lunedì 18 novembre 2019 09:27

$f, g : (a,b) \to \mathbb{R}$ $\quad x_0 \in (a,b)$ $\quad f(x_0) = g(x_0) = 0$

Con f e g derivabili una $g(x) \neq 0$ per $x \neq x_0$

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f(x)-f(x_0)}{x-x_0} \cdot \frac{x-x_0}{g(x)-g(x_0)} =$$

$$= \frac{\dfrac{f(x)-f(x_0)}{x-x_0}}{\dfrac{g(x)-g(x_0)}{x-x_0}} \xrightarrow{x \to x_0} \frac{f'(x_0)}{g'(x_0)} \qquad \text{se } g'(x_0) \neq 0$$

Cioè nelle ipotesi fatte $\exists \lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{f'(x_0)}{g'(x_0)}$

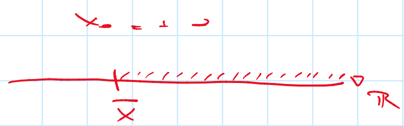
TEOREMA DE L'HÔPITAL

Siano f e g due funzioni derivabili t.c.

$\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = 0 \qquad (x_0 \in \mathbb{R},\ x_0 = +\infty \text{ o } x_0 = -\infty)$

Supponiamo che

1) Se $x_0 \in \mathbb{R}$ $\quad \exists \delta > 0$ t.c. $g'(x) \neq 0$ per $0 < |x - x_0| < \delta$
2) Se $x_0 = +\infty$ $(o\ -\infty)$ $\quad \exists \bar{x}$ t.c. $g'(x) \neq 0$ $\forall x > \bar{x}$ $(\forall x < \bar{x})$



Se $\exists \lim\limits_{x \to x_0} \dfrac{f'(x)}{g'(x)}$

Allora $\quad \exists \lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to x_0} \dfrac{f'(x)}{g'(x)}$

N.B. Si può estendere al caso in cui $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)}$ sia una forma indeterminata del tipo $\dfrac{\infty}{\infty}$.

$$\lim_{x \to +\infty} x\left(\frac{\pi}{2} - \arctan(x)\right) = 1$$

$\frac{\pi}{2} - \arctan(x) \to 0$ "$\infty \cdot 0$"

$$x\left(\frac{\pi}{2} - \arctan(x)\right) = \frac{\frac{\pi}{2} - \arctan(x)}{x^{-1}} \quad \text{"}\frac{0}{0}\text{"}$$

$f(x) = \frac{\pi}{2} - \arctan(x)$ $\quad g(x) = x^{-1}$

$$\frac{f'(x)}{g'(x)} = \frac{\frac{-1}{1+x^2}}{-x^{-2}} = \frac{1}{1+x^2}x^2 = \frac{\cancel{x^2}\,1}{\cancel{x^2}\left(\underbrace{\frac{1}{x^2}}_{\to 0}+1\right)} \to 1$$

$$\lim_{x \to +\infty} \frac{\ln(x)}{x^\alpha} = 0 \qquad \alpha > 0$$

$f(x) = \ln(x)$ $\quad g(x) = x^\alpha$

$$\frac{f'(x)}{g'(x)} = \frac{\frac{1}{x}}{\alpha x^{\alpha-1}} = \frac{1\,\cancel{x^{-1}}}{\alpha x^{\alpha \cancel{-1}}} = \frac{1}{\alpha x^\alpha} \longrightarrow 0 \quad \text{per } x \to +\infty$$

$$\lim_{x \to +\infty} \frac{e^x}{x^\alpha} \qquad \alpha > 0$$

"$\frac{\infty}{\infty}$" $\quad f(x) = e^x \quad g(x) = x^\alpha$

$$\frac{f'(x)}{g'(x)} = \frac{e^x}{\alpha x^{\alpha-1}}$$

$0 < \alpha \leq 1$
- $\alpha \in (0,1)$ $\quad \alpha x^{\alpha-1} \to 0$ $\quad \frac{f'}{g'} \to +\infty$
- $\alpha = 1$ $\quad \alpha x^{\alpha-1} = \alpha$ $\quad \frac{f'}{g'} \to +\infty$

