad \forall n \in \mathbb{N}, \ \forall \delta > 0$$

$$\lim_{n \to \infty} \sup_{t \in \mathbb{R}} \mathbb{P}\left(\left| \frac{1}{n} T_{n,t} - F_X(t) \right| > \delta \right) = 0 \quad \forall \delta > 0$$

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CONVERGENZA PUNTUALE $\quad f_n : \mathbb{R} \to \mathbb{R} \quad f : \mathbb{R} \to \mathbb{R}$

$\forall x \in \mathbb{R} \quad \exists \lim_{n \to \infty} f_n(x) = f(x)$

$a_n = f_n(x)$

$\forall x \in \mathbb{R} \ \forall \varepsilon > 0 \ \exists \bar{n} = \bar{n}(\varepsilon, x)$ t.c. $\forall n > \bar{n} \ |f_n(x) - f(x)| < \varepsilon$

CONVERGENZA UNIFORME

$\forall \varepsilon > 0 \ \exists \bar{n} = \bar{n}(\varepsilon)$ t.c. $\forall n > \bar{n}$ e $\forall x \in \mathbb{R}$

$|f_n(x) - f(x)| < \varepsilon$

$\forall \varepsilon > 0 \ \exists \bar{n} = \bar{n}(\varepsilon)$ t.c. $\forall n > \bar{n} \quad \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| < \varepsilon$

$f_n : x \in [0, 1] \to x^n \in \mathbb{R}$

f_n converge puntualmente a $\begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases}$

Si dimostra che la convergenza non è uniforme