martedì 3 marzo 2020 09:43

FUNZIONI ELEMENTARI

Funzioni potenza $f(x) = kx^{\alpha}$ $k \in \mathbb{R}$, $\alpha \in \mathbb{R}$

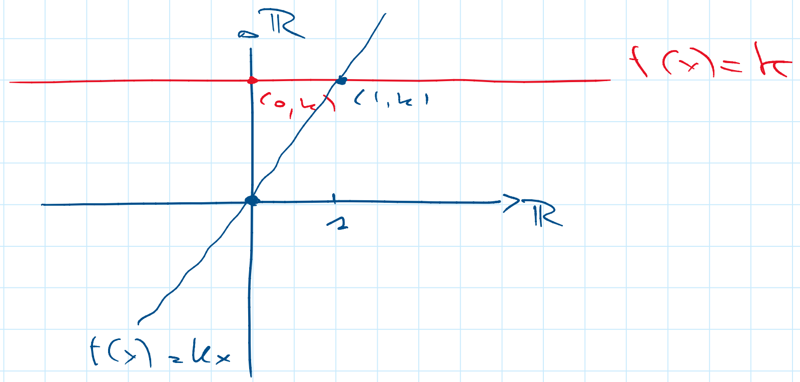
1) $\alpha = 0$ $f(x) = k$ $D = \mathbb{R}$

2) $\alpha = 1$ $f(x) = kx$ $D = \mathbb{R}$ funzioni lineari

$$\frac{f(x)}{x} = k \quad \forall x \in \mathbb{R} \setminus \{0\}$$

$f(x) = k$

$f(x) = kx$

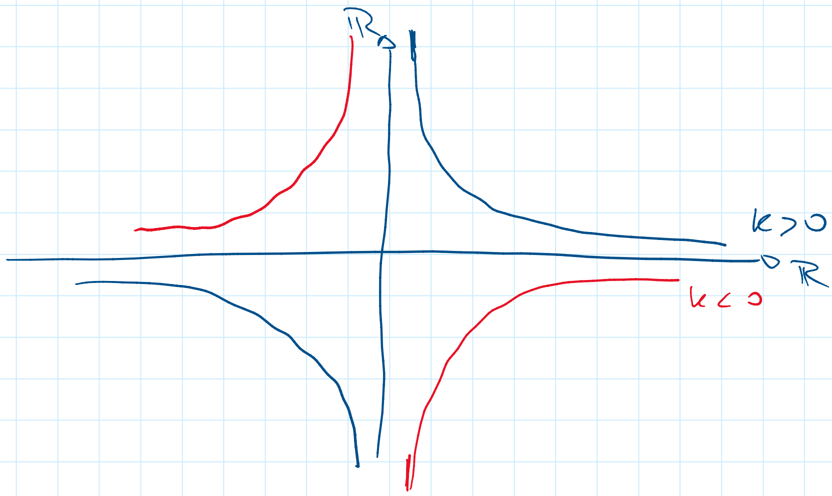


3) $\alpha = -1$ $f(x) = \frac{k}{x}$ $D = \mathbb{R} \setminus \{0\}$ DISPARI

$f(-x) = -f(x)$

$k > 0$ $f(x) > 0$ sse $x > 0$

$k < 0$ $f(x) > 0$ sse $x < 0$



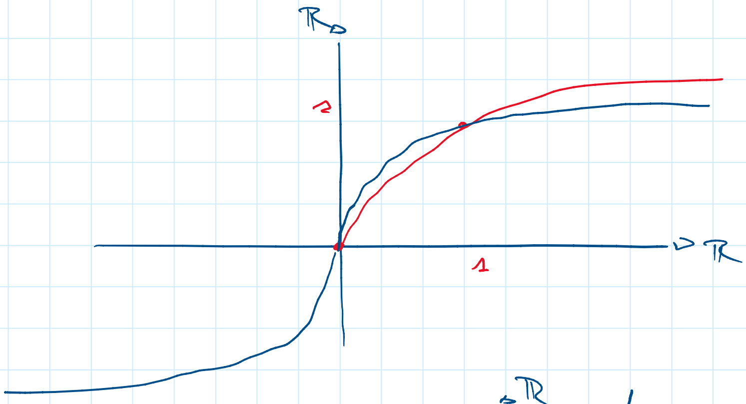
4) $\alpha \in \mathbb{Q}$

$\alpha = \frac{1}{2}$ $f(x) = k\sqrt{x}$ $D = [0, +\infty)$

$\alpha = \frac{1}{3}$ $f(x) = k\sqrt[3]{x}$ ~~$D = [0, +\infty)$~~ $D = \mathbb{R}$ ←

$\alpha = \frac{1}{3}$ $\quad f(x) = k\sqrt[3]{x}$ $\quad \cancel{D = [0, +\infty)}$ $\quad D = \mathbb{R} \leftarrow$

$\alpha = -\frac{1}{2}$ $\quad f(x) = k\, x^{-1/2} = \frac{k}{\sqrt{x}}$ $\quad D = (0, +\infty)$

