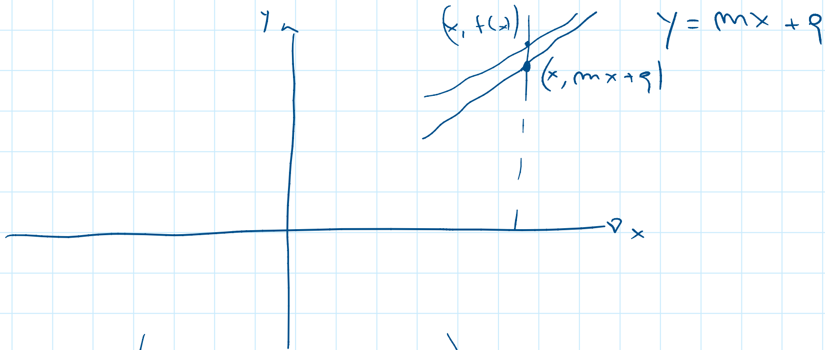
venerdì 15 novembre 2019 09:29

$f: \mathbb{R} \to \mathbb{R}$ t.c. $\lim_{x\to+\infty} f(x) = +\infty$ $(o\ -\infty)$



$$\lim_{x\to+\infty}\left(f(x) - (mx+q)\right) = 0$$

$$\lim_{x\to+\infty}\frac{f(x)-(mx+q)}{x} = 0$$

$$\frac{f(x)-(mx+q)}{x} = \frac{f(x)}{x} - m + \underbrace{\frac{q}{x}}_{\to 0} \longrightarrow 0 \quad \text{per } x\to+\infty$$

SSE $\lim_{x\to+\infty}\frac{f(x)}{x} = m$ $\leftarrow$ $\boxed{m \neq 0}$

$f(x) - mx + q \to 0$ SSE $f(x) - mx \to q$

$\lim_{x\to+\infty}(f(x)-mx) = q$ $\leftarrow$ $\Rightarrow$ dico che $y = mx+q$ è

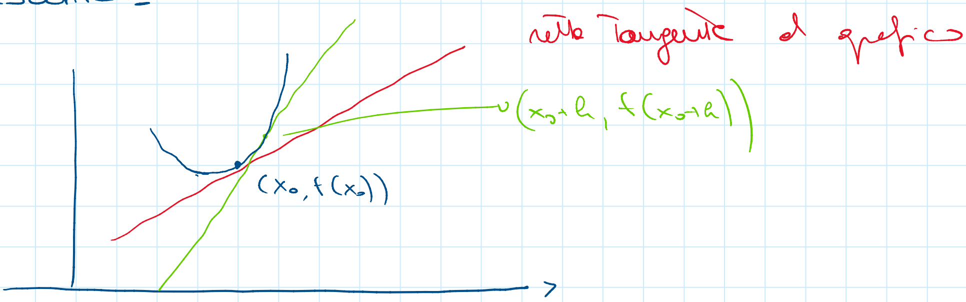
ASINTOTO OBLIQUO
PER $x\to+\infty$

— o —

$f:(a,b)\to\mathbb{R}$ T.c. $\exists f'': x\in(a,b) \to f''(x)\in\mathbb{R}$

con $f''(x) > 0$ $\forall x \in (a,b)$

Di conseguenza $f': x\in(a,b)\to f'(x)\in\mathbb{R}$ è strettamente crescente.



Questo implica che il grafico della funzione ha una concavità

Questo implica che il grafico della funzione ha una concavità rivolta verso l'alto. Si dice che la funzione f è CONVESSA.

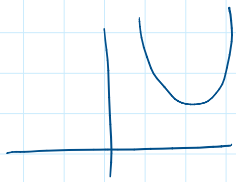
Analogamente, se $f''(x) < 0 \quad \forall x \in (a,b) \Rightarrow$ il grafico della funzione ha una concavità rivolta verso il basso. Si dice che la funzione f è concava.

