venerdì 6 dicembre 2019 09:27

INTEGRALE PER PARTI

Sappiamo che se f e g sono due funzioni derivabili, allora

$$\frac{d}{dx}(fg)(x) = f(x)g'(x) + f'(x)g(x)$$

$$\int (f(x)g(x))' dx = \int (f(x)g'(x) + f'(x)g(x))\, dx$$

$$f(x)g(x) + C = \int f(x)g'(x)\,dx + \int f'(x)g(x)\,dx$$

$$\int f(x)g'(x)\,dx = f(x)g(x) + \cancel{C} - \int f'(x)g(x)\,dx$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$\frac{d}{dx}(fg)(x) = f(x)g'(x) + f'(x)g(x) \qquad \forall x \in (a,b)$$

$$\int_a^b \frac{d}{dx}(fg)(x)\,dx = \int_a^b (f(x)g'(x) + f'(x)g(x))\,dx$$

$$(fg)(x)\Big|_{x=a}^{x=b} = \int_a^b f(x)g'(x)\,dx + \int_a^b f'(x)g(x)\,dx$$

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx$$

ESEMPIO $\int \underset{f}{t}\,\underset{g'}{\sin(t)}\,dt$ $\qquad f(t) = t \quad g'(t) = \sin(t)$

$f'(t) = 1 \quad g(t) = -\cos(t)$

$$= -t\cos(t) - \int 1(-\cos(t))\,dt$$

$$= -t\cos(t) + \int \cos(t)\,dt = -t\sin(t) + \cos(t) + C \quad C \in \mathbb{R}.$$

— ∘ —

$$\int \ln(t)\,dt = \int \underbrace{1}_{g'} \cdot \underbrace{\ln(t)}_{f}\,dt \qquad g'(t) = 1 \quad f(t) = \ln(t)$$

$$g(t) = t \qquad f'(t) = \frac{1}{t}$$

$$= t\ln(t) - \int t \cdot \frac{1}{t}\,dt = t\ln(t) - \int 1\,dt = t\ln(t) - t + C$$

INTEGRAZIONE PER SOSTITUZIONE

Se F e g sono due funzioni derivabili, allora

$$\frac{d}{dt} F(g(t)) = F'(g(t))\,g'(t)$$

Indico F' con f

$$\int \frac{d}{dt} F(g(t))\,dt = \int F'(g(t))\,g'(t)\,dt = \int f(g(t))\,g'(t)\,dt$$

$$\underline{F(g(t)) + C} = \int f(g(t))\,g'(t)\,dt$$

$$\int f(y)\,dy \Big|_{y=g(t)}$$

$$\int f(\overbrace{g(t)})\,\overbrace{g'(t)\,dt} = $$

$$= \int f(y)\,dy \Big|_{y=g(t)}$$

$$\frac{d}{dt}(F \circ g)(t) = f(g(t))\,g'(t)$$

$$\int_a^b \frac{d}{dt}(F\circ g)(t)\,dt = \int_a^b f(g(t))\,g'(t)\,dt$$

$$\int_a^b f(g(t))\,g'(t)\,dt = F(g(t))\Big|_{t=a}^{t=b} = F(g(b)) - F(g(a))$$

$$= \int_{g(a)}^{g(b)} f(x)\,dx$$

$$\boxed{\int_a^b f(g(t))\,g'(t)\,dt = \int_{g(a)}^{g(b)} f(x)\,dx}$$

$$\int_a \ldots g(a)$$

ESEMPIO

$$\int t\cos(t^2)\,dt = \int \frac{1}{2}(2t)\cos(t^2)\,dt =$$

$$= \frac{1}{2}\int (2t)\cos(t^2)\,dt \qquad y = g(t) \qquad dy = g'(t)\,dt$$

$$= \frac{1}{2}\int g'(t)\cos(g(t))\,dt \qquad y = t^2 \qquad dy = 2t\,dt$$

$$= \frac{1}{2}\int \cos(y)\,dy\Big|_{y=t^2} = \frac{1}{2}\sin(y) + C\Big|_{y=t^2}$$

$$= \frac{1}{2}\sin(t^2) + C$$

$$\int x\exp(-x^2)\,dx \qquad \int x^n\exp(-x^2)\,dx$$

$$g(x) = -x^2 = y \qquad g'(x)\,dx = dy \qquad dy = -2x\,dx$$

$$\int x\exp(-x^2)\,dx = \int -\frac{1}{2}(-2x)\exp(-x^2)\,dx = -\frac{1}{2}\int e^y\,dy\Big|_{y=-x^2}$$

