1° TEOREMA FONDAMENTALE DEL CALCOLO INTEGRALE

Se $f : [a,b] \to \mathbb{R}$ è una funzione continua, allora $\forall x \in [a,b]$ la funzione f è integrabile sull'intervallo $[a,x]$ e

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x) \qquad \forall x \in [a,b]$$

— o —

Se $f : [a,b] \to \mathbb{R}$ è una funzione continua, allora la funzione

$$F : x \in [a,b] \mapsto \int_a^x f(t)\,dt$$

è una funzione derivabile e $F'(x) = f(x) \qquad \forall x \in [a,b]$

2° TEOREMA FONDAMENTALE DEL CALCOLO INTEGRALE

Sia $f : [x_0, x_1] \to \mathbb{R}$ una funzione derivabile con derivata $f' : [x_0, x_1] \to \mathbb{R}$ continua.

Allora

$$\int_{x_0}^x f'(t)\,dt = f(x) - f(x_0) \qquad \forall x \in [x_0, x_1]$$

DIM Basta considerare $A(x) = \int_{x_0}^x f'(t)\,dt$

Sappiamo che A è derivabile e che $A'(x) = f'(x)$ $\forall x \in [x_0, x_1]$ $\Rightarrow$ $(A-f)'(x) = A'(x) - f'(x) \equiv 0$

$\Rightarrow \exists\, C \in \mathbb{R}$ t.c. $\underline{(A-f)(x) = C}$ $\forall x \in [x_0, x_1]$

In particolare $(A-f)(x_0) = C$ cioè $A(x_0) - f(x_0) = C$

$\Rightarrow C = -f(x_0) \Rightarrow A(x) - f(x) = -f(x_0) \quad \forall x \in [x_0, x_1]$

cioè $A(x) = f(x) - f(x_0) \quad \forall x \in [x_0, x_1]$

$$\int_{x_0}^x f'(t)\,dt = f(x) - f(x_0) \qquad \forall x \in [x_0, x_1]$$

cioè $\quad A(x) = f(x) - f(x_0) \quad \forall x \in [x_0, x_1]$

$$\Rightarrow \int_{x_0}^{x} f'(t)\,dt = \underline{f(x) - f(x_0)} \quad \forall x \in [x_0, x_1]$$ (A)

$f(x) - f(x_0)$ si scrive anche $f(t)\Big|_{t=x_0}^{t=x_1}$ o $f(t)\Big|_{x_0}^{x_1}$

— o —

$$\int_{x_0}^{x} f'(t)\,dt = f(x) - f(x_0)$$

$\int_{x_0}^{x} g(t)\,dt$ Mi basta trovare una funzione $F : [x_0, x_1] \to \mathbb{R}$ t.c. $g(t) = f'(t)$ $\forall t \in [x_0, x_1]$

$$\Rightarrow \int_{x_0}^{x} g(t)\,dt = \int_{x_0}^{x} f'(t)\,dt = f(x) - f(x_0)$$

N.B. Se $g : [x_0, x_1] \to \mathbb{R}$ è una funzione continua e e $f : [x_0, x_1] \to \mathbb{R}$ è una funzione derivabile t.c.

$f'(t) = g(t) \quad \forall t \in [x_0, x_1] \Rightarrow$

$\forall C \in \mathbb{R}$ la derivata di $f_C : t \in [x_0, x_1] \to f(t) + C$ è anche $g(t)$

Introduciamo quindi la nozione di integrale indefinito.

Col simbolo $\int g(t)\,dt$ indico l'insieme delle funzioni la cui derivata è g.