$f(x) = ax^2 + bx + c \qquad a \neq 0$

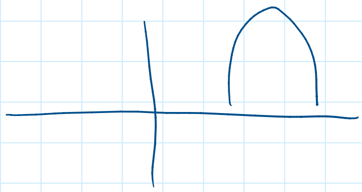
$f'(x) = 2ax + b$

$f''(x) = 2a$

Se $a > 0$



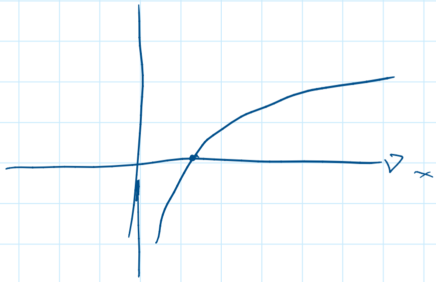
Se $a < 0$



$f(x) = \ln(x) \qquad x > 0$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2} < 0$



ESERCIZIO $\quad f(x) = \frac{x^2+3}{x-1}$

$x - 1 \neq 0 \qquad x \neq 1 \qquad D = \mathbb{R} \setminus \{1\} = (-\infty, +1) \cup (1, +\infty)$

$\lim_{x \to 1^+} \frac{x^2+3}{x-1} = +\infty$

$x = 1$ ASINTOTO VERTICALE

$\lim_{x \to 1^-} \frac{x^2+3}{x-1} = -\infty$

$\lim_{x \to +\infty} \frac{x^2+3}{x-1} = \lim_{x \to +\infty} \frac{\cancel{x^2}\left(1 + \cancel{\frac{3}{x^2}}\right)}{\cancel{x}\left(1 - \cancel{\frac{1}{x}}\right)} = +\infty$

$\lim \quad f(x) \quad \lim \quad \frac{x^2+3}{} \quad \lim \quad \frac{\cancel{x^2}\left(1+\frac{3}{x}\right)}{} \quad - \quad 1$

$$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{x^2+3}{x(x-1)}=\lim_{x\to+\infty}\frac{\cancel{x^2}\left(1+\frac{3}{x}\right)}{\cancel{x}\cdot\cancel{x}\left(1-\frac{1}{x}\right)}=1$$

Se c'è asintoto obliquo $\Rightarrow m=1$

$$\lim_{x\to+\infty} f(x)-1\cdot x=\lim_{x\to+\infty}\left(\frac{x^2+3}{x-1}-x\right)=$$

$$=\lim_{x\to+\infty}\frac{x^2+3-x(x-1)}{x-1}=\lim_{x\to+\infty}\frac{\cancel{x^2}+3-\cancel{x^2}+x}{x-1}=$$

$$=\lim_{x\to+\infty}\frac{x+3}{x-1}=\lim_{x\to+\infty}\frac{\cancel{x}\left(1+\frac{3}{x}\right)}{\cancel{x}\left(1-\frac{1}{x}\right)}=1$$

Per $x\to+\infty$ ho ASINTOTO OBLIQUO $y=x+1$

$$\lim_{x\to-\infty}\frac{x^2+3}{x-1}=\lim_{x\to-\infty}\frac{x^{\cancel{2}}\left(1+\frac{3}{x}\right)}{\cancel{x}\left(1-\frac{1}{x}\right)}=-\infty$$

$$\lim_{x\to-\infty}\frac{f(x)}{x}=\lim_{x\to-\infty}\frac{x^2+3}{x(x-1)}=\lim_{x\to-\infty}\frac{\cancel{x^2}\left(1+\frac{3}{x}\right)}{\cancel{x}\cdot\cancel{x}\left(1-\frac{1}{x}\right)}=1$$

$$\lim_{x\to-\infty} f(x)-1\cdot x=\lim_{x\to-\infty}\left(\frac{x^2+3}{x-1}-x\right)=$$

$$=\lim_{x\to-\infty}\frac{\cancel{x^2}+3-x(\cancel{x}-1)}{x-1}=\lim_{x\to-\infty}\frac{x+3}{x-1}=$$

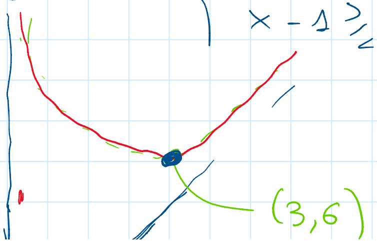
$$=\lim_{x\to-\infty}\frac{\cancel{x}\left(1+\frac{3}{x}\right)}{\cancel{x}\left(1-\frac{1}{x}\right)}=1$$