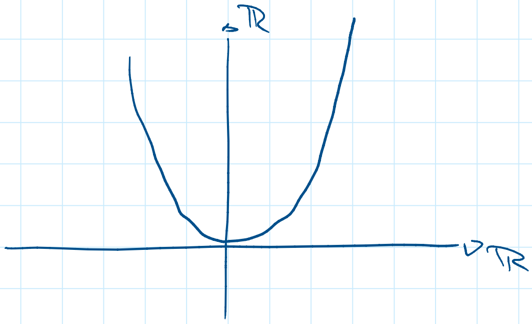


$f(x) = \sqrt{x}$

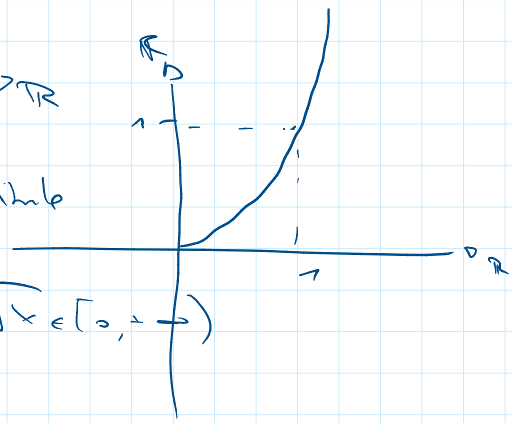
$f(x) = x^{1/3}$

$(k = 1)$

$g(x) = x^2$



$f(1) = f(-1) = 1$

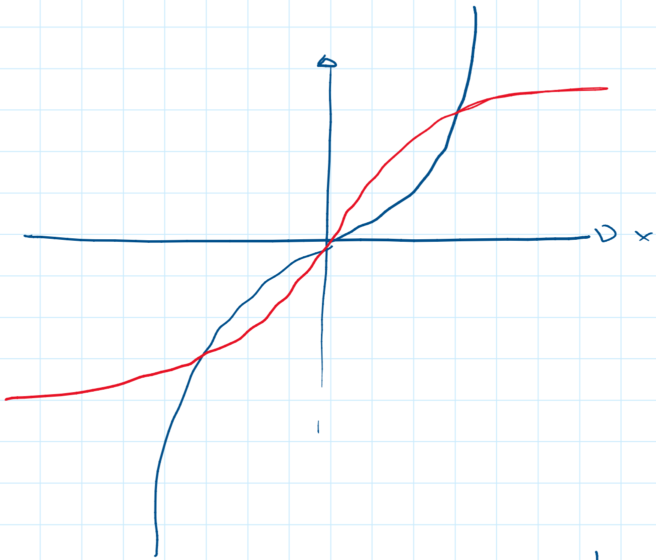


$\tilde{g}: x \in [0, +\infty) \mapsto x^2 \in [0, +\infty)$ è invertibile

La sua inversa è $f: x \in [0, +\infty) \mapsto \sqrt{x} \in [0, +\infty)$

$g(x) = x^3$

è DISPARI

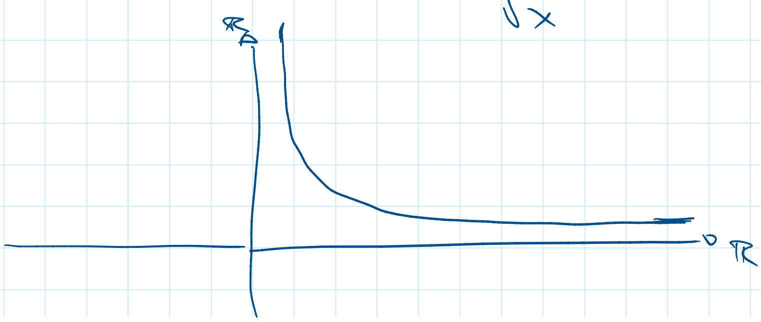


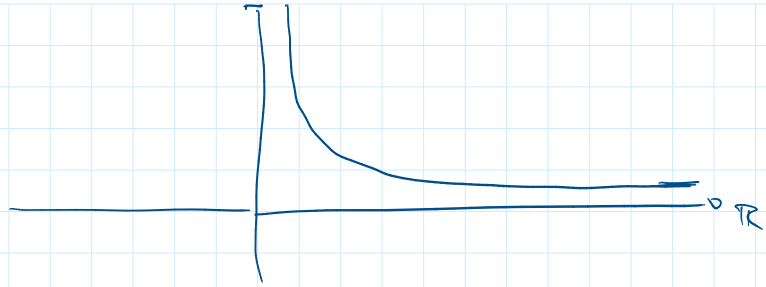
$g: \mathbb{R} \to \mathbb{R}$ invertibile

$f: x \in \mathbb{R} \mapsto x^{1/3} \in \mathbb{R}$

è solo la sua inversa

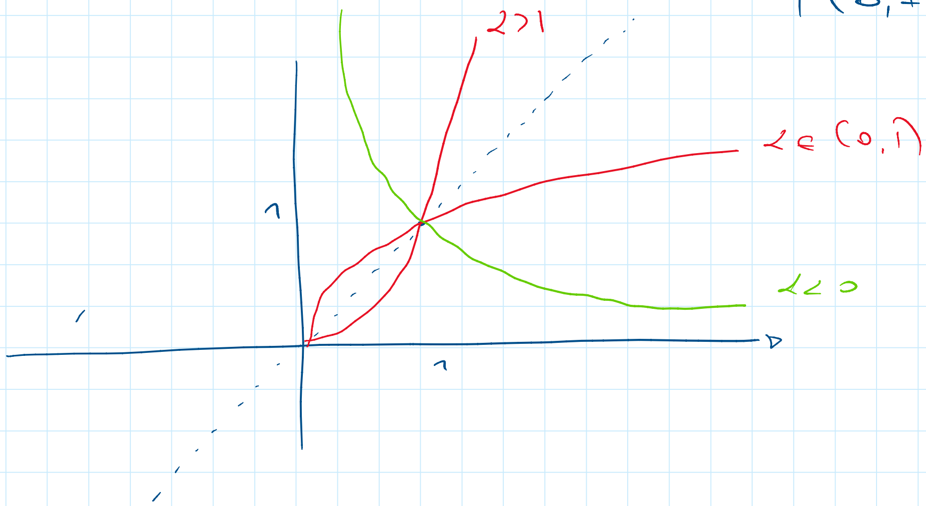
3) $\alpha \in \mathbb{Q}$ $\quad \alpha < 0$ $\quad f(x) = x^{-1/2} = \frac{1}{\sqrt{x}}$ $\quad D = (0, +\infty)$





4) $\alpha \in \mathbb{R}$ $\quad f(x) = kx^\alpha$ $\quad D = \begin{cases} [0, +\infty) & \text{se } \alpha > 0 \\ (0, +\infty) & \text{se } \alpha < 0 \end{cases}$

$k = 1$



FUNZIONI LOGARITMICHE e FUNZIONI ESPONENZIALI

$a > 0$ e $a \neq 1$ $\quad f(x) = \log_a(x)$ $\quad D = (0, +\infty)$

$g(x) = a^x$ $\quad D = \mathbb{R}$

$g(x) = a^x$

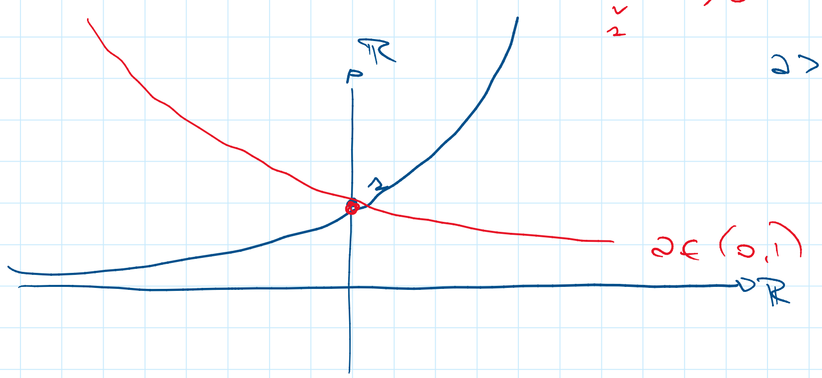
$a > 1$ e $x_1 < x_2$ $\Rightarrow$ $a^{x_1} < a^{x_2}$ str. monotona crescente

$a^{x_2} > a^{x_1}$ sse $\dfrac{a^{x_2}}{a^{x_1}} > \dfrac{a^{x_1}}{a^{x_1}}$ poiché $a^{x_1} > 0$

cioè $a^{\underbrace{x_2 - x_1}_{> 0}} > 1$ vera

$a > 1$

$a^x = \left(\frac{1}{a}\right)^{-x}$

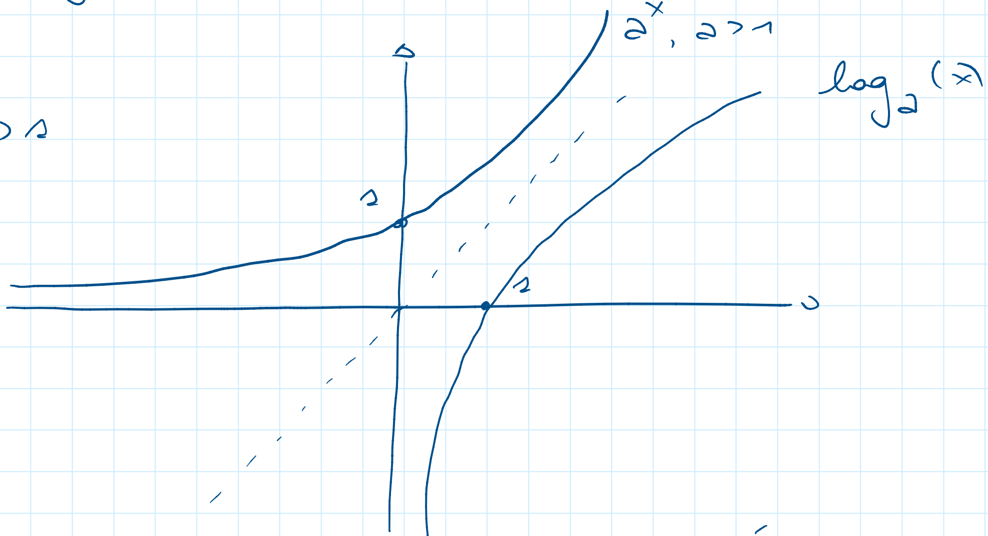


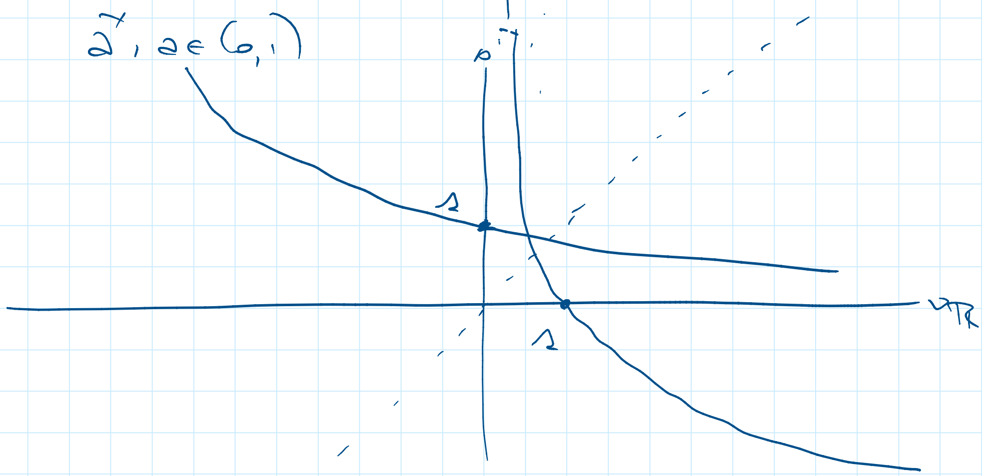
Se $a \in (0,1)$ $\quad g(x) = a^x = \left(\frac{1}{a}\right)^{-x} \quad \frac{1}{a} > 1$