$$= -\frac{1}{2}e^y\Big|_{y=-x^2} + C = -\frac{1}{2}e^{-x^2} + C$$

$$\int \tan(x)\,dx \qquad \tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$= \int \frac{\sin(x)}{\cos(x)}\,dx \qquad g(x) = \cos(x) = y$$

$$dy = g'(x)\,dx = -\sin(x)\,dx$$

$$= \int -1\cdot\frac{1}{y}\,dy\Big|_{y=\cos(x)}$$

$$= -\ln|y|\Big|_{y=\cos(x)} + C = -\ln|\cos(x)| + C$$

ESERCIZIO

$$\to \int \sin^2(x)\,dx = \int \overset{f}{\sin(x)} \cdot \overset{g'}{\sin(x)}\,dx$$

$f(x) = \sin(x)$

$g'(x) = \sin(x)$

$f'(x) = \cos(x)$

$g(x) = -\cos(x)$

$$= \sin(x)(-\cos(x)) - \int \cos(x)(-\cos(x))\,dx$$

$$= -\sin(x)\cos(x) + \int \cos^2(x)\,dx$$

$$\cos^2(x) + \sin^2(x) = 1 \qquad \cos^2(x) = 1 - \sin^2(x)$$

$$\to \int \sin^2(x)\,dx = -\sin(x)\cos(x) + \int (1 - \sin^2(x))\,dx =$$

$$= -\sin(x)\cos(x) + x - \int \sin^2(x)\,dx \leftarrow$$

$$2\int \sin^2(x)\,dx = -\sin(x)\cos(x) + x + C$$

$$\int \sin^2(x)\,dx = -\frac{1}{2}\sin(x)\cos(x) + \frac{x}{2} + C \qquad C \in \mathbb{R}$$

$$\int \cos^2(x)\,dx = \int (1 - \sin^2(x))\,dx = \int 1\,dx - \int \sin^2(x)\,dx$$

$$= x - \left(-\frac{1}{2}\sin(x)\cos(x) + \frac{x}{2}\right) + C = \frac{x}{2} + \frac{1}{2}\sin(x)\cos(x) + C$$

$$\int \underset{g'}{x^2}\,\underset{f}{\ln(x)}\,dx$$

$g'(x) = x^2$ $\qquad f(x) = \ln(x)$

$g(x) = \frac{1}{3}x^3$ $\qquad f'(x) = \frac{1}{x}$

$$= \frac{1}{3}x^3\ln(x) - \int \frac{1}{3}x^3 \cdot \frac{1}{x}\,dx = \frac{1}{3}x^3\ln(x) - \frac{1}{3}\int x^2\,dx =$$

$$= \frac{1}{3}x^3\ln(x) - \frac{1}{3}\cdot\frac{1}{3}x^3 + C$$

$$\alpha \neq -1 \quad \int x^\alpha\,dx = \frac{1}{\alpha+1}x^{\alpha+1}$$

$$= \frac{1}{3}x^3 \ln(x) - \frac{1}{3}\frac{1}{3}x^3 + C$$

$$= \frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3 + C$$

$\alpha \neq -1 \quad \int x^\alpha dx = \frac{1}{\alpha+1} x^{\alpha+1}$

$$\int \underset{g'}{x}\, \underbrace{\ln(x^2+x)}_{f}\, dx$$

$g'(x) = x \qquad f(x) = \ln(x^2+x)$

$g(x) = \frac{1}{2}x^2 \qquad f'(x) = \frac{2x+1}{x^2+x}$

$$= \frac{1}{2}x^2 \ln(x^2+x) - \int \frac{1}{2}x^2 \frac{2x+1}{x^2+x} dx$$

$$= \frac{1}{2}x^2 \ln(x^2+x) - \frac{1}{2}\int x^{\cancel{2}} \frac{2x+1}{\cancel{x}(x+1)} dx$$

$$\int \frac{x(2x+1)}{x+1} dx = \int \frac{2x^2+x}{x+1} dx \quad ♡$$

$$\begin{array}{rr|l} \cancel{2x^2} & +x & x+1 \\ \hline -\cancel{2x^2} & -2x & 2x-1 \\ \hline & \cancel{-x} & \\ & \cancel{+x}+1 & \\ \hline & 1 & \end{array}$$