$\boxed{\alpha > 1}$ $\quad \frac{f'(x)}{g'(x)}$ "$\frac{\infty}{\infty}$"

$$\frac{f''(x)}{g''(x)} = \frac{e^x}{\alpha(\alpha-1)x^{\alpha-2}}$$

$$1<\alpha\le 2 \quad \begin{cases} \alpha\in(1,2) & x^{\alpha-2}\to 0 \quad \to \infty \\ \alpha=2 & 2\cdot 1=2 \quad \to \infty \end{cases}$$

$$\alpha>2 \qquad \frac{f'''(x)}{g'''(x)} \qquad f'''=(f'')' \quad g'''=(g'')'$$

$$\forall \alpha>0 \quad \lim_{x\to+\infty} \frac{e^x}{x^\alpha} = +\infty$$

$$\lim_{x\to 1} \frac{\frac{1}{x}-2+x}{\sin^2(\pi x)}$$

$$\frac{1}{x}-2+x \to 1-2+1=0 \qquad \text{"}\frac{0}{0}\text{"}$$

$$\sin^2(\pi x) \to 0$$

$$f(x)=\frac{1}{x}-2+x \qquad g(x)=\sin^2(\pi x)$$

$$\frac{f'(x)}{g'(x)} = \frac{-x^{-2}+1}{2\sin(\pi x)\cdot\cos(\pi x)\cdot\pi} =$$

$$= \frac{-\frac{1}{x^2}+1}{2\pi\sin(\pi x)\cos(\pi x)} = \frac{\frac{x^2-1}{x^2}}{2\pi\sin(\pi x)\cos(\pi x)}$$

$$= \underbrace{\frac{1}{x^2\,2\pi\cos(\pi x)}}_{\to\, -\frac{1}{2\pi}} \cdot \underbrace{\frac{x^2-1}{\sin(\pi x)}}_{\text{"}\frac{0}{0}\text{"} \;\to\; \frac{-2}{\pi}} \longrightarrow -\frac{1}{2\pi}\cdot\frac{-2}{\pi} = \frac{1}{\pi^2}$$

$$f_1(x)=x^2-1 \qquad g_1(x)=\sin(\pi x)$$

$$f_1(x) = x^2 - 1 \qquad g_1(x) = \sin(\pi x)$$

$$\frac{f_1'(x)}{g_1'(x)} = \frac{2x}{\cos(\pi x)\cdot\pi} \xrightarrow{x \to 1} \frac{2}{-\pi}$$

$$\lim_{x\to 0} \left(\frac{\sin(x)}{x}\right)^{1/x} \qquad 2^x = e^{x\ln(2)}$$

$$\left(\frac{\sin(x)}{x}\right)^{1/x} = \exp\left(\underbrace{\frac{1}{x}\cdot\ln\left(\frac{\sin(x)}{x}\right)}\right) \longrightarrow e^0 = 1$$

$$\text{esponente} = \frac{\ln\left(\frac{\sin(x)}{x}\right)}{x} \qquad \text{“}\frac{0}{0}\text{”}$$

$$f(x) = \ln\left(\frac{\sin(x)}{x}\right) \qquad g(x) = x$$

$$\frac{f'(x)}{g'(x)} = \frac{\frac{1}{\frac{\sin(x)}{x}}\cdot\frac{\cos(x)\cdot x - \sin(x)\cdot 1}{x^2}}{1} =$$

$$= \frac{\cancel{x}}{\sin(x)}\,\frac{x\cos(x) - \sin(x)}{x^{\cancel{2}}} = \frac{x\cos(x) - \sin(x)}{x\sin(x)} \qquad \text{“}\frac{0}{0}\text{”}$$

$$f_1(x) = x\cos(x) - \sin(x) \qquad g_1(x) = x\sin(x)$$

$$\frac{f_1'(x)}{g_1'(x)} = \frac{\cancel{\cos(x)} + x(-\sin(x)) - \cancel{\cos(x)}}{\sin(x) + x\cos(x)} =$$

$$= \frac{-x\sin(x)}{\sin(x) + x\cos(x)} \qquad \text{“}\frac{0}{0}\text{”}$$

$$f_2(x) = -x\sin(x) \qquad g_2(x) = \sin(x) + x\cos(x)$$

$$\frac{f_2'(x)}{} = \frac{-\sin(x) - x\cos(x)}{} = \frac{-\sin(x) - x\cos(x)}{}$$