$f(x) = \log_a(x) \quad D = (0, +\infty)$

$g : y \in \mathbb{R} \mapsto a^y \in (0, +\infty) \implies f$ è l'inversa di g

$a > 1$



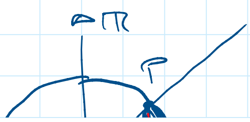


OSSERVAZIONE $\quad \log_a(a^x) = x \quad \forall x \in \mathbb{R}$

$a^{\log_a(x)} = x \quad \forall x \in (0, +\infty)$

FUNZIONI TRIGONOMETRICHE

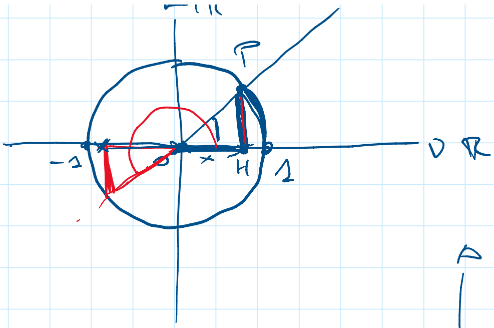
$f(x) = \sin(x) \quad g(x) = \cos(x)$



$PH = \sin(x) \quad D = \mathbb{R}$

$OH = \cos(x) \quad$ Periodiche di

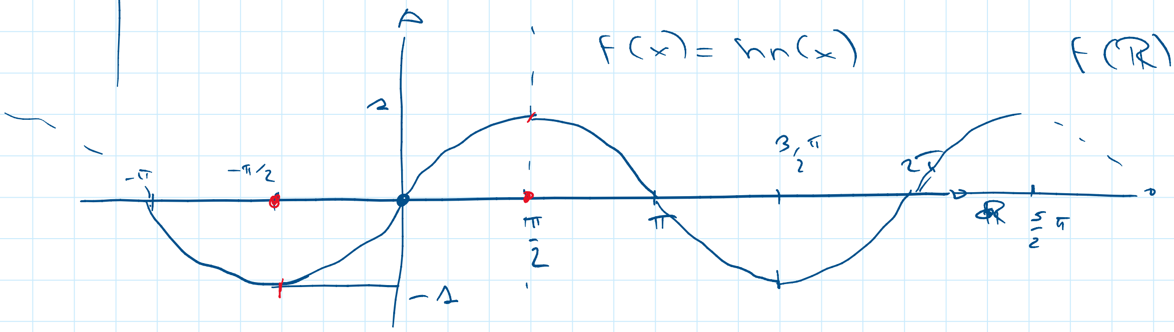
$PH = \sin(x)$ $\quad$ $D = \mathbb{R}$



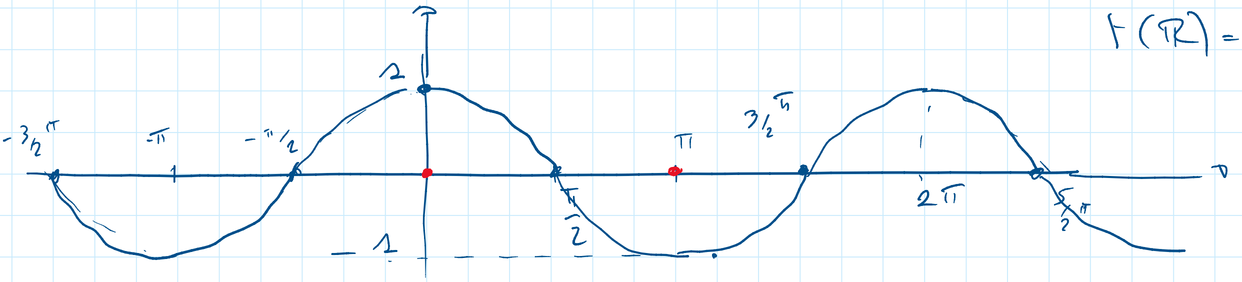
$OH = \cos(x)$

Periodiche di periodo 2π

$f(x) = \sin(x)$ $\qquad$ $f(\mathbb{R}) = [-1, 1]$



$f(\mathbb{R}) = [-1, 1]$



$$\cos^2(x) + \sin^2(x) = 1 \qquad \forall x \in \mathbb{R}$$

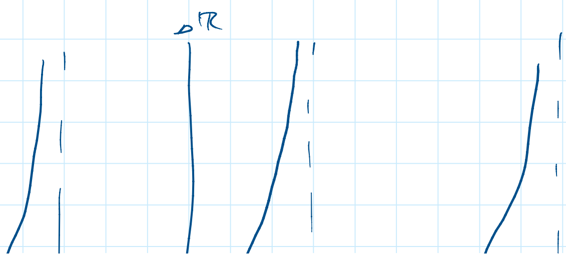
$$h(x) = \frac{\sin(x)}{\cos(x)} \qquad D = \{x \in \mathbb{R} : \cos(x) \neq 0\} =$$

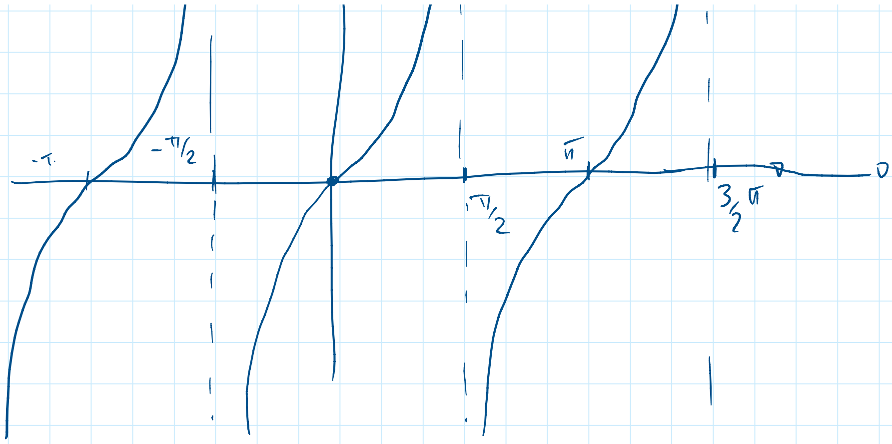
$$= \mathbb{R} \setminus \left\{(2k+1)\frac{\pi}{2} : k \in \mathbb{Z}\right\}$$

$$h(x+\pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{-\sin(x)}{-\cos(x)} = h(x) \Rightarrow \text{periodica di periodo } \pi$$

La funzione h si indica col simbolo tg e si chiama Tangente di x

$$tg(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin(x)}{\cos(x)} = -Tg(x)$$



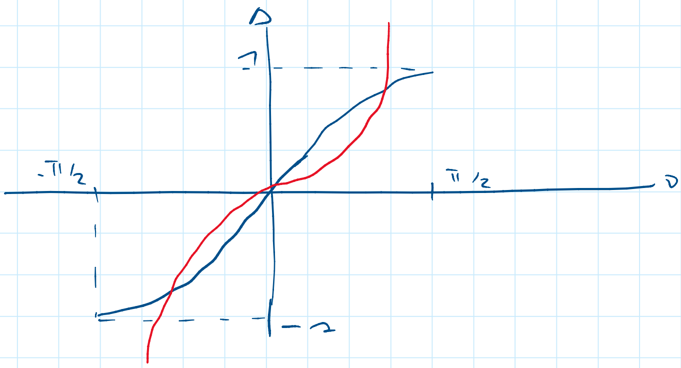


$$K(x) = \mathrm{cotg}(x) := \frac{\cos(x)}{\sin(x)}$$ Per esercizio

— o —

FUNZIONI TRIGONOMETRICHE INVERSE

$\tilde{f} : x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \longmapsto \sin(x) \in [-1, 1]$ è invertibile