Per $x\to-\infty$ $y=x+1$ è ASINTOTO OBLIQUO

$$f(x)=\frac{x^2+3}{x-1}$$

$f(x)\gtrless 0$ SSE $\begin{cases} x\in D\\ x-1\gtrless 0\end{cases}$ $\begin{cases} x\neq 1\\ x\gtrless 1\end{cases}$

f Fto sempre



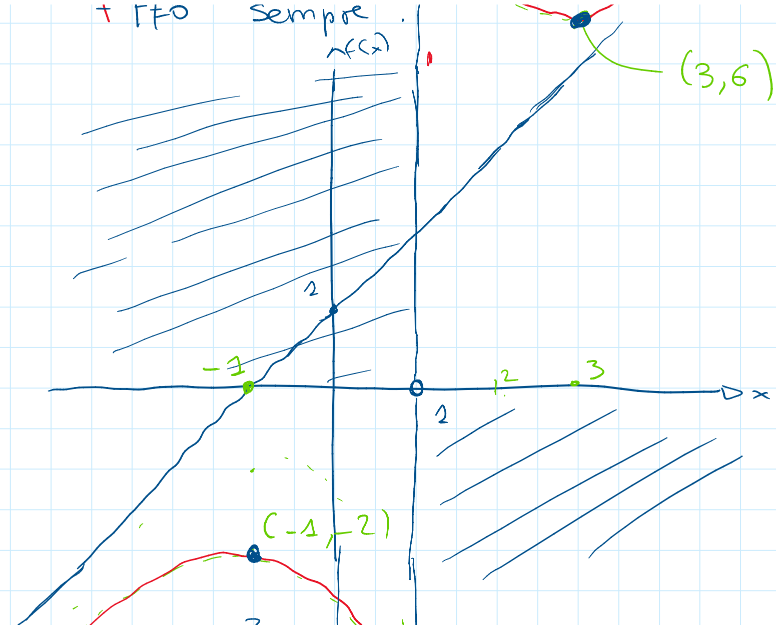
$f(x)>0$ SSE $x>1$

$f(x)<0$ SSE $x<1$

+ $\Gamma f \circ$ sempre .

$f(x) > 0$ SSE $x > 1$

$f(x) < 0$ SSE $x < 1$



$$f(x) = \frac{x^2+3}{x-1}$$

$$f'(x) = \frac{(x^2+3)'(x-1) - (x^2+3)(x-1)'}{(x-1)^2} =$$

$$= \frac{2x(x-1) - (x^2+3)\cdot 1}{(x-1)^2} = \frac{2x^2 - 2x - x^2 - 3}{(x-1)^2} =$$

$$= \frac{x^2 - 2x - 3}{(x-1)^2}$$

$f'(x) \gtrless 0$ SSE $\begin{cases} x \in D & x \neq 1 \\ x^2 - 2x - 3 \gtrless 0 \end{cases}$

$x^2 - 2x - 3 = 0 \qquad \frac{\Delta}{4} = 1 + 3 = 4$

$$x_{1,2} = \frac{1 \pm 2}{1} \quad \begin{matrix} 3 \\ -1 \end{matrix}$$

$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
↗ -1	↘ 1	↘ 3	↗

$\cancel{x^2} \cancel{2x}$

$$x^2 - 2x - 3 = (x-3)(x+1)$$

$f(x) = \frac{x^2+3}{x-1} \Rightarrow f(3) = \frac{9+3}{3-1} = 6$ MIN RELATIVO

$f(-1) = \frac{1+3}{-1-1} = -2$ MAX RELATIVO

$$f'(x) = \frac{x^2-2x-3}{(x-1)^2}$$

$$f''(x) = \frac{(x^2-2x-3)'(x-1)^2 - (x^2-2x-3)\left((x-1)^2\right)'}{(x-1)^4} =$$

$$= \frac{(2x-2)(x-1)^2 - (x^2-2x-3)\left(2(x-1)\cdot 1\right)}{(x-1)^4}$$

$$= \frac{2(x-1)(x-1)^2 - 2(x-1)(x^2-2x-3)}{(x-1)^4} =$$

$$= \frac{2\cancel{(x-1)}\left(\cancel{x^2}-\cancel{2x}+1-\cancel{x^2}+\cancel{2x}+3\right)}{(x-1)^{\cancel{4}\,3}} = \frac{8}{(x-1)^3}$$