Se $f : [x_0, x_1] \to \mathbb{R}$ è una di queste funzioni

$$\Rightarrow \int g(t)\,dt = f(t) + C \qquad C \in \mathbb{R}$$

ESEMPI $\quad n \in \mathbb{N} \quad \int x^n\,dx$

$$x^n = \frac{1}{n+1}(n+1)x^n \qquad \int x^n\,dx = \frac{1}{n+1}\int (n+1)x^n\,dx =$$

$$\frac{d}{dx}x^{n+1} = (n+1)x^n \qquad = \frac{1}{n+1}\left(x^{n+1}+C\right) = \frac{x^{n+1}}{n+1} + C$$

$C \in \mathbb{R}$

$$\alpha \in \mathbb{R} \qquad \int x^\alpha\,dx$$

$$\alpha \neq -1 \qquad x^\alpha = \frac{1}{\alpha+1}(\alpha+1)x^\alpha = \frac{1}{\alpha+1}\frac{d}{dx}x^{\alpha+1}$$

$$\int x^\alpha\,dx = \frac{1}{\alpha+1}\int(\alpha+1)x^\alpha\,dx = \frac{1}{\alpha+1}\left(x^{\alpha+1}+C\right) = \frac{x^{\alpha+1}}{\alpha+1} + C$$

$C \in \mathbb{R}$

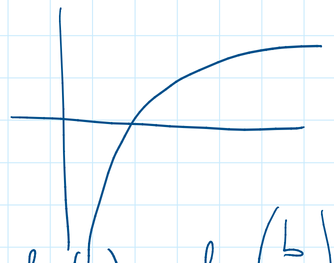
$$\alpha = -1 \qquad \int \frac{1}{x}\,dx$$

$$x \in (0,+\infty) \qquad \frac{d}{dx}\ln(x) = \frac{1}{x} \qquad \frac{1}{x} = \frac{d}{dx}\ln(|x|)$$

$$x \in (-\infty, 0) \qquad \frac{1}{x} = \frac{-1}{-x} = \frac{d}{dx}\ln(-x)$$

$$\int \frac{1}{x}\,dx = \ln(|x|) + C$$

$$\int_a^b \frac{1}{x}\,dx$$



$$0 < a < b \qquad \int_a^b \frac{1}{x}\,dx = \ln(x)\Big|_a^b = \ln(b) - \ln(a) = \ln\left(\frac{b}{a}\right)$$

$$a < b < 0 \qquad \int_a^b \frac{1}{x}\,dx = \ln(-x)\Big|_a^b = \ln(-b) - \ln(-a) = \ln\left(\frac{+b}{+a}\right)$$

$$0 < -b < -a \qquad \ln\left(\frac{-b}{-a}\right)$$

$$\int e^x\,dx = e^x + C \qquad C \in \mathbb{R}$$

$$\int a^x\,dx \qquad a > 0 \quad a \neq 1$$

$$a = e^{\ln(a)} \qquad a^x = \left(e^{\ln(a)}\right)^x = e^{x\ln(a)}$$

$$a^x = \frac{1}{\ln(a)} \ln(a)\, e^{x\ln(a)} = \frac{1}{\ln(a)} \frac{d}{dx} e^{x\ln(a)} = \frac{1}{\ln(a)} \frac{d}{dx} a^x$$

$$\int a^x \, dx = \frac{1}{\ln(a)} \int \ln(a)\, e^{x\ln(a)} dx = \frac{1}{\ln(a)} \int \left(\frac{d}{dx} a^x\right) dx$$

$$= \frac{1}{\ln(a)} \left(a^x + C \right) = \frac{1}{\ln(a)} a^x + C \qquad C \in \mathbb{R}.$$

$$\int \sin(x)\, dx \qquad \frac{d}{dx} \cos(x) = -\sin(x)$$

$$\sin(x) = -1 \cdot (-\sin(x))$$

$$\int \sin(x)\, dx = -1 \cdot \int (-\sin(x))\, dx = -1 \left(\cos(x) + C \right) =$$

$$= -\cos(x) + C \qquad C \in \mathbb{R}.$$

$$\int \cos(x)\, dx = \sin(x) + C \qquad C \in \mathbb{R}. \qquad \frac{d}{dx} \sin(x) = \cos(x)$$

$$\int \frac{1}{\sin^2(x)} dx$$

$$f(x) = \frac{\cos(x)}{\sin(x)} =: \operatorname{cotan}(x) \qquad f'(x) = \frac{-\sin(x) \cdot \sin(x) - \cos(x)\cos(x)}{\sin^2(x)}$$