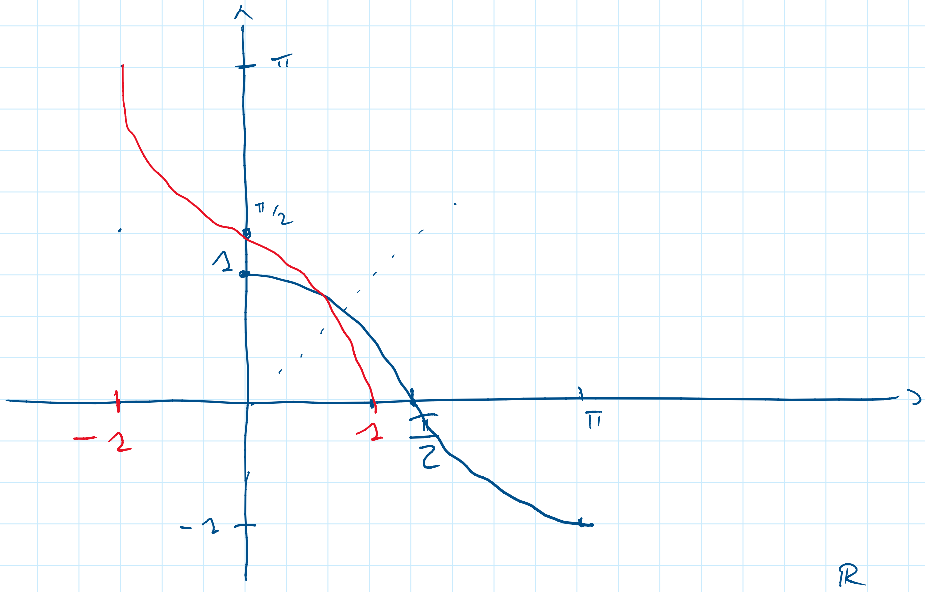
La funzione inversa $g : x \in [-1, 1] \longmapsto g(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ si chiama ARCO IL CUI SENO e si indica

$$\arcsin : x \in [-1, 1] \longmapsto \arcsin(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

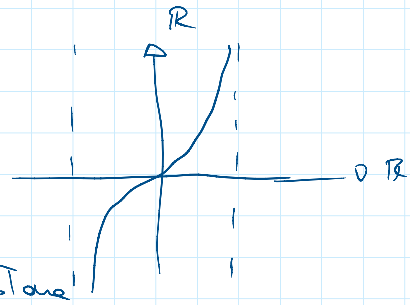
$\tilde{f} : x \in [0, \pi] \longmapsto \cos(x) \in [-1, 1]$ è strettamente monotona decrescente e invertibile.

La funzione inversa si chiama ARCO IL CUI COSENO e si scrive e si scrive

$$\arccos : x \in [-1,1] \longmapsto \arccos(x) \in [0,\pi]$$



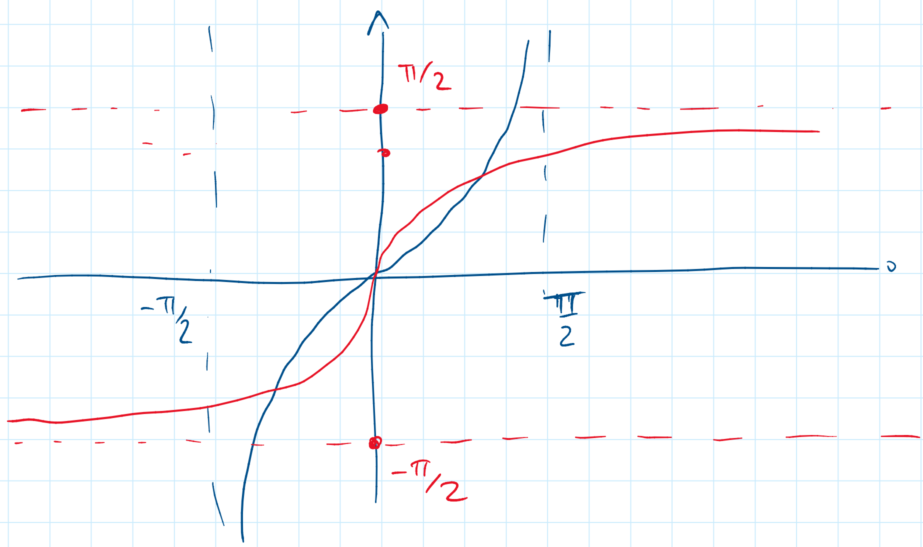
$$\tilde{f} : x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow Tg(x) \in \mathbb{R}$$



È invertibile e strettamente monotona crescente.

La sua inversa si chiama ARCO LA CUI TANGENTE e si indica

$$\operatorname{arctg} : x \in \mathbb{R} \longmapsto \operatorname{arcTg}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

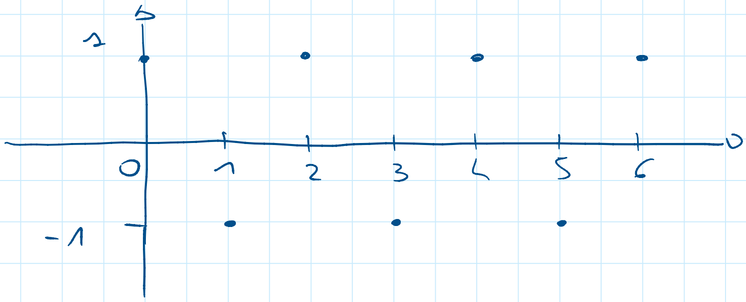


SUCCESSIONI

Una funzione $f : \mathbb{N} \to \mathbb{R}$ si dice una SUCCESSIONE

Si indicano $\{a_n\}_{n\in\mathbb{N}}$ dove $a_n := f(n)$

ESEMPIO $a_n = (-1)^n$

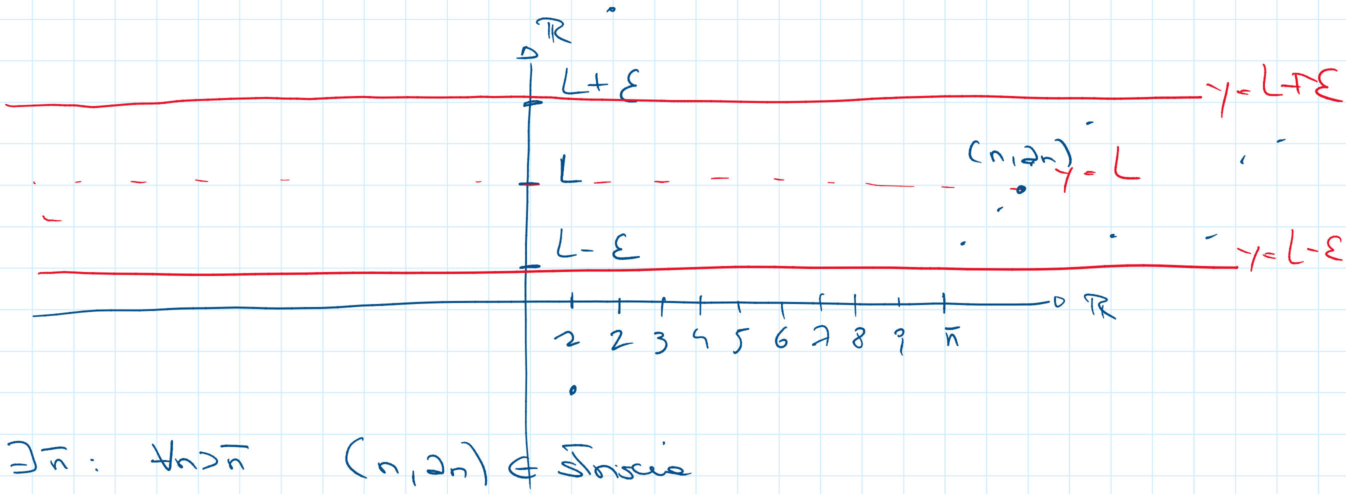


Sia $\{a_n\}_{n\in\mathbb{N}}$ una successione a valori in $\mathbb{R}$ e sia $L \in \mathbb{R}$.