$f''(x) < 0$ $\qquad$ 1 $\qquad$ $f''(x) > 0$

$\cap$ $\qquad\qquad$ $\cup$

$$f(x) = \sqrt{x^2+x} - x$$

$$D = \{ x \in \mathbb{R} : x^2 + x \geq 0 \}$$

$x^2 + x = 0 \qquad x(x+1) = 0$

$+ \quad -1 \quad - \quad 0 \quad +$

$$D = (-\infty, -1] \cup [0, +\infty$$

$f(x) \geq 0$ SSE $\sqrt{x^2+x} - x \geq 0$

SSE $\sqrt{x^2+x} \geq x \qquad x \in (-\infty, -1] \cup [0, +\infty)$

Se $x \leq -1$, $x < 0$ $\Rightarrow$ vero banalmente

cioè $f(x) > 0 \quad \forall x \leq -1$

Se $x \geq 0$ le disuguaglianze si conserva passando ai quadrati perché entrambi i membri della disuguaglianza sono nonnegativi

$$f(x) \geq 0 \quad \text{SSE} \quad \left(\sqrt{x^2+x}\right)^2 \geq x^2 \quad , \quad x \geq 0$$

$$\text{SSE} \quad \cancel{x^2} + x \geq \cancel{x^2} \quad , \quad x \geq 0$$

Quindi $f(x) \geq 0 \quad \forall x \geq 0$ cioè $f(x) \geq 0 \quad \forall x \in D$

$$D = (-\infty, -1] \cup [0, +\infty)$$

$$f(0) = \sqrt{x^2+x} - x \Big|_{x=0} = 0$$

$$f(-1) = \sqrt{x^2+x} - x \Big|_{x=-1} = 1$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \sqrt{x^2+x} - x_{\sqrt{x^2}} \qquad \sqrt{f(x)} - \sqrt{g(x)}$$

Supponiamo $\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} h(x) = +\infty$

$$\lim_{x \to +\infty} \sqrt{g(x)} - \sqrt{h(x)} \qquad \frac{\left(\sqrt{g(x)} - \sqrt{h(x)}\right)\left(\sqrt{g(x)} + \sqrt{h(x)}\right)}{\sqrt{g(x)} + \sqrt{h(x)}}$$

$$= \frac{g(x) - h(x)}{\sqrt{g(x)} + \sqrt{h(x)}}$$

$$(a-b)(a+b) = a^2 - \cancel{ab} + \cancel{ba} - b^2$$

$$x \to +\infty \qquad f(x) = \sqrt{x^2+x} - x = \sqrt{x^2+x} - \sqrt{x^2} =$$

$$= \frac{\left(\sqrt{x^2+x} - \sqrt{x^2}\right)\left(\sqrt{x^2+x} + \sqrt{x^2}\right)}{\sqrt{x^2+x} + \sqrt{x^2}} =$$

$$= \frac{(\cancel{x^2}+x) - \cancel{x^2}}{\sqrt{x^2+x} + x} = \frac{x}{\sqrt{x^2+x} + x}$$

$$= \frac{\cancel{x}\ 1}{\ldots} = \frac{1}{\ldots} =$$

$$= \frac{\cancel{x}\ 1}{\cancel{x}\left(\underbrace{\frac{1}{x}}_{=\sqrt{x^2}}\sqrt{x^2+x}+1\right)} = \frac{1}{\sqrt{\frac{x^2+x}{x^2}}+1} =$$

$$= \frac{1}{\sqrt{1+\frac{1}{x}}+1} \longrightarrow \frac{1}{2}$$

AS ORIZZONTALE $y = \frac{1}{2}$ per $x \to +\infty$

$$\lim_{x\to-\infty}\left(\sqrt{x^2+x}-x\right) = +\infty$$

$$\sqrt{x^2+x}-x = \sqrt{\underbrace{x^2}_{\to+\infty}\underbrace{\left(1+\frac{1}{x}\right)}_{\to 1}}+(-x) \to +\infty \quad \text{per } x\to-\infty$$

$$\lim_{x\to-\infty}\frac{f(x)}{x} = \lim_{x\to-\infty}\frac{\sqrt{x^2+x}-x}{x} = -2$$

$$\frac{\sqrt{x^2+x}-x}{x} = \frac{\sqrt{x^2+x}}{x}-1$$

$x \to -\infty$, $x<0$ $\quad \cancel{x = \sqrt{x^2}} \quad -x = \sqrt{x^2}$

$$\frac{\sqrt{x^2+x}}{x}-1 = \frac{\sqrt{x^2+x}}{-\sqrt{x^2}}-1 = -\sqrt{\frac{x^2+x}{x^2}}-1$$

