$x_0 \in \mathbb{R}$, $x_0 = +\infty$ o $x_0 = -\infty$

Sia f funzione definita in un intorno di x_0.

Se $\lim_{x \to x_0} f(x) = 0$ dico che f è INFINITESIMA IN x_0

Se $\lim_{x \to x_0} f(x) = +\infty$ o $-\infty$, dico che f è INFINITA IN x_0

Se f e g sono due funzioni infinitesime in x_0, dico che g è un INFINITESIMO DI ORDINE SUPERIORE RISPETTO A x_0 se

$$\lim_{x \to x_0} \frac{g(x)}{f(x)} = 0$$

ESEMPIO $x_0 = 0$

$$g(x) = 1 - \cos(x) \qquad \lim_{x \to 0} g(x) = 0$$

$$f(x) = x \qquad \lim_{x \to 0} f(x) = 0$$

$$\lim_{x \to 0} \frac{g(x)}{f(x)} = \lim_{x \to 0} \frac{1 - \cos(x)}{x} =$$

$$= \lim_{x \to 0} \underbrace{\frac{1-\cos(x)}{x^2}}_{\to 1/2} \cdot \underbrace{\frac{\cancel{x^2}}{\cancel{x}}}_{\to 0} = 0$$

$1 - \cos(x) = o(x)$ in $x_0 = 0$

Se g è un infinitesimo di ordine superiore rispetto a f in un pto x_0 scrivo $g = o(f)$

ESEMPIO $0 < \alpha < \beta$

$$g(x) = x^\beta \qquad \lim_{x \to 0^+} g(x) = 0$$

$$f(x) = x^\alpha \qquad \lim_{x \to 0^+} f(x) = 0$$

$$\frac{g(x)}{f(x)} = \frac{x^\beta}{x^\alpha} = x^{\beta - \alpha}$$

$\beta - \alpha > 0 \Rightarrow \lim_{x \to 0} \frac{x^\beta}{x^\alpha} = 0$ $\qquad x^\beta = o(x^\alpha)$ in $x_0 = 0$

ESEMPIO

$$g(x) = \exp\left(-\frac{1}{x}\right)$$

$$f(x) = x^\alpha \qquad \alpha > 0$$

$g(x) = \exp(-\frac{1}{x})$

$f(x) = x^\alpha \qquad \alpha > 0$

$$\lim_{x \to 0} g(x) = \begin{cases} \lim_{x \to 0^+} \exp\left(-\frac{1}{x}\right) = 0 \\ \lim_{x \to 0^-} \exp\left(-\frac{1}{x}\right) = +\infty \end{cases}$$

$$\lim_{x \to 0} f(x) = \begin{cases} \lim_{x \to 0^-} x^\alpha & \text{non esiste tranne che per alcuni valori di } \alpha \\ \lim_{x \to 0^+} x^\alpha = 0 \end{cases}$$

Per $x > 0$ $\quad \frac{g(x)}{f(x)} = \frac{\exp\left(-\frac{1}{x}\right)}{x^\alpha} \qquad y = \frac{1}{x}$

$x \to 0^+$ SSE $y \to +\infty$

$$\frac{g(x)}{f(x)} = \frac{\exp(-y)}{y^{-\alpha}} = \frac{y^\alpha}{e^y} \xrightarrow[\substack{\text{visto con} \\ \text{Hôpital}}]{} 0 \quad \text{quando } y \to +\infty$$

$\exp\left(-\frac{1}{x}\right) = o(x^\alpha)$ per $x \to 0^+$ $\quad \forall \alpha > 0$

Siano f e g t.c. $\lim_{x \to x_0} f(x) = +\infty$ $(o -\infty)$

$\lim_{x \to x_0} g(x) = +\infty$ $(o -\infty)$

dico che g è un infinito di ordine inferiore rispetto a f in x_0 se $\lim_{x \to x_0} \frac{g(x)}{f(x)} = 0$

ESEMPIO $\quad x_0 = +\infty \qquad g(x) = \ln(x)$

$f(x) = x^\alpha \qquad \alpha > 0$

Abbiamo visto (con L'Hôpital) che $\lim_{x \to +\infty} \frac{\ln(x)}{x^\alpha} = 0 \quad \forall \alpha > 0$

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ESEMPIO $\quad x_0 = +\infty \qquad g(x) = x^\alpha \quad \alpha > 0$

$f(x) = e^x$

$\frac{g(x)}{f(x)} = \frac{x^\alpha}{e^x}$ Abbiamo visto (con L'Hôpital) che

$\lim_{x \to +\infty} \frac{x^\alpha}{e^x} = 0 \quad \forall \alpha > 0$.