$$= \frac{-(\cos^2(x) + \sin^2(x))}{\sin^2(x)}$$

$$= \frac{-1}{\sin^2(x)}$$

$$\frac{1}{\sin^2(x)} = -1 \cdot \frac{-1}{\sin^2(x)}$$

$$\int \frac{1}{\sin^2(x)} dx = -1 \int \frac{-1}{\sin^2(x)} dx =$$

$$= -1 \left(\operatorname{cotan}(x) + C \right) = -\operatorname{cotan}(x) + C \qquad C \in \mathbb{R}$$

$$\int \frac{1}{\cos^2(x)}\,dx = \tan(x) + C \quad C\in\mathbb{R} \qquad \frac{d}{dx}\tan(x) = \frac{1}{\cos^2(x)}$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C \quad C\in\mathbb{R} \qquad \frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C \qquad \frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\arccos(x) = \frac{1}{(\cos(y))'}\Big|_{y=\arccos(x)} = \frac{-1}{\sin(y)}\Big|_{y=\arccos(x)}$$

$$\sin^2(y) + \cos^2(y) = 1 \quad \Rightarrow \quad \sin^2(y) + x^2 = 1$$

$$\sin^2(y) = 1 - x^2 \qquad \sin(y) = \sqrt{1-x^2}$$

$$\sin(y) = -\sqrt{1-x^2}$$

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos(x) + C$$

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = -1\int \frac{1}{\sqrt{1-x^2}}\,dx = -\arcsin(x) + C$$

$$f(x) = \arccos(x) + \arcsin(x) \qquad x\in[-1,1)$$

$$f'(x) = \frac{-1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} = 0 \qquad \forall x\in[-1,1)$$

$$f(x) \equiv C \qquad f(x) = f(0) \quad \forall x\in[-1,1)$$

$$f(0) = \frac{\pi}{2} + 0 \quad \Rightarrow \quad \boxed{\arccos(x) + \arcsin(x) = \frac{\pi}{2} \quad \forall x\in[-1,1]}$$

ESERCIZIO

$$\int \frac{1}{ax+b}\,dx \qquad a \neq 0 \qquad\qquad \int \frac{1}{x}\,dx = \ln(|x|) + C$$

$$\frac{1}{ax+b} = \frac{1}{a\left(x+\frac{b}{a}\right)}$$

$$\int \frac{1}{ax+b}\,dx = \int \frac{1}{a}\cdot\frac{1}{x+\frac{b}{a}}\,dx = \frac{1}{a}\int \frac{1}{x+\frac{b}{a}}\,dx$$

$$= \frac{1}{a}\left(\ln\left(\left|x+\frac{b}{a}\right|\right)+C\right) = \frac{1}{a}\ln\left(\left|x+\frac{b}{a}\right|\right) + C \qquad C \in \mathbb{R}$$

— o —

ESERCIZIO

$$\int \frac{x^2+1}{2x+3}\,dx$$

$$\begin{array}{rrr|l}
\cancel{x^2} & +\,0\cdot x & +\,1 & 2x+3 \\ \hline
-\cancel{x^2} & -\frac{3}{2}x & & \frac{x}{2}-\frac{3}{4} \\ \hline
 & -\cancel{\frac{3}{2}x} & +\,1 & \\
 & \cancel{\frac{3}{2}x} & +\,\frac{9}{4} & \\ \hline
 & & \frac{13}{4} &
\end{array}$$

$$x^2+1 = (2x+3)\left(\frac{x}{2}-\frac{3}{4}\right) + \frac{13}{4}$$

$$\frac{x^2+1}{2x+3} = \frac{(2x+3)\left(\frac{x}{2}-\frac{3}{4}\right)+\frac{13}{4}}{2x+3} = \frac{x}{2} - \frac{3}{4} + \frac{13/4}{2x+3}$$

$$\int \frac{x^2+1}{2x+3}\,dx = \int \frac{x}{2}\,dx - \int \frac{3}{4}\,dx + \frac{13}{4}\int \frac{1}{2x+3}\,dx$$