Dico che $\lim_{n\to\infty} a_n = L$ se

$$\forall \varepsilon > 0 \quad \exists \bar{n} \in \mathbb{N} \quad \text{t.c.} \quad \forall n \geq \bar{n} \quad |a_n - L| < \varepsilon$$

$|a_n - L| < \varepsilon$ si scrive anche $L - \varepsilon < a_n < L + \varepsilon$

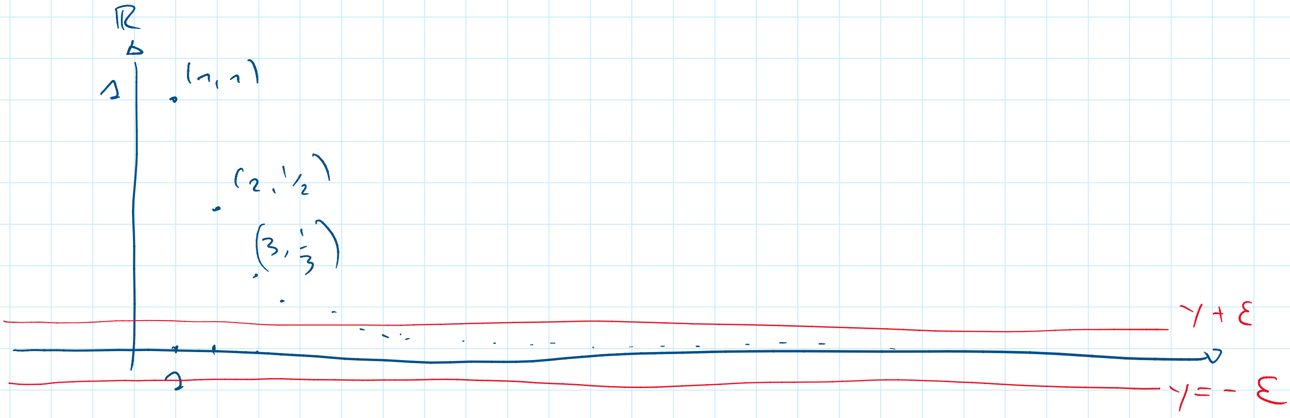


$\exists \bar{n} : \forall n > \bar{n} \quad (n, a_n) \in$ striscia

$a_n = \frac{1}{n} \quad n \geq 1$

$L = 0$

$\varepsilon > 0 \quad \exists \bar{n} \quad \text{t.c.} \quad \forall n \geq \bar{n} \quad 0 - \varepsilon < \frac{1}{n} < 0 + \varepsilon \quad \longleftarrow$



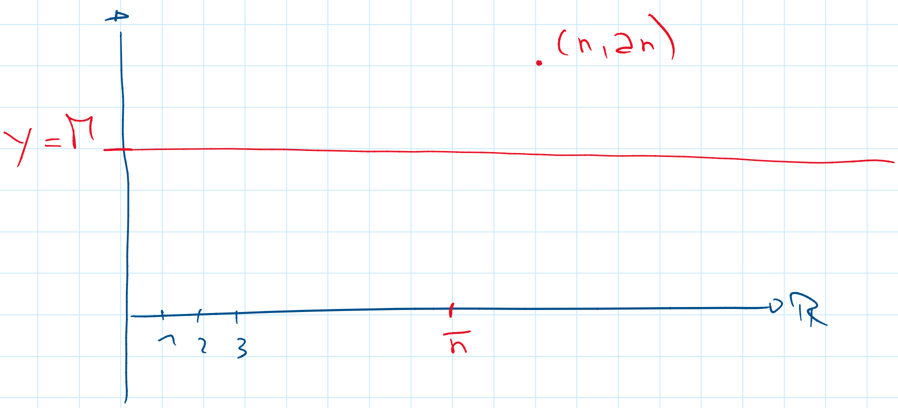
$\frac{1}{n} > -\varepsilon$ vero $\forall n \geq 1$

$\frac{1}{n} < \varepsilon$ è vero sse $n > \frac{1}{\varepsilon}$ basta prendere

$\bar{n} = 1 + \left\lfloor \frac{1}{\varepsilon} \right\rfloor$

Scrivo che $\lim_{n \to \infty} a_n = +\infty$ e dico che a_n DIVERGE A $+\infty$ se

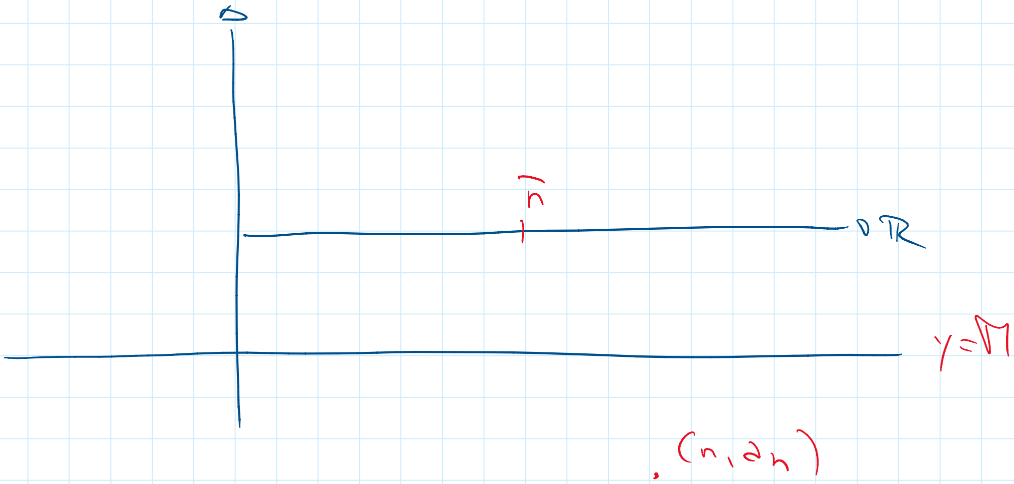
$\forall M \in \mathbb{R} \quad \exists \bar{n} : \forall n \geq \bar{n} \quad a_n > M$



$a_n = n \qquad a_n = n^2 \qquad a_n = 2^n$

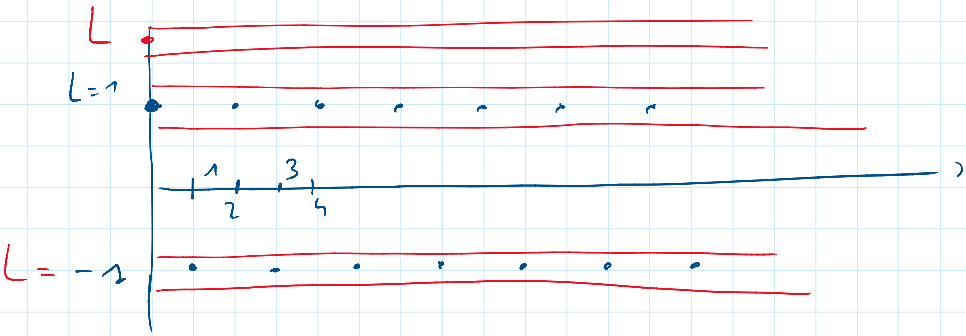
Scrivo che $\lim_{n \to \infty} a_n = -\infty$ e dico che a_n DIVERGE A $-\infty$.

se $\forall M \in \mathbb{R} \quad \exists \bar{n} \in \mathbb{N} : \quad \forall n \geq \bar{n} \quad a_n < M$



$a_n = -n \qquad a_n = -n^2 \qquad a_n = -2^n$

$a_n = (-1)^n$



TEOREMA (NO DIM) Se $(a_n)_{n \in \mathbb{N}}$ è una successione monotona crescente (cioè se $a_n \leq a_{n+1} \quad \forall n \in \mathbb{N}$) allora