$2x^2+x = (x+1)(2x-1)+1$

$$\frac{2x^2+x}{x+1} = \frac{(x+1)(2x-1)+1}{x+1}$$

$$= 2x-1+\frac{1}{x+1}$$

$$♡ = \int\left(2x-1+\frac{1}{x+1}\right)dx = x^2 - x + \ln|x+1| + C$$

$\Rightarrow$ L'integrale da calcolare vale

$$\frac{1}{2}x^2 \ln(x^2+x) - \frac{1}{2}\left(x^2 - x + \ln\underbrace{|x+1|}\right) + C$$

$x^2+x>0 \qquad x(x+1)>0 \qquad \begin{cases} x>0 \\ x+1>0 \end{cases} \qquad \begin{cases} x<0 \\ x+1<0 \end{cases}$

$$\begin{cases} x>0 \\ x>-1 \end{cases} \qquad \begin{cases} x<0 \\ x<-1 \end{cases}$$

$$(0, +\infty) \qquad (-\infty, -1)$$

ESERCIZIO

$$\int \frac{\ln(x)}{x(1+\ln(x))}\,dx \qquad \ln(x) = g(x) = y \qquad dy = \frac{1}{x}\,dx$$

$$\int \frac{y+1-1}{1+y}\,dy \Big|_{y=\ln(x)} = \int \left(1 - \frac{1}{1+y}\right) dy \Big|_{y=\ln(x)}$$

$$= \left(y - \ln|1+y|\right)\Big|_{y=\ln(x)} + C =$$

$$= \ln(x) - \ln|1+\ln(x)| + C = \ln\left(\frac{x}{|1+\ln(x)|}\right) + C$$

— o —

$$\int \frac{x = \frac{1}{2} 2x}{(1+x^2)^3}\,dx \qquad y = g(x) = x^2 \qquad dy = 2x\,dx$$

$$= \int \frac{\frac{1}{2}}{(1+y)^3}\,dy \Big|_{y=x^2} = \frac{1}{2} \int (1+y)^{-3}\,dy \Big|_{y=x^2} =$$

$$= \frac{1}{2} \frac{1}{-3+1} (1+y)^{-3+1} \Big|_{y=x^2} + C$$

$$= -\frac{1}{4} (1+y)^{-2} \Big|_{y=x^2} + C =$$

$$= -\frac{1}{4} \frac{1}{(1+x^2)^2} + C$$

— o —

$$\int \frac{x = \frac{1}{2} 2x}{(1+x^2)^3}\,dx \qquad y = g(x) = x^2+1 \qquad dy = 2x\,dx$$

$$\int \frac{\ldots}{(1+x^2)^3}$$

$$= \int \frac{\frac{1}{2}}{y^3}\, dy \,\Big|_{y=x^2+1} = \frac{1}{2}\int y^{-3}\, dy \,\Big|_{y=x^2+1}$$

$$= \frac{1}{2}\,\frac{1}{-3+1}\, y^{-3+1}\Big|_{y=x^2+1} + C = -\frac{1}{4}\,\frac{1}{(x^2+1)^2} + C$$

— o —

$$\int x^3 \exp(-x^2)\,dx =$$

$$\frac{d}{dx}\exp(-x^2) = \exp(-x^2)\,(-2x)$$

$$= \int \overset{f}{-\frac{1}{2}x^2}\,\underbrace{(-2x)\,\exp(-x^2)}_{g'}\,dx$$

$f(x) = -\frac{1}{2}x^2$

$g'(x) = -2x\exp(-x^2)$

$$= -\frac{1}{2}x^2\exp(-x^2) - \int -x\exp(-x^2)\,dx$$

$f'(x) = -x$

$g(x) = \exp(-x^2)$

$$= -\frac{1}{2}x^2\exp(-x^2) - \frac{1}{2}\int -2x\exp(-x^2)\,dx$$

$$= -\frac{1}{2}x^2\exp(-x^2) - \frac{1}{2}\exp(-x^2) + C$$

— o —

$$\int x^3\exp(-x^2)\,dx$$

$y = g(x) = -x^2$

$dy = g'(x)\,dx = -2x\,dx$

$\frac{x^3}{-2x} = -\frac{1}{2}x^2$

$$= \int -\frac{1}{2}x^2\,\underbrace{(-2x)}\exp(-x^2)\,dx$$

$$= \int \frac{1}{2}\,\underset{f}{y}\;\underset{g'}{e^y}\,dy\,\Big|_{y=-x^2}$$

— o

$f(y) = y \qquad g'(y) = e^{+y}$

$f'(y) = 1 \qquad g(y) = +e^{+y}$

$$= \frac{1}{2}\left(ye^y - \int 1\,e^y\,dy\right)\Big|_{y=-x^2}$$

$$= \frac{1}{2}\left(y e^{y} - e^{y} + C \right)\Big|_{y=-x^2} = \frac{1}{2}\left(-x^2\exp(-x^2) - \exp(-x^2)\right) + C$$