$f : (a,b) \to \mathbb{R}$

$x_0 \in (a,b) \qquad p_1(x) = mx + q$

$E_1(x) = f(x) - p_1(x)$

Cerco m e q t.c. $\lim_{x \to x_0} \frac{E_1(x)}{x - x_0} = 0$

Affinché sia vero questo limite è necessario che

$\lim_{x \to x_0} E_1(x) = 0$

$\lim_{x \to x_0} E_1(x) = \lim_{x \to x_0} f(x) - (mx + q) =$ Suppongo f continua in x_0

$= f(x_0) - (mx_0 + q)$

$\Rightarrow \lim_{x \to x_0} E_1(x) = 0 \quad$ SSE $\quad mx_0 + q = f(x_0)$

SSE $\quad q = f(x_0) - mx_0$

SSE $\quad p_1(x) = mx + q = mx + f(x_0) - mx_0 =$

$= f(x_0) + m(x - x_0)$

Voglio $\lim_{x \to x_0} \frac{E_1(x)}{x - x_0} = 0$ cioè che

$$\lim_{x \to x_0} \frac{f(x) - (f(x_0) + m(x - x_0))}{x - x_0} = 0$$

Ho $\dfrac{f(x)-(f(x_0)+m(x-x_0))}{x-x_0} = \dfrac{f(x)-f(x_0)}{x-x_0} - m$

Suppongo che f sia derivabile in x_0 $\Rightarrow$

$$\lim_{x\to x_0} \frac{E_1(x)}{x-x_0} = f'(x_0) - m = 0 \quad \text{SSE} \quad m = f'(x_0)$$

SSE $\quad p_1(x) = f'(x_0)(x-x_0) + f(x_0)$

SSE il grafico di $p_1(x)$ è la retta tangente al grafico in x_0.

Cerco $p_2(x) = ax^2 + bx + c$ t.c. $\displaystyle\lim_{x\to x_0} \frac{f(x)-p_2(x)}{(x-x_0)^2} = 0$

cioè cerco $p_2(x)$ t.c. $E_2(x) := f(x) - p_2(x)$ sia un $o((x-x_0)^2)$

$E_2(x) = f(x) - (ax^2+bx+c)$

$$p_2(x) = ax^2+bx+c = a(x^2-2x_0x+x_0^2)\underbrace{-a(-2x_0x+x_0^2)+bx+c}_{\tilde{p}_1(x)}$$
$$= a(x-x_0)^2 + \tilde{p}_1(x)$$

con $\tilde{p}_1(x)$ polinomio di 1° grado

Voglio $\displaystyle\lim_{x\to x_0} \frac{E_2(x)}{(x-x_0)^2} = 0$

Condizione necessaria perché questo accada è che sia

$$\lim_{x\to x_0} \frac{E_2(x)}{x-x_0} = 0$$

Infatti $\quad \dfrac{E_2(x)}{(x-x_0)^2} = \dfrac{E_2(x)}{x-x_0}\left(\dfrac{1}{x-x_0}\right) \to \begin{matrix} +\infty & \text{se } x\to x_0^+ \\ -\infty & \text{se } x\to x_0^- \end{matrix}$

$$\frac{E_2(x)}{x-x_0} = \frac{f(x) - a(x-x_0)^2 - \tilde{p}_1(x)}{x-x_0} =$$

$$= \frac{f(x)-\tilde{p}_1(x)}{x-x_0} \underbrace{-a(x-x_0)}_{\to\ \text{converge a 0 quando } x\to x_0}$$