1) Se la successione è limitata superiormente (cioè se $\exists M \in \mathbb{R}$ t.c. $a_n \leq M \quad \forall n \in \mathbb{N}$) allora

$$\exists \lim_{n \to \infty} a_n = \sup \{ a_n : n \in \mathbb{N} \}$$

2) Se la successione non è limitata superiormente, allora $\lim_{n \to \infty} a_n = +\infty$

$n \to \infty$

Se $(a_n)_{n\in\mathbb{N}}$ è una successione monotona decrescente, allora

1) Se la successione è limitata inferiormente, allora

$$\exists \lim_{n\to\infty} a_n = \inf\{a_n : n\in\mathbb{N}\}$$

2) Se la successione non è limitata inferiormente, allora $\exists \lim_{n\to\infty} a_n = -\infty$

ESEMPIO $a_n = \frac{1}{n}$

LIMITE DI FUNZIONE

Sia $I \subset \mathbb{R}$, I intervallo, o una semiretta o retta

Sia $c \in \mathbb{R}$ t.c.

se I è un intervallo o una semiretta, allora c appartiene ad I o è un estremo di I

$I = [a,b)$ $\qquad$ $I = (a,b)$

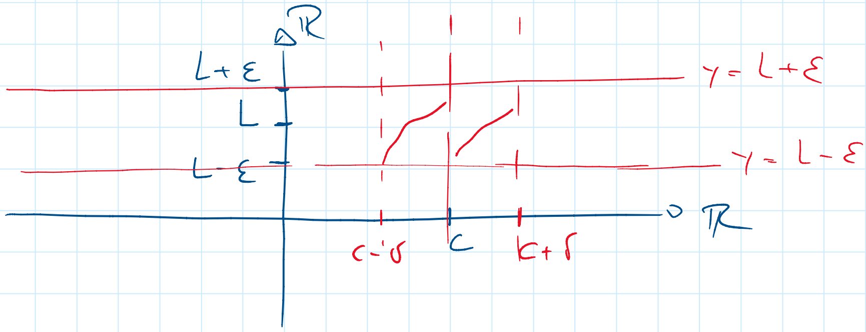
Sia $f : I \setminus \{c\} \to \mathbb{R}$ o $f : I \to \mathbb{R}$

Sia $L \in \mathbb{R}$

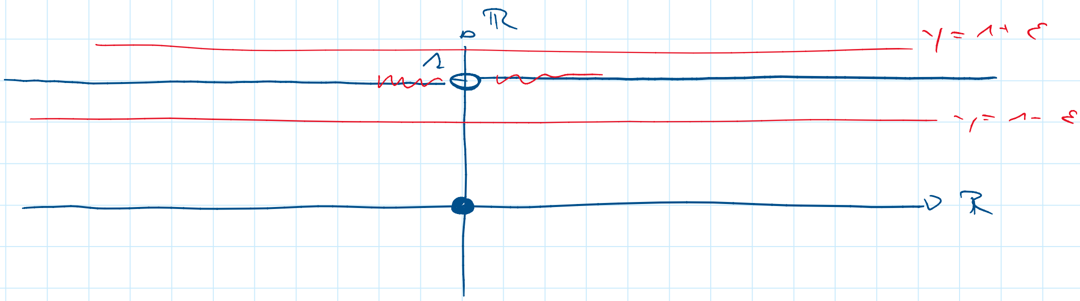
Scrivo che $\lim_{x\to c} f(x) = L$ e dico che $f(x)$ converge a L quando x tende a c se

$\forall \varepsilon > 0 \quad \exists \delta > 0$ t.c. $\forall x \in (c-\delta, c+\delta) \cap I$, $x \neq c$

si ha $|f(x) - L| < \varepsilon$

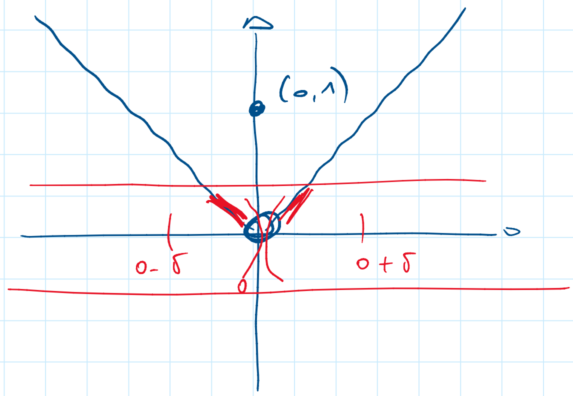


$$f: x\in\mathbb{R} \longmapsto \begin{cases} 1 & x\neq 0 \\ 0 & x=0 \end{cases}$$



$$\lim_{x\to 0} f(x)=1$$

$$f(x)=\begin{cases} |x| & x\neq 0 \\ 1 & x=0 \end{cases}$$



$$\lim_{x\to 0} f(x)=0$$

Scrivo che $\lim_{x\to c} f(x)=+\infty$ e dico che $f(x)$ diverge a $+\infty$ per x che tende a c se

$\forall M\in\mathbb{R}$ $\exists\delta>0$ t.c. $\forall x\in I\cap(c-\delta,c+\delta)$, $x\neq c$ si ha $f(x)>M$

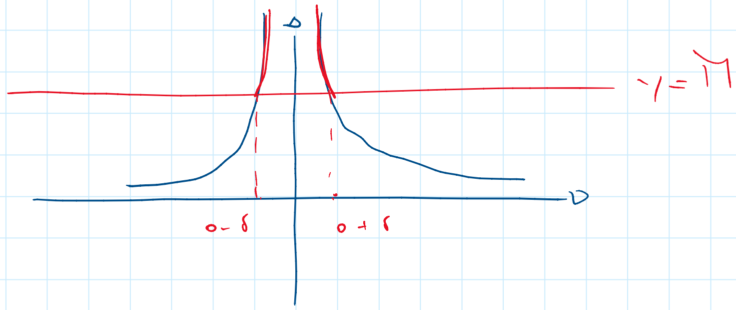
Scrivo che $\lim_{x\to c} f(x)=-\infty$ e dico che $f(x)$ diverge a $-\infty$ per x che tende a c se

$\forall M \in \mathbb{R} \quad \exists \delta > 0 \quad$ t.c. $\quad \forall x \in I \cap (c-\delta, c+\delta), \; x \neq c$

si ha $\quad f(x) < -M$

$f(x) = \frac{1}{x^2} \qquad D = \mathbb{R} \setminus \{0\} = I \qquad$ è pari

$x \neq 0$



$\lim_{x \to 0} \frac{1}{x^2} = +\infty$

Fisso $M \in \mathbb{R}$ e considero la disequazione $\frac{1}{x^2} > M$

1) $M \leq 0$ è vero $\forall x \in \mathbb{R} \setminus \{0\}$

2) $M > 0 \quad \frac{1}{x^2} > M$ è equivalente a $\begin{cases} x^2 < \frac{1}{M} \\ x \neq 0 \end{cases}$

cioè $\begin{cases} -\frac{1}{\sqrt{M}} < x < +\frac{1}{\sqrt{M}} \\ x \neq 0 \end{cases}$

cioè $\quad x \in \left(-\frac{1}{\sqrt{M}}, \frac{1}{\sqrt{M}}\right), \; x \neq 0$

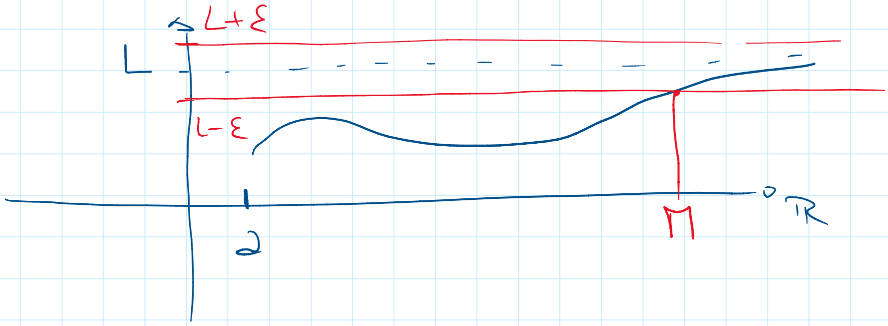
$\Rightarrow$ è soddisfatta la definizione di limite divergente a $+\infty$ con $\delta = \frac{1}{\sqrt{M}}$.