$g_2'(x)$ $\cos(x)+\cos(x)-x\sin(x)$ $2\cos(x)-x\sin(x)$

$$\longrightarrow \frac{0}{2} = 0$$

Al variare del parametro $k\in\mathbb{R}$ determinare il numero e il segno delle soluzioni dell'equazione $f(x)=k$ dove $f(x)= x-2\arctan(x)$

$$\begin{cases} x\in D \\ f(x)=k \end{cases}$$

$D=\mathbb{R}$ $\qquad f(-x)=-f(x)$

$f(0)=0$ $\qquad \lim_{x\to+\infty} f(x) = +\infty$

$f'(x)=0$ $\qquad f'(x) = 1-\frac{2}{1+x^2} = \frac{1+x^2-2}{1+x^2} = \frac{x^2-1}{x^2+1}$

$f'(x)\gtrless 0$ SSE $x^2-1\gtrless 0$

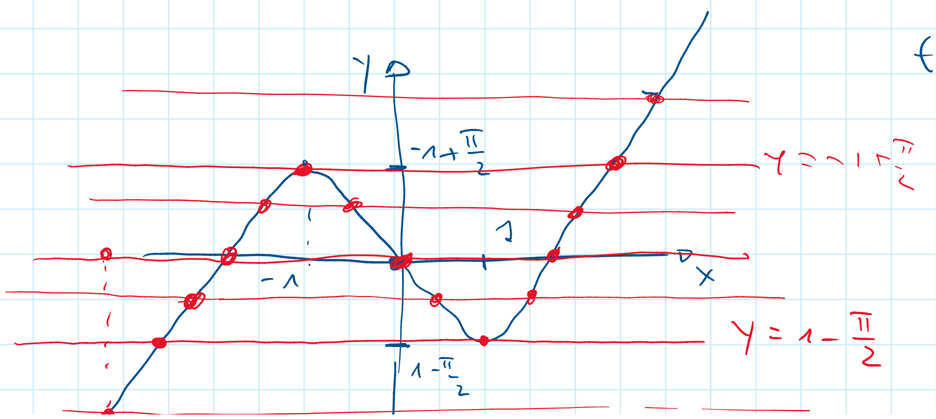
SSE $(x-1)(x+1)\gtrless 0$

$>0\ \forall x>0$

$f'<0$ $\quad f'>0$

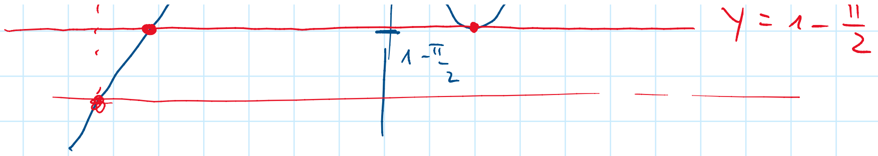
0 $\quad$ 1

$$f(1) = x-2\arctan(x)\Big|_{x=1} = 1-2\arctan(1) = 1-2\cdot\frac{\pi}{4} = 1-\frac{\pi}{2}$$



$f(x)=k$

$$\begin{cases} y=f(x) \leftarrow \\ y=k \leftarrow \end{cases}$$



$k < 1 - \frac{\pi}{2}$ $\quad \exists !$ soluzione ed è negativa

$k = 1 - \frac{\pi}{2}$ $\quad$ 1 soluz negativa e 1 soluz positiva

$k \in \left(1-\frac{\pi}{2}, 0\right)$ $\quad$ 1 soluz negativa e 2 soluz positive

$k = 0$ $\quad$ 1 soluz negativa, 1 soluz. $x=0$ e 1 soluz positiva

$k \in \left(0, \frac{\pi}{2} - 1\right)$ $\quad$ 2 soluz negative e 1 soluz positiva

$k = \frac{\pi}{2} - 1$ $\quad$ 1 soluz negativa e 1 soluz positiva

$k > \frac{\pi}{2} - 1$ $\quad$ 1 soluz positiva.