$$\lim_{x\to x_0} \frac{E_2(x)}{x-x_0} = 0 \quad \text{SSE} \quad \lim_{x\to x_0} \frac{f(x)-\tilde{p}_1(x)}{x-x_0} = 0$$

$$\text{SSE} \quad \tilde{p}_1(x) = p_1(x) = f'(x_0)(x-x_0) + f(x_0)$$

$$\Rightarrow p_2(x) = a(x-x_0)^2 + f'(x_0)(x-x_0) + f(x_0)$$

$$\lim_{x\to x_0} \frac{E_2(x)}{(x-x_0)^2} = \lim_{x\to x_0} \frac{f(x) - a(x-x_0)^2 - f'(x_0)(x-x_0) - f(x_0)}{(x-x_0)^2}$$

$$= \lim_{x\to x_0} \left(-a + \underbrace{\frac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{(x-x_0)^2}}_{\text{FORMA INDETERMINATA } \frac{0}{0}} \right)$$

$$\frac{\left(f(x)-f(x_0)-f'(x_0)(x-x_0)\right)'}{\left((x-x_0)^2\right)'} = \frac{f'(x)-f'(x_0)}{2(x-x_0)}$$

$$\lim_{x\to x_0} \frac{f'(x)-f'(x_0)}{2(x-x_0)} = \frac{1}{2} f''(x_0)$$

Suppongo che f sia derivabile 2 volte in x_0

$$\Rightarrow p_2(x) = \frac{1}{2} f''(x_0)(x-x_0)^2 + f'(x_0)(x-x_0) + f(x_0)$$

Cerco $p_3(x) = a_3 x^3 + b_3 x^2 + c_3 x + d$ t.c.

$$\lim_{x\to x_0} \frac{f(x)-p_3(x)}{(x-x_0)^3} = 0 \quad \text{cioè t.c. } E_3(x) = f(x) - p_3(x) = o\left((x-x_0)^3\right)$$

$$p_3(x) = a_3 \left(\overbrace{x^3 - 3x_0x^2 + 3x_0^2 x - x_0^3}^{(x-x_0)^3} \right) - a_3\left(-3x_0x^2 + 3x_0^2x - x_0^3\right)$$
$$+ b_3 x^2 + c_3 x + d$$
$$= a_3(x-x_0)^3 + \tilde{p}_2(x)$$

Condizione necessaria affinché $\lim_{x \to x_0} \frac{f(x) - P_3(x)}{(x-x_0)^3} = 0$

è che sia $\lim_{x \to x_0} \frac{f(x) - P_3(x)}{(x-x_0)^2} = 0$

$$\frac{f(x) - P_3(x)}{(x-x_0)^2} = \frac{f(x) - a_3(x-x_0)^3 - \tilde{P}_2(x)}{(x-x_0)^2} =$$

$$= \underbrace{-a_3(x-x_0)}_{\to 0 \text{ quando } x \to x_0} + \frac{f(x) - \tilde{P}_2(x)}{(x-x_0)^2}$$

Quindi $\lim_{x \to x_0} \frac{f(x) - P_3(x)}{(x-x_0)^2} = 0$ SSE $\lim_{x \to x_0} \frac{f(x) - \tilde{P}_2(x)}{(x-x_0)^2} = 0$

SSE $\tilde{P}_2(x) = P_2(x) = \frac{1}{2} f''(x_0)(x-x_0)^2 + f'(x_0)(x-x_0) + f(x_0)$