— o —

Sia $f : (a, +\infty) \to \mathbb{R}$ e sia $L \in \mathbb{R}$

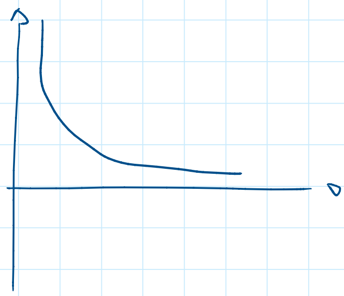
Scrivo che $\lim_{x \to +\infty} f(x) = L$ e dico che f converge ad L quando x diverge a $+\infty$ se

$\forall \varepsilon > 0 \quad \exists M \in \mathbb{R} \quad$ t.c. $\quad \forall x > M \quad |f(x) - L| < \varepsilon$



ESEMPIO $\quad f(x) = \frac{1}{x^2} \qquad D = (0, +\infty)$

$$\lim_{x \to +\infty} \frac{1}{x^2} = 0$$



$\forall \varepsilon > 0 \ \exists M \in \mathbb{R}$ T.c. $\forall x > M \qquad \left| \frac{1}{x^2} - 0 \right| < \varepsilon$

cioè

$\forall \varepsilon > 0 \ \exists M \in \mathbb{R}$ T.c. $\forall x > M \qquad -\varepsilon < \frac{1}{x^2} < \varepsilon$

$\frac{1}{x^2} > -\varepsilon \qquad$ è vera $\ \forall x \neq 0$

$\frac{1}{x^2} < \varepsilon \qquad$ è equivalente a $\quad \begin{cases} x^2 > \frac{1}{\varepsilon} \\ x \neq 0 \end{cases} \quad$ equivalente a

$$\begin{cases} x > \sqrt{\frac{1}{\varepsilon}} \ \vee \ x < -\sqrt{\frac{1}{\varepsilon}} \\ x \neq 0 \end{cases}$$

È soddisfatta la definizione di $\lim_{x \to +\infty} \frac{1}{x^2} = 0$

Basta prendere $M = \sqrt{\frac{1}{\varepsilon}} \quad \forall \varepsilon > 0$.

Sia $f : (a, +\infty) \to \mathbb{R}$. Scrivo che $\lim_{x \to +\infty} f(x) = +\infty$

e dico che f diverge a $+\infty$ quando x diverge a $+\infty$

se

$\forall K \in \mathbb{R} \quad \exists M \in \mathbb{R} \quad$ T.c. $\quad \forall x > M \quad f(x) > K$

se

$\forall L \in \mathbb{R} \quad \exists M \in \mathbb{R} \quad$ t.c. $\quad x \geq M \Rightarrow f(x) > L$

$f(x) = x^2 \qquad f(x) = \log_2(x) \qquad f(x) = 2^x$

Scrivo che $\lim_{x \to +\infty} f(x) = -\infty$ e dico che f diverge a $-\infty$ quando x diverge a $+\infty$ se

$\forall L \in \mathbb{R} \quad \exists M \in \mathbb{R} \quad$ t.c. $\quad x \geq M \Rightarrow f(x) < L$

$f(x) = -x \qquad f(x) = -x^2$

LEMMA DI COLLEGAMENTO (no dim) Sia $I \subset \mathbb{R}$ e sia
intervallo
$f: I \to \mathbb{R}$. e sia c un estremo di I.

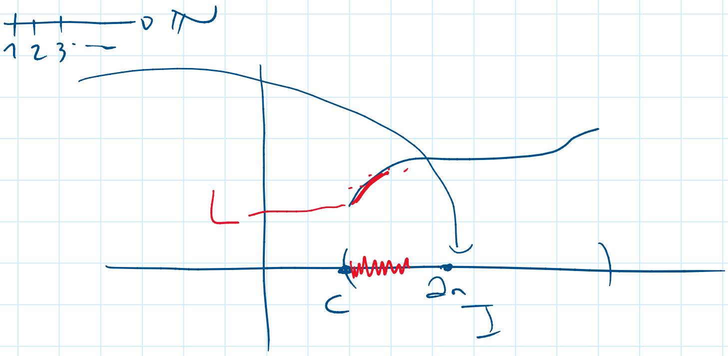
Allora $\lim_{x \to c} f(x)$ esiste ed è uguale a un valore L

SSE

$\forall (a_n)_{n \in \mathbb{N}}$ successione a valori in $I \setminus \{c\}$

cioè $\quad a_n \in I \ \forall n \in \mathbb{N}$ e $a_n \neq c \ \forall n \in \mathbb{N}$

t.c. $\lim_{n \to \infty} a_n = c$ si ha $\lim_{n \to \infty} f(a_n) = L$



$b_n := f(a_n)$

$\lim_{n \to \infty} b_n = L$

— o —

Sia $f: (a, +\infty) \to \mathbb{R}$

Allora $\lim_{x\to+\infty} f(x)$ esiste ed è uguale a L

SSE

$\forall (a_n)_{n\in\mathbb{N}}$ successione a valori in $(a,+\infty)$

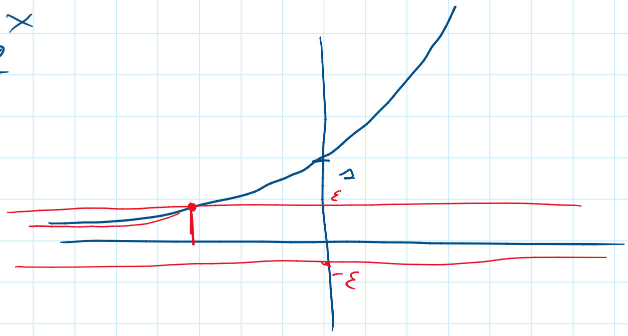
T.c. $\lim_{n\to\infty} a_n = +\infty$ si ha $\lim_{n\to\infty} f(a_n) = L$

Sia $f:(-\infty,b)\to\mathbb{R}$ funzione e sia $L\in\mathbb{R}$.

Scrivo che $\lim_{x\to-\infty} f(x)=L$ e dico che $f(x)$ converge a L quando x diverge a $-\infty$ se

$\forall\varepsilon>0\ \exists M\in\mathbb{R}$ T.c. $\forall x<M$ si ha $|f(x)-L|<\varepsilon$

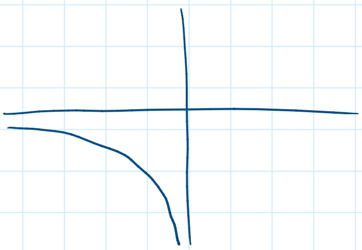
$f(x)=2^x$



$\lim_{x\to-\infty} 2^x = 0$

$f(x)=\frac{1}{x}$ $(-\infty,0)$

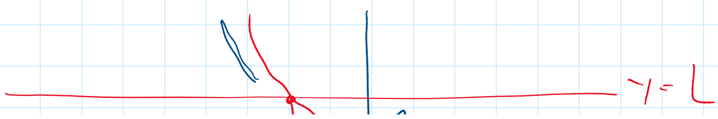
$\lim_{x\to-\infty}\frac{1}{x}=0$



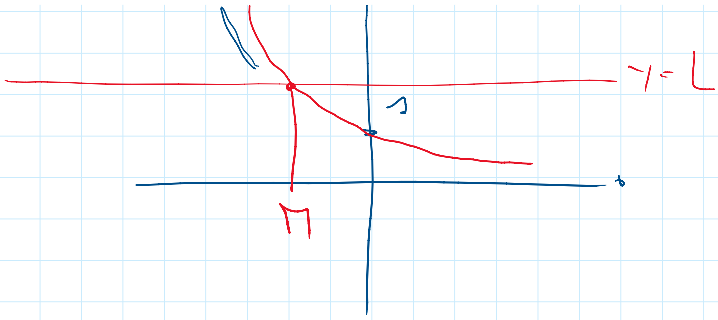
Scrivo che $\lim_{x\to-\infty} f(x)=+\infty$ e dico che f diverge a $+\infty$ per x che diverge a $-\infty$ se