$1 - \frac{\pi}{2} < k < 0$

$f(x) = \dfrac{\ln(x)}{x^2}$, # di soluz dell'eq $f(x) = k$ al variare di $k \in \mathbb{R}$

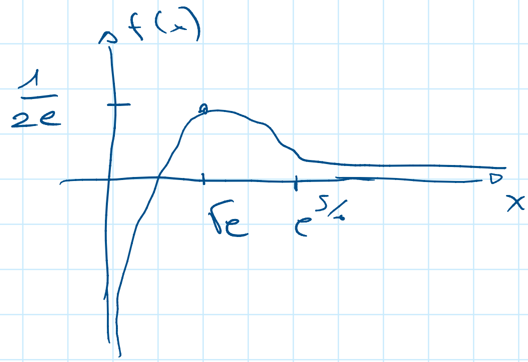
$D = (0, +\infty)$

$$\lim_{x \to 0^+} \frac{\ln(x)}{x^2} = -\infty \qquad \lim_{x \to +\infty} \frac{\ln(x)}{x^2} = 0$$

$$f'(x) = \frac{\frac{1}{\cancel{x}} \cdot x^{\cancel{2}} - \ln(x) \cdot 2x}{x^4}$$

$$= \frac{x(1 - 2\ln(x))}{x^4} =$$

$$= \frac{1 - 2\ln(x)}{x^3} \gtrless 0$$

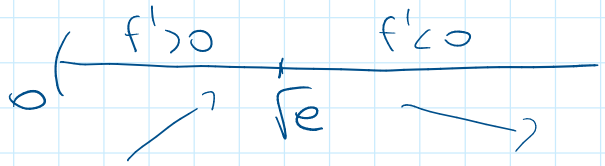


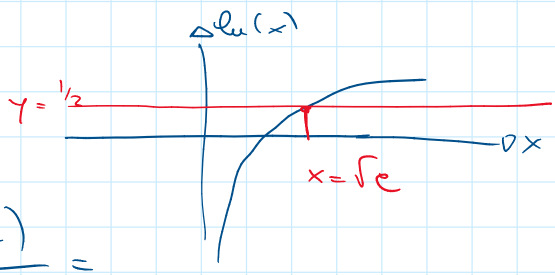
$$\begin{cases} y = f(x) \\ y = k \end{cases}$$

SSE $\quad 1 - 2\ln(x) \gtrless 0$

SSE $\quad \ln(x) \lesseqgtr \frac{1}{2}$ $\qquad \ln(x) = \frac{1}{2}$ $\qquad x = e^{1/2} = \sqrt{e}$

SSE $\ln(x) \lesseqgtr \frac{1}{2}$ $\qquad \ln(x) = \frac{1}{2} \qquad x = e^{\frac{1}{2}} = \sqrt{e}$





$$f(\sqrt{e}) = \left.\frac{\ln(x)}{x^2}\right|_{x=\sqrt{e}} = \frac{\ln(\sqrt{e})}{(\sqrt{e})^2} =$$

$$= \frac{\frac{1}{2}}{e} = \frac{1}{2e}$$

$K \leq 0$ $\quad \exists!$ soluzione

$0 < K < \frac{1}{2e}$ $\quad 2$ soluzione

$k = \frac{1}{2e}$ $\quad \exists!$ soluzione

$K > \frac{1}{2e}$ $\quad \nexists$ soluzione

Tracciare il grafico di f

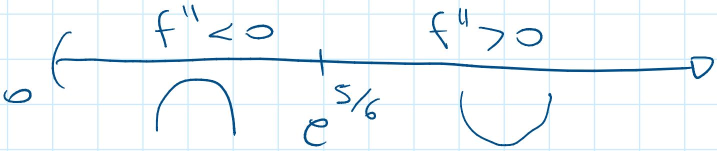
$$f'(x) = \frac{1 - 2\ln(x)}{x^3}$$

$$f''(x) = \frac{x^3 \cdot \left(\frac{-2}{x}\right) - (1 - 2\ln(x))\, 3x^2}{x^6}$$

$$= \frac{-2\cancel{x^2} + 3\cancel{x^2}(2\ln(x) - 1)}{x^{\cancel{6}\,4}}$$

$$= \frac{6\ln(x) - 5}{x^4} \gtreqless 0 \qquad \text{SSE} \quad 6\ln(x) \gtreqless 5$$