— o —

$$\int \frac{x+1}{\sqrt{1-x^2}}\,dx = \int \frac{x+1}{(1-x^2)^{1/2}}\,dx =$$

$$= \int \frac{x}{\sqrt{1-x^2}}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \int -\frac{1}{2}(-2x)(1-x^2)^{-1/2}\,dx + \arcsin(x)$$

$$= -\cancel{\frac{1}{2}}\,\frac{1}{\cancel{-\frac{1}{2}+1}}(1-x^2)^{-\frac{1}{2}+1} + \arcsin(x) + C$$

$$= -\sqrt{1-x^2} + \arcsin(x) + C$$

— o —

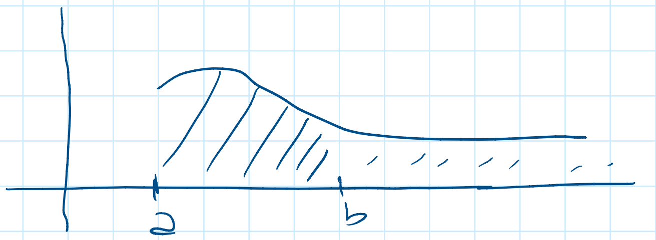
$$\int \arcsin(x)\,dx = \int \underbrace{1}_{g'} \cdot \underbrace{\arcsin(x)}_{f}\,dx$$

— o —

$f : [a,b] \to \mathbb{R}$ continua $\qquad \int_a^b f(t)\,dt$

$f : [a,+\infty) \to \mathbb{R}$ continua

$$\int_a^{+\infty} f(t)\,dt = ?$$



$$\int_a^{+\infty} f(t)\,dt := \lim_{b \to +\infty} \int_a^b f(t)\,dt$$

se tale limite esiste ed è finito.

ESEMPIO ① $\int^{+\infty} \cos(x)\,dx$

ESEMPIO

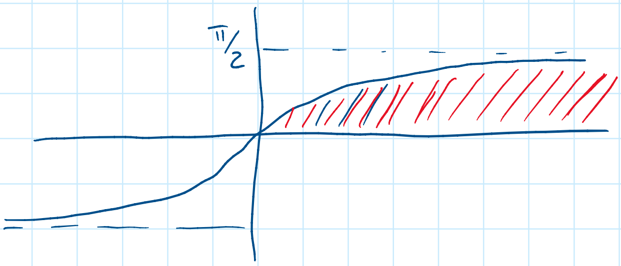
① $\int_0 \cos(x)\,dx$

$$\forall b>0 \quad \int_0^b \cos(x)\,dx = \sin(x)\Big|_{x=0}^{x=b} = \sin(b) - \sin(0) = \sin(b)$$

$$\lim_{b\to+\infty} \sin(b) \quad \text{NON ESISTE}$$

② $\int_0^{+\infty} \frac{1}{1+x^2}\,dx$

$$b>0 \quad \int_0^b \frac{1}{1+x^2}\,dx = \arctan(x)\Big|_{x=0}^{x=b} = \arctan(b) - \cancel{\arctan(0)}^{=0}$$



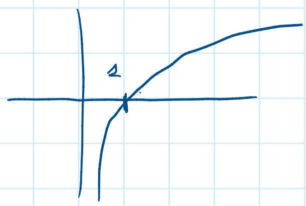
$$\lim_{b\to+\infty} \int_0^b \frac{1}{1+x^2}\,dx =$$

$$= \lim_{b\to+\infty} \arctan(b) = \frac{\pi}{2}$$

③ $\int_1^{+\infty} x^\alpha\,dx \qquad \int_1^b x^\alpha\,dx \qquad b>1$

$$\alpha = -1 \qquad \int_1^b \frac{1}{x}\,dx = \ln(|x|)\Big|_{x=1}^{x=b} = \ln(b) - \ln(1) =$$

$$\ln(b) \to +\infty$$



$$\alpha \neq -1 \qquad \int_1^b x^\alpha\,dx = \frac{1}{\alpha+1}\,x^{\alpha+1}\Big|_{x=1}^{x=b}$$

$$= \frac{1}{\alpha+1}\left(b^{\alpha+1} - 1\right)$$

$$\lim_{b\to+\infty} \frac{1}{\alpha+1}\left(b^{\alpha+1}-1\right) = \begin{cases} +\infty & \alpha+1>0 \quad \alpha>-1 \\ \dfrac{-1}{\alpha+1} & \alpha+1<0 \quad \alpha<-1 \end{cases}$$