$\forall L\in\mathbb{R}\ \exists M\in\mathbb{R}$ T.c. $\forall x<-M$ si ha $f(x)>L$

$f(x)=\left(\frac{1}{2}\right)^x$



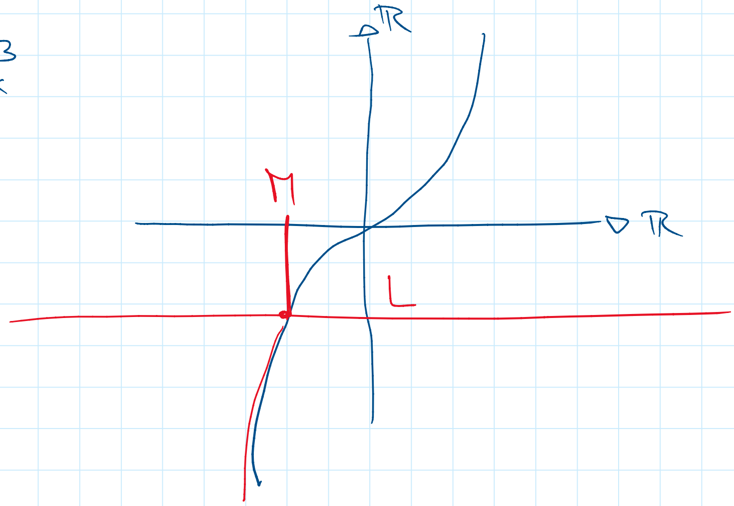
$f(x) = \left(\frac{1}{2}\right)$



Scrivo che $\lim_{x\to-\infty} f(x) = -\infty$ e dico che $f(x)$ diverge a $-\infty$ quando x diverge a $-\infty$ se

$\forall L \in \mathbb{R} \quad \exists M \in \mathbb{R} \quad$ t.c. $\quad x < M \Rightarrow f(x) < L$

$f(x) = x^3$



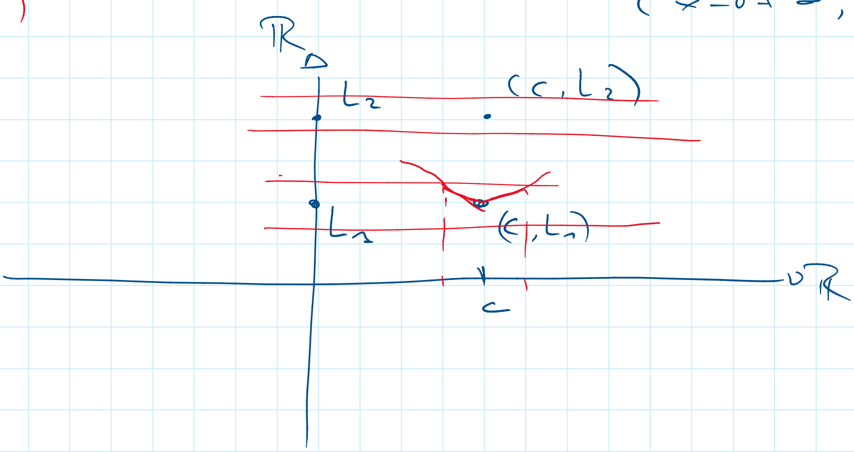
$x^3 < L$

se $L \geq 0$, $x^3 < L$ è vero $\forall x < 0$

se $L < 0$ $\quad x < -\sqrt[3]{|L|}$

TEOREMA (NO DIM) Se $\lim_{x\to c} f(x)$ esiste, allora è unico

$(x \to +\infty, x \to -\infty)$



$c \in \mathbb{R}$ e $\delta > 0$ $(c-\delta, c+\delta)$ si dice intorno di c

Una qualsiasi semiretta destra aperta $(a, +\infty)$ si dice intorno di $+\infty$

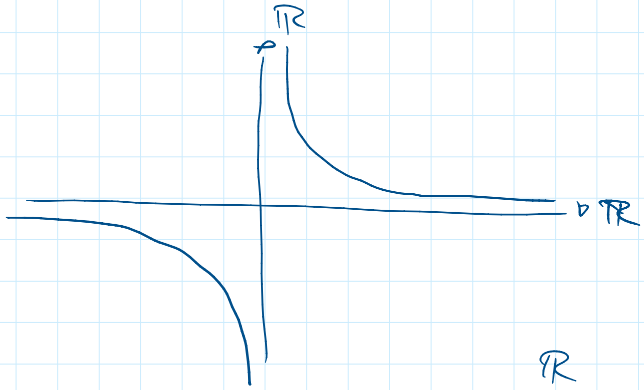
Una qualsiasi semiretta sinistra aperta $(-\infty, b)$ si dice intorno di $-\infty$.

Si dice che una proprietà vale definitivamente per $x \to c$ se vale in un intorno di c.

— o —

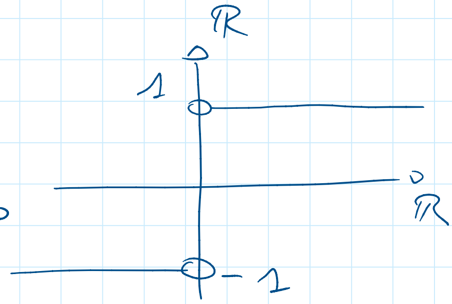
$c = 0$ $\quad f(x) = \frac{1}{x}$

$\nexists \lim\limits_{x \to 0} f(x)$



$c = 0$ $\quad f(x) = \frac{x}{|x|}$

$D = \mathbb{R} \setminus \{0\}$ $\quad f(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$



$\nexists \lim\limits_{x \to 0} f(x)$

$\lim\limits_{x \to 0^+} f(x) = 1$

$\lim\limits_{x \to 0^-} f(x) = -1$

Sia $f : I \to \mathbb{R}$ e sia $c \in I$ o un estremo di I.

Scrivo che $\lim\limits_{x \to c^+} f(x) = L$ se

$\forall \varepsilon > 0 \ \exists \delta > 0$ t.c. $\forall x \in (c, c+\delta) \cap I$ si ha

$$|f(x) - L| < \varepsilon$$

Scrivo che $\lim\limits_{x \to c^-} f(x) = L$ se

$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad t.c. \quad \forall x \in (c-\delta, c) \cap I$ si ha

$$|f(x) - L| < \varepsilon$$

Scrivo che $\lim_{x \to c^+} f(x) = \pm\infty$ e dico che f diverge a $\pm\infty$ quando x tende a c da destra se

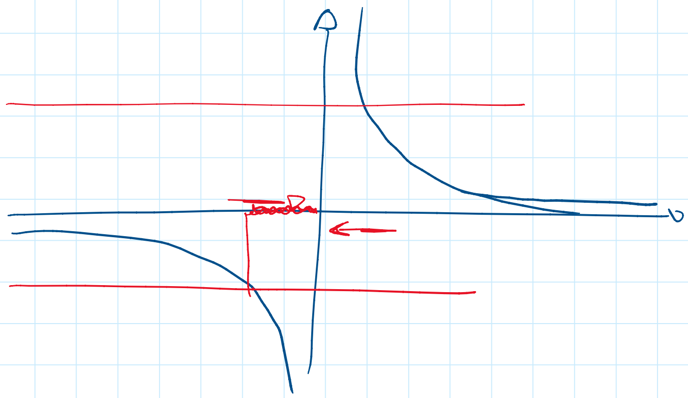
$\forall M \in \mathbb{R} \quad \exists \delta > 0 \quad t.c. \quad \forall x \in (c, c+\delta) \cap I$ si ha

$$f(x) > M$$
$$f(x) < M$$

Scrivo che $\lim_{x \to c^-} f(x) = \pm\infty$ e dico che $f(x)$ diverge a $\pm\infty$ quando x tende a c da sinistra se

$\forall M \in \mathbb{R} \ \exists \delta > 0 \ t.c. \ \forall x \in (c-\delta, c) \cap I$ si ha $f(x) > M$

$f(x) < M$



$f(x) = \frac{1}{x}$

$\lim_{x \to 0^+} f(x) = +\infty$

$\lim_{x \to 0^-} f(x) = -\infty$