$$= -\sqrt{1+\underbrace{\frac{1}{x}}_{\to 0}}-1 \longrightarrow -2$$

$$\lim_{x\to-\infty}\left(f(x)-(-2x)\right) = \lim_{x\to-\infty}\left(\sqrt{x^2+x}-x+2x\right) =$$

$$= \lim_{x\to-\infty}\left(\sqrt{x^2+x}+x\right) = -\frac{1}{2}$$

$$\sqrt{x^2+x}+x = \sqrt{x^2+x}-\overset{>0}{(-x)} = \sqrt{x^2+x}-\sqrt{x^2} =$$

$$= \frac{\left(\sqrt{x^2+x}-\sqrt{x^2}\right)\left(\sqrt{x^2+x}+\sqrt{x^2}\right)}{\sqrt{x^2+x}+\underbrace{\sqrt{x^2}}_{-x}} = \frac{(\cancel{x^2}+x)-\cancel{x^2}}{\sqrt{x^2+x}-x} =$$

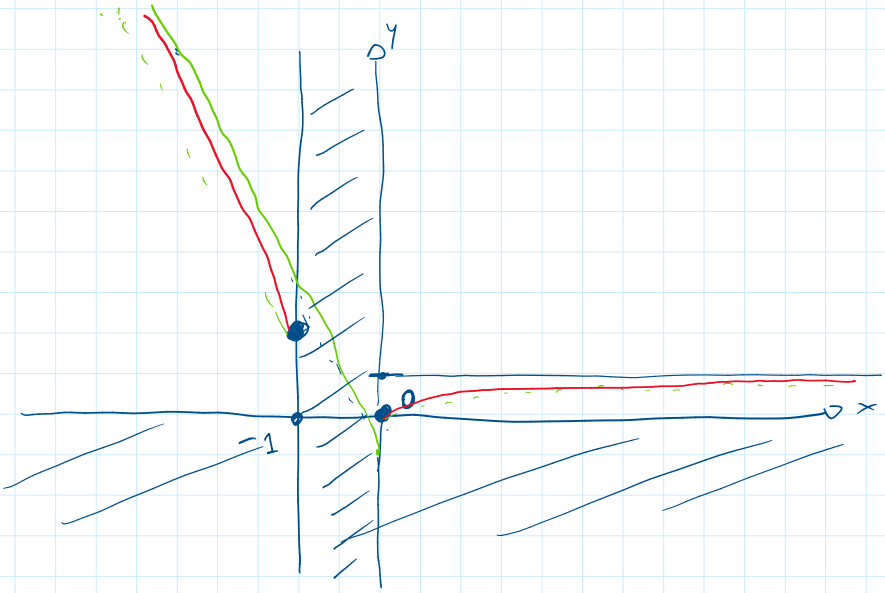
$$= \frac{x}{\sqrt{x^2+x}-x} = \frac{\cancel{x}\ 1}{\cancel{x}\left(\frac{1}{x}\sqrt{x^2+x}-1\right)} =$$

$$-(-x) = -\sqrt{x^2}$$

$$= \frac{1}{-\sqrt{\frac{x^2+x}{x^2}}-1} =$$

$$= \frac{-1}{\sqrt{1+\frac{1}{x}}+1} \longrightarrow \frac{-1}{2}$$

Per $x \to -\infty$ AS OBLIQUO $y=-2x-\frac{1}{2}$



$$f(x)=\sqrt{x^2+x}-x = (x^2+x)^{1/2}-x$$

$$f'(x) = \frac{1}{2}(x^2+x)^{\frac{1}{2}-1}(2x+1)-1 = \frac{1}{2}\frac{2x+1}{\sqrt{x^2+x}}-1 =$$

$$= \frac{2x+1-2\sqrt{x^2+x}}{2\sqrt{x^2+x}}$$

$f'(x) \gtrless 0$ SSE $\begin{cases} 2x+1-2\sqrt{x^2+x} \gtrless 0 \\ \ldots \end{cases}$

$$f'(x) \gtrless 0 \iff \begin{cases} 2x+1-2\sqrt{x^2+x} \gtrless 0 \\ x\in D \