$$= \frac{1}{2}\int x^1 dx - \frac{3}{4}\int 1\,dx + \frac{13}{4}\int \frac{1}{2\left(x+\frac{3}{2}\right)}\,dx$$

$$= \frac{1}{2}\,\frac{1}{1+1}\,x^{1+1} - \frac{3}{4}x + \frac{13}{4}\cdot\frac{1}{2}\ln\left(\left|x+\frac{3}{2}\right|\right) + C$$

$$= \frac{x^2}{4} - \frac{3}{4}x + \frac{13}{8}\ln\left(\left|x+\frac{3}{2}\right|\right) + C$$

ESERCIZIO

$$\int \frac{1}{x^2-5x+6}\,dx$$

$$x^2-5x+6=0 \qquad \Delta = 25-24=1 \qquad x_{1,2}=\frac{5\pm 1}{2} = \begin{cases} 3 \\ 2 \end{cases}$$

$$x^2-5x+6 = (x-2)(x-3)$$

$$\frac{1}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$\frac{1}{(x-2)(x-3)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$$\underline{0x+1} = \underline{(A+B)x} + \underline{(-3A-2B)} \qquad \begin{cases} -3A-2B=1 \\ A+B=0 \end{cases}$$

$$\begin{cases} B=-A \\ -3A+2A=1 \end{cases} \qquad \begin{cases} B=-A \\ -A=1 \end{cases} \qquad \begin{cases} A=-1 \\ B=1 \end{cases}$$

$$\int \frac{1}{(x-2)(x-3)}\,dx = \int\left(\frac{-1}{x-2} + \frac{1}{x-3}\right)dx =$$

$$= -1\int\frac{1}{x-2}\,dx + \int\frac{1}{x-3}\,dx = -\ln(|x-2|) + \ln(|x-3|) + C$$

$$= \ln\left(\left|\frac{x-3}{x-2}\right|\right) + C$$

$$\int \frac{1}{x^2+2x+1}\,dx \qquad x^2+2x+1 = (x+1)^2$$

$$\int (x+1)^{-2}\,dx = \frac{1}{-2+1}(x+1)^{-2+1} + C$$

— ∘ — $(|x-2|)$

$$\int \frac{1}{x^2+2x+1}\,dx \qquad x^2+2x+1=(x+1)^2$$

$$\int (x+1)^{-2}\,dx = \frac{1}{-2+1}(x+1)^{-2+1}+C$$

$$= -\frac{1}{x+1}+C$$

— ∘ —

$$\int \frac{1}{x^2+2x+4}\,dx \qquad x^2+2x+4=0 \qquad \frac{\Delta}{4}=1-4=-3<0$$

$$\to\ x^2+2x+\alpha=(x+c)^2 \qquad \cancel{x^2}+2x+\alpha=\cancel{x^2}+2cx+c^2$$

$$\begin{cases} 2=2c \\ \alpha=c^2 \end{cases} \qquad \begin{cases} c=1 \\ \alpha=1 \end{cases}$$

$$x^2+2x+4=$$

$$x^2+2x+1+3=(x+1)^2+3$$

$$\int \frac{1}{(x+1)^2+3}\,dx \qquad \int \frac{1}{x^2+1}\,dx=\arctan(x)+C$$

$$(x+1)^2+3=3\left(\frac{(x+1)^2}{3}+1\right)=3\left(\left(\frac{x+1}{\sqrt{3}}\right)^2+1\right)$$

$$\int \frac{1}{x^2+2x+4}\,dx=\int \frac{1}{3\left(\left(\frac{x+1}{\sqrt{3}}\right)^2+1\right)}\,dx=\frac{1}{3}\int \frac{\frac{1}{\sqrt{3}}\sqrt{3}}{\left(\frac{x+1}{\sqrt{3}}\right)^2+1}\,dx$$

$$=\frac{\sqrt{3}}{3}\int \frac{\frac{1}{\sqrt{3}}}{\left(\frac{x+1}{\sqrt{3}}\right)^2+1}\,dx=\frac{\sqrt{3}}{3}\int \frac{d}{dx}\arctan\left(\frac{x+1}{\sqrt{3}}\right)dx$$

$$=\frac{\sqrt{3}}{3}\left(\arctan\left(\frac{x+1}{\sqrt{3}}\right)+C\right)=\frac{\sqrt{3}}{3}\arctan\left(\frac{x+1}{\sqrt{3}}\right)+C$$