SSE $\ln(x) \gtreqless \frac{5}{6}$ $\qquad$ SSE $x \gtreqless e^{5/6}$



Tracciare il grafico della funzione $f(x)=\sqrt{\dfrac{x^3}{x+3}}$

$$\begin{cases} x+3\neq 0 \\ \dfrac{x^3}{x+3}\geq 0 \end{cases} \qquad \begin{cases} x^3\geq 0 \\ x+3>0 \end{cases} \cup \begin{cases} x^3\leq 0 \\ x+3<0 \end{cases}$$

$$\begin{cases} x\geq 0 \\ x>-3 \end{cases} \cup \begin{cases} x\leq 0 \\ x<-3 \end{cases}$$

$$[0,+\infty) \qquad\qquad (-\infty,-3)$$

$$D=(-\infty,-3)\cup[0,+\infty)$$

$$\lim_{x\to-\infty}\sqrt{\frac{x^3}{x+3}}=\lim_{x\to-\infty}\sqrt{\frac{x^3}{x\left(1+\frac{3}{x}\right)}}=\lim_{x\to-\infty}\sqrt{\frac{x^2}{1+\frac{3}{x}}}=\underline{+\infty}$$

$$\lim_{x\to-\infty}\frac{f(x)}{x}=-1$$

$$\frac{f(x)}{x}=\frac{1}{x}\sqrt{\frac{x^3}{x+3}}=\frac{-1}{\underbrace{-x}_{=\sqrt{x^2}}}\sqrt{\frac{x^3}{x+3}}=-\sqrt{\frac{\cancel{x^3}\ 1}{\cancel{x^2}\,\cancel{x}\left(1+\frac{3}{x}\right)}}$$

$$\downarrow$$

$$-1$$

$$\lim_{x\to-\infty}f(x)-(-1\cdot x)=\lim_{x\to-\infty}\left(f(x)+x\right)$$

$$f(x)+x=\sqrt{\frac{x^3}{x+3}}+x \qquad\qquad \frac{(a+b)(a-b)}{a-b}=\frac{a^2-b^2}{a-b}$$

$$\frac{\dfrac{x^3}{x+3}-x^2}{}=\frac{\dfrac{\cancel{x^3}-\cancel{x^3}-3x^2}{x+3}}{}$$

$$\sqrt{\frac{x^3}{x+3}} + x \qquad\qquad \sqrt{\frac{x^3}{x+3}} + x$$

$$\frac{\dfrac{-3x^{\cancel{2}}}{\cancel{x}\left(1+3/x\right)}}{\sqrt{\dfrac{x^{\cancel{3}2}}{\cancel{x}\left(1+\frac{3}{x}\right)}} + x} =$$

$\sqrt{x^2} = |x| = -x$

$$= \frac{-3}{1+\frac{3}{x}} \cdot \frac{x}{-x\sqrt{\dfrac{1}{1+3/x}} + x} =$$

$$= \underbrace{\frac{-3}{1+\frac{3}{x}}}_{\to -3} \cdot \frac{1}{1-\sqrt{\dfrac{1}{1+\frac{3}{x}}}} \longrightarrow 0^- \qquad \longrightarrow +\infty$$

$$\lim_{x\to -3^-} \sqrt{\frac{x^3}{x+3}} = +\infty$$

$$f(0) = \sqrt{\frac{x^3}{x+3}}\Bigg|_{x=0} = 0$$

$$\lim_{x\to +\infty} \sqrt{\frac{x^3}{x+3}} = \lim_{x\to +\infty} \sqrt{\frac{x^{\cancel{3}2}}{\cancel{x}\left(1+\frac{3}{x}\right)}} = +\infty$$

$$\lim_{x\to +\infty} \frac{f(x)}{x} = \lim_{x\to +\infty} \frac{1}{x}\sqrt{\frac{x^3}{x+3}} =$$