SSE $P_3(x) = a_3(x-x_0)^3 + \frac{1}{2} f''(x_0)(x-x_0)^2 + f'(x_0)(x-x_0) + f(x_0)$

$$\frac{f(x) - P_3(x)}{(x-x_0)^3} = \frac{f(x) - f(x_0) - f'(x_0)(x-x_0) - \frac{1}{2} f''(x_0)(x-x_0)^2 - a_3(x-x_0)^3}{(x-x_0)^3}$$

$$= -a_3 + \underbrace{\frac{f(x) - f(x_0) - f'(x_0)(x-x_0) - \frac{1}{2} f''(x_0)(x-x_0)^2}{(x-x_0)^3}}_{\text{per } x \to x_0 \text{ forma indeterminata } \frac{0}{0}} \to \frac{1}{6} f''(x_0)$$

$$\frac{\left(f(x) - f(x_0) - f'(x_0)(x-x_0) - \frac{1}{2} f''(x_0)(x-x_0)^2\right)'}{((x-x_0)^3)'} =$$

$$= \frac{f'(x) - f'(x_0) - \cancel{\frac{1}{2}} f''(x_0) \cancel{2}(x-x_0)}{3(x-x_0)^2}$$

Forma indeterminata $\frac{0}{0}$

$$\frac{\left(f'(x) - f'(x_0) - f''(x_0)(x-x_0)\right)'}{(\ldots)'} =$$

$$\frac{(\ldots)'}{\left(3(x-x_0)^2\right)'} =$$

$$= \frac{f''(x) - f''(x_0)}{3 \cdot 2\,(x-x_0)} = \frac{1}{6}\,\frac{f''(x) - f''(x_0)}{x - x_0}$$

Supponiamo che la funzione $f''(x)$ sia derivabile in x_0

$$\left(f''(x_0)\right)' =: f'''(x_0)$$

In questo caso $\exists \lim_{x \to x_0} \frac{1}{6}\,\frac{f''(x) - f''(x_0)}{x - x_0} = \frac{1}{6} f'''(x_0)$

$$\lim_{x \to x_0} \frac{f(x) - P_3(x)}{(x-x_0)^3} = -a_3 + \frac{1}{6} f'''(x_0) = 0 \quad \text{SSE} \quad a_3 = \frac{1}{6} f'''(x_0)$$

$$\text{SSE} \quad P_3(x) = \frac{1}{6} f'''(x_0)(x-x_0)^3 + \frac{1}{2} f''(x_0)(x-x_0)^2 + f'(x_0)(x-x_0) + f(x_0)$$

Si dimostra che se f è derivabile almeno n volte in (a,b), allora $\exists!\, p_n(x)$ polinomio di grado n t.c.

$$f(x) - p_n(x) = o\left((x-x_0)^n\right) \quad \text{in } x_0.$$

Inoltre

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k =$$

$$= f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Inoltre si può dimostrare che se la funzione f è derivabile almeno $n+1$ volte, allora la differenza

$$E_n(x) := f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

dove ξ è un qualche pto compreso nell'intervallo di estremi x_0 e x.

$f(x) = e^x$ $\qquad x_0$

$f'(x) = e^x \qquad f''(x) = e^x \qquad \Rightarrow \quad f^{(k)}(x) = e^x \qquad \forall k \in \mathbb{N}$

$$p_n(x) = \sum_{k=0}^{n} \frac{e^{x_0}}{k!}(x-x_0)^k = e^{x_0} \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!}$$

In particolare per $x_0 = 0$ $\quad p_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$

$f(x) = \sin(x)$

$f'(x) = \cos(x) \qquad f''(x) = -\sin(x) \qquad f'''(x) = -\cos(x)$

$f^{(4)}(x) = \sin(x) = f(x)$

$f^{(4j)}(x) = f(x) = \sin(x) \qquad f^{(4j+2)}(x) = -\sin(x)$

$f^{(4j+1)}(x) = \cos(x) \qquad f^{(4j+3)}(x) = -\cos(x)$

$x_0 = 0 \qquad f^{(2j)}(x) = 0$

$f^{(4j+1)}(0) = 1 \qquad f^{(4j+3)}(0) = -1$

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

$n = 2m \qquad p_{2m}(x) = \sum_{k=0}^{2m} \frac{f^{(k)}(0)}{k!} x^k$

$k = 0, \dots 2m \qquad k = 2j \quad j = 0, \dots m$

$\qquad k = 2j+1 \quad j = 0, \dots m-1$

$$p_{2m}(x) = \sum_{j=0}^{m} \cancel{\frac{f^{(2j)}(0)}{(2j)!}}^{=0} x^{2j} + \sum_{j=0}^{m-1} \frac{f^{(2j+1)}(0)}{(2j+1)!} x^{2j+1} =$$