— ∘ —

$$\int \frac{3x+1}{x^2+2x+4}\,dx$$

$$\frac{d}{dx}\left(x^2+2x+4\right) = 2x+2$$

$$\frac{3}{2}\cdot\frac{2}{3}$$

$$3x+1 = \frac{3}{2}2x+1 = \frac{3}{2}\left(2x+\frac{2}{3}\right) = \frac{3}{2}\left(2x+2\underbrace{-2+\frac{2}{3}}_{-\frac{4}{3}}\right)$$

$$= \frac{3}{2}\left(2x+2\right)+\frac{3}{2}\cdot\frac{-4}{3} = \frac{3}{2}\left(2x+2\right)-2$$

$$\frac{3x+1}{x^2+2x+4} = \frac{\frac{3}{2}(2x+2)}{x^2+2x+4} - \frac{2}{x^2+2x+4}$$

$$\int \frac{3x+1}{x^2+2x+4}\,dx = \frac{3}{2}\int \underbrace{\frac{2x+2}{x^2+2x+4}}_{\frac{(x^2+2x+4)'}{x^2+2x+4}}\,dx - 2\int \frac{1}{x^2+2x+4}\,dx$$

$g(x) = \ln f(x)$

$g'(x) = \frac{1}{f(x)} f'(x)$

$g(x) = \ln(|f(x)|)$

$f(x)>0 \quad g(x) = \frac{f'(x)}{f(x)}$

$f(x)<0 \quad g(x) = \ln(-f(x))$

$g'(x) = \frac{1}{-f(x)}\cdot(-f'(x)) = \frac{f'(x)}{f(x)}$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln(|f(x)|)+C$$

$$\Rightarrow \quad = \frac{3}{2}\ln\left(x^2+2x+4\right) - 2\cdot\frac{\sqrt{3}}{3}\arctan\left(\frac{x+1}{\sqrt{3}}\right)+C$$

— o —

$$\int \frac{x^2+2}{x^2+2x+4}\,dx$$

$$\begin{array}{rrr|l} \cancel{x^2} & +0x & +2 & x^2+2x+4 \\ \hline \cancel{-x^2} & -2x & -4 & 1 \\ \hline \end{array}$$

$$-2x-2$$

$$\frac{x^2+2}{x^2+2x+4} = \frac{1\left(x^2+2x+4\right)+\left(-2x-2\right)}{x^2+2x+4} =$$

$$= 1 - \frac{2x+2}{x^2+2x+4}$$

$$\int \frac{x^2+2}{x^2+2x+4}\,dx = \int\left(1 - \frac{\left(x^2+2x+4\right)'}{x^2+2x+4}\right)dx$$

$$= x - \ln\left(x^2+2x+4\right) + C$$

$$\int \frac{x^2+x+3}{x^2+2x+4}$$

$$\begin{array}{r|l} x^2+x+3 & x^2+2x+4 \\ \hline -x^2-2x-4 & 1 \\ \hline -x-1 & \end{array}$$

$$x^2+x+3 = 1\left(x^2+2x+4\right) - \left(x+1\right)$$

$$\frac{x^2+2x+3}{x^2+2x+4} = 1 - \frac{x+1}{x^2+2x+4} = \otimes$$

$$x+1 = \frac{1}{2}\left(2x+2\right)$$

$$\otimes \quad 1 - \frac{1}{2}\,\frac{2x+2}{x^2+2x+4}$$

$$\int \frac{x^2+x+3}{x^2+2x+4}\,dx = \int\left(1 - \frac{1}{2}\,\frac{\left(x^2+2x+4\right)'}{x^2+2x+4}\right)dx$$

$$= x - \frac{1}{2}\ln\left(x^2+2x+4\right) + C$$