$x > 0$

$x = \sqrt{x^2}$

$$= \lim_{x\to +\infty} \sqrt{\frac{\cancel{x^3}}{\cancel{x^2}\cdot\cancel{x}\left(1+\frac{3}{x}\right)}} = 1$$

$$\lim_{x\to+\infty} f(x) - 1\cdot x = \lim_{x\to+\infty}\left(\sqrt{\frac{x^3}{x+3}} - x\right)$$

$$\frac{(a-b)(a+b)}{a+b}$$

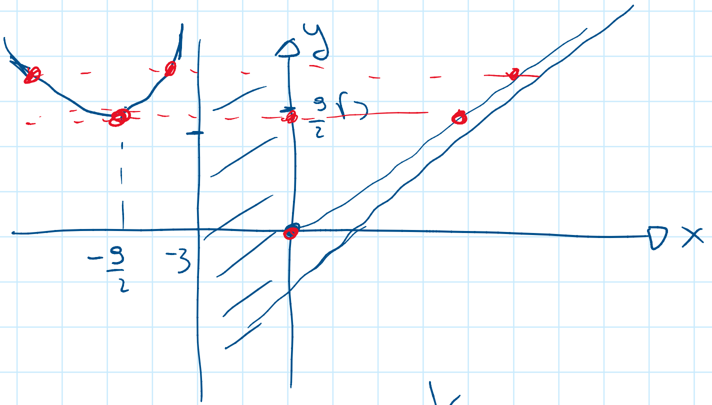
$$\sqrt{\frac{x^3}{x+3}} - x = \frac{\left(\sqrt{\frac{x^3}{x+3}}-x\right)\left(\sqrt{\frac{x^3}{x+3}}+x\right)}{\sqrt{\frac{x^3}{x+3}}+x} =$$

$$= \frac{\frac{x^3}{x+3} - x^2}{\sqrt{\frac{x^3}{x+3}}+x} = \frac{\frac{\cancel{x^3}-\cancel{x^3}-3x^2}{x+3}}{\sqrt{\frac{x^{\cancel{3}\,2}}{\cancel{x}\left(1+\frac{3}{x}\right)}}+x}$$

$$= \frac{-3x^2}{x\left(1+\frac{3}{x}\right)} \cdot \frac{1}{x\sqrt{\frac{1}{1+\frac{3}{x}}}+x} =$$

$$= \frac{-3\cancel{x^2}}{\cancel{x^2}\underbrace{\left(1+\frac{3}{x}\right)}_{\downarrow\,1}\underbrace{\left(\frac{1}{\sqrt{1+\frac{3}{x}}}+1\right)}_{\downarrow\,1}} \longrightarrow \frac{-3}{2}$$

Per $x\to+\infty$ AS. OBLIQUO $\boxed{y = x - \frac{3}{2}}$



$k<0$ ∄ soluz

$k=0$ ∃! $x=0$

$0<k<\frac{9}{2}\sqrt{3}$ ∃! soluz ed è positiva

$k=\frac{9}{2}\sqrt{3}$ 1 soluz positiva e 1 soluz negativa

$k>\frac{9}{2}\sqrt{3}$ 1 soluz positiva

$$f(x) = \left(\frac{x^3}{x+3}\right)^{1/2}$$

$$f(x) = \left(\frac{x}{x+3}\right)$$

$k > \frac{9}{2}\sqrt{3}$ 1 soluz positiva e 2 negative

$$f'(x) = \frac{1}{2}\left(\frac{x^3}{x+3}\right)^{\frac{1}{2}-1} \frac{3x^2(x+3) - x^3 \cdot 1}{(x+3)^2} =$$

$$= \frac{1}{2}\left(\frac{x+3}{x^3}\right)^{1/2} \frac{2x^3 + 9x^2}{(x+3)^2} =$$

$$= \frac{1}{2} \frac{x^2(2x+9)}{x^{3/2}(x+3)^{3/2}} \geq 0 \qquad 2x+9 \geq 0$$

$$x \geq -\frac{9}{2}$$

$f' < 0$ $f' > 0$ $f' > 0$

$-\frac{9}{2}$ -3 0

$$f\left(-\frac{9}{2}\right) = \left(\frac{\left(-\frac{9}{2}\right)^3}{-\frac{9}{2}+3}\right)^{1/2} = \left(\frac{-9^3}{8\left(-\frac{9}{2}+\frac{6}{2}\right)}\right)^{1/2} =$$

$$= \left(\frac{\cancel{-}\,9^3}{\cancel{8}^{\,4} \cdot \frac{\cancel{+}3}{\cancel{2}}}\right)^{1/2} = \frac{3^3}{\sqrt{4 \cdot 3}} = \frac{3^2 \cdot 3}{2\, 3^{1/2}} = \frac{9\sqrt{3}}{2}$$