$f^{(2j+1)}(0) = (-1)^j \qquad = \sum_{j=0}^{m-1} \frac{(-1)^j}{(2j+1)!} x^{2j+1}$

$n = 2m+1 \qquad p_{2m+1}(x) = \sum_{k=0}^{2m+1} \frac{f^{(k)}(0)}{k!} x^k$

$k = 0, 1, \dots 2m+1 \qquad k = 2j \qquad j = 0 \dots m$

$k = 0, 1 \ldots 2m+1$ $\quad K = 2j$ $\quad j = 0 — m$

$k = 2j+1$ $\quad j = 0 — m$

$$P_{2m+1}(x) = \sum_{j=0}^{m} \frac{\cancel{f^{(2j)}(0)}^{=0}}{(2j)!} x^{2j} + \sum_{j=0}^{m} \frac{f^{(2j+1)}(0)}{(2j+1)!} x^{2j+1}$$

$$= \sum_{j=0}^{m} \frac{(-1)^j}{(2j+1)!} x^{2j+1}$$

$$P_5(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{f^{(6)}(\xi)}{6!}x^6$$

$$\left| f(x) - P_5(x) \right| = \left| \frac{f^{(6)}(\xi)}{6!} x^6 \right| \le \underbrace{\frac{1}{6!} x^6} \qquad E_5(x) = o(x^5)$$

$f(x) = \cos(x)$ $\quad x_0 = 0$

$f'(x) = -\sin(x)$ $\quad f''(x) = -\cos(x)$ $\quad f'''(x) = \sin(x)$

$f^{(4)}(x) = \cos(x) = f(x)$

Come prima:

$f^{(4j)}(x) = f(x) = \cos(x)$ $\quad\quad x_0 = 0$ $\quad f^{(4j)}(0) = 1$

$f^{(4j+1)}(x) = f'(x) = -\sin(x)$ $\quad f^{(4j+1)}(0) = 0 \leftarrow$

$f^{(4j+2)}(x) = f''(x) = -\cos(x)$ $\quad f^{(4j+2)}(0) = -1$

$f^{(4j+3)}(x) = f'''(x) = \sin(x)$ $\quad f^{(4j+3)}(0) = 0$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

$n = 2m$

$$P_{2m}(x) = \sum_{k=0}^{2m} \frac{f^{(k)}(0)}{k!} x^k =$$

$k = 2j$ $\quad j = 0, — m$ $\quad k = 2j+1$ $\quad j = 0, — m-1$

$$P_{2m}(x) = \sum_{j=0}^{m} \frac{f^{(2j)}(0)}{(2j)!} x^{2j} + \sum_{j=0}^{m-1} \frac{\cancel{f^{(2j+1)}(0)}^{=0}}{(2j+1)!} x^{2j+1}$$

$$f^{(2j)}(0) = (-1)^j$$

$$P_{2m}(x) = \sum_{j=0}^{m} \frac{(-1)^j}{(2j)!} x^{2j}$$

$$P_{2m+1}(x) = P_{2m}(x) + \frac{\overbrace{\cancel{f^{(2m+1)}}(0)}^{=0}}{(2m+1)!} x^{2m+1} = \sum_{j=0}^{m} \frac{(-1)^j}{(2j)!} x^{2j}$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^5)$$

$f(x) = \frac{1}{1-x}$ $\qquad x_0 = 0$

$$f(x) = (1-x)^{-1}$$

$$f'(x) = +(1-x)^{-2}$$

$$f''(x) = +2(1-x)^{-3}$$

$$f'''(x) = +3\cdot 2(1-x)^{-4}$$

$$f^{(4)}(x) = +4\cdot 3\cdot 2(1-x)^{-5}$$

$$f^{(k)}(x) = k!\,(1-x)^{-(k+1)} \qquad f^{(k)}(0) = k!$$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} x^k$$

$$P_5(x) = 1 + x + x^2 + x^3 + x^4 + x^5$$

$f(x) = \frac{1}{1+x}$ $\qquad x_0 = 0$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = g(-x) \qquad g(x) = \frac{1}{1-x}$$

$P_n(x)$ = polinomio della funzione g calcolato in $-x$

$$= \sum_{k=0}^{n} (-x)^k = \sum_{k=0}^{n} (-1)^k x^k$$

$$P_4(x) = 1 - x + x^2 - x^3 + x^4$$

$$f(x) = \ln(1-x) \qquad x_0 = 0$$

$$f'(x) = \frac{1}{1-x}\cdot(-1) = -g(x) \qquad g(x) = \frac{1}{1-x}$$

$$f^{(k)}(x) = -g^{(k-1)}(x) = -(k-1)!\,(1-x)^{-(k-1)-1} = -(k-1)!\,(1-x)^{-k}$$

$$f^{(k)}(0) = -(k-1)! \quad k \geq 1 \qquad f^{(0)}(0) = \ln(1-0) = 0$$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k = \sum_{k=1}^{n} \frac{-(k-1)!}{k!}x^k = \sum_{k=1}^{n} \frac{-1}{k}x^k$$

$$k! = k\cdot\underbrace{(k-1)\cdot(k-2)\cdots 2\cdot 1} = (k-1)!\cdot k$$

$$P_n(x) = -\sum_{k=1}^{n}\frac{x^k}{k}$$

$$P_4(x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}\right) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$$

PER ESERCIZIO $\quad f(x) = \ln(1+x)$ in $x_0 = 0$

$$f(x) = \frac{1}{1+x^2} \qquad x_0 = 0$$

$$\frac{1}{1+(x^2)} = g(x^2) \qquad g(x) = \frac{1}{1+x} \qquad P_n(x,g) = \sum_{k=0}^{n}(-1)^k x^k$$

$$P_n(x^2, g) = \sum_{k=0}^{n}(-1)^k(x^2)^k = \sum_{k=0}^{n}(-1)^k x^{2k} = P_{2n}(x)$$

$$= P_{2n+1}(x)$$

$$f(x) = \arctan(x) \qquad x_0 = 0$$

$$f'(x) = \frac{1}{1+x^2} \qquad f^{(k)}(x) = \left(\frac{1}{1+x^2}\right)^{(k-1)}$$

$$g(x) = \frac{1}{1+x^2} \qquad P_{2n}(x,g) = P_{2n+1}(x,g) = \sum_{k=0}^{n}(-1)^k x^{2k}$$

$$g^{(2k+1)}(0) = 0 \qquad \frac{g^{(2k)}(0)}{(2k)!} = (-1)^k$$

$$g^{(2k+1)}(0) = 0 \qquad \frac{g^{(2k)}(0)}{(2k)!} = (-1)^k$$

$$g^{(2k)}(0) = (-1)^k (2k)!$$

$$f^{(2k+2)}(0) = 0 \qquad \text{cioè} \qquad f^{(2k)}(0) = 0 \quad \forall k \geq 1$$

$$f^{(2k+1)}(0) = (-1)^k (2k)!$$

$$P_{2n+1}(x) = \sum_{j=0}^{2n+1} \frac{f^{(j)}(0)}{j!} x^j = \qquad \begin{array}{ll} j = 2k & k = 0, \ldots n \\ j = 2k+1 & k = 0, \ldots n \end{array}$$

$$= \sum_{k=0}^{n} \frac{\underbrace{\cancel{f^{(2k)}(0)}}_{=0}}{(2k)!} x^{2k} + \sum_{k=0}^{n} \frac{f^{(2k+1)}(0)}{(2k+1)!} x^{2k+1}$$

$$= \sum_{k=0}^{n} \frac{(-1)^k \cancel{(2k)!}}{\underbrace{\cancel{(2k+1)!}}_{2k+1}} x^{2k+1} = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} x^{2k+1}$$

$$P_5(x) = x - \frac{1}{3} x^3 + \frac{1}{5} x^5$$