$\int^b x^\alpha \quad -1$

$$\int_1^b x^\alpha dx = \frac{-1}{\alpha+1} \quad \text{se} \quad \alpha+1<0$$

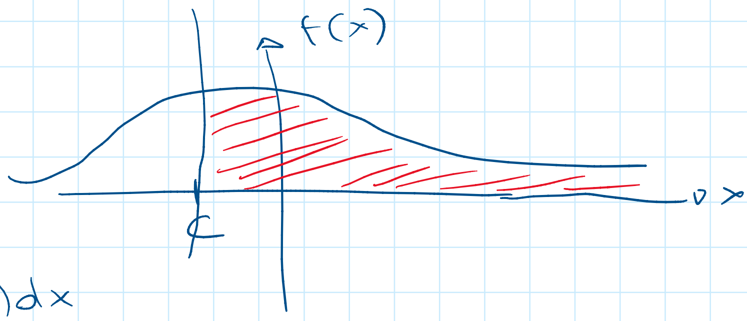
— o —

$f : (-\infty, b] \to \mathbb{R}$ continua

$$\int_{-\infty}^b f(x)dx := \lim_{a\to-\infty} \int_a^b f(x)dx$$ se questo limite esiste finito

— o —

$f : \mathbb{R} \to \mathbb{R}$ continua

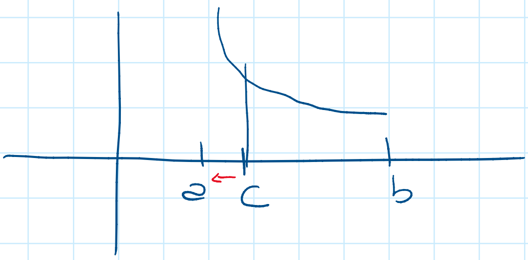


$$\int_{-\infty}^{+\infty} f(x)dx := \underbrace{\int_c^{+\infty} f(x)dx} + \underbrace{\int_{-\infty}^c f(x)dx}$$

se entrambi gli integrali esistono finiti

— o —

$f : (a,b] \to \mathbb{R}$ continua $\quad \lim_{x\to a^+} f(x) = ??$



f è continua in $[c,b]$

$\forall c\in(a,b)$

$$\lim_{c\to a^+} \int_c^b f(x)dx =: \int_a^b f(x)dx$$

— o —

$$\int_0^1 x^\alpha dx \qquad \alpha<0 \qquad c\in(0,1) \quad \int_c^1 x^\alpha dx$$

$$\alpha = -1 \qquad \int_c^1 x^{-1}\,dx = \ln|x| \Big|_{x=c}^{x=1} = \ln(1) - \ln(c) = -\ln(c)$$

$$\lim_{c \to 0^+} (-\ln(c)) = +\infty$$

$$\alpha \neq -1 \qquad \int_c^1 x^{\alpha}\,dx = \frac{1}{\alpha+1} x^{\alpha+1} \Big|_{x=c}^{x=1} = \frac{1}{\alpha+1}\left(1 - c^{\alpha+1}\right)$$

$$\lim_{c \to 0^+} \frac{1}{\alpha+1}\left(1 - c^{\alpha+1}\right) = \begin{cases} \dfrac{1}{\alpha+1} & \alpha+1>0 \quad \alpha>-1 \\ +\infty & \alpha+1<0 \quad \alpha<-1 \end{cases}$$

— o —

$$\int \frac{1}{1+\sin(x)}\,dx$$

$$y = tg\left(\frac{x}{2}\right) = \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}$$

$$dy = \frac{1}{\cos^2\left(\frac{x}{2}\right)} \cdot \frac{1}{2}\,dx$$

$$\cos(x) = \frac{1-y^2}{1+y^2}$$

$$\sin(x) = \frac{2y}{1+y^2}$$

$$\cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) = 1$$

$$1 + \frac{\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)} = \frac{1}{\cos^2\left(\frac{x}{2}\right)}$$

$$\frac{1}{\cos^2\left(\frac{x}{2}\right)} = 1 + y^2$$

$$\frac{1}{1+\sin(x)} = \frac{1}{1+\frac{2y}{1+y^2}} = \frac{1+y^2}{1+2y+y^2} = \frac{1+y^2}{(1+y)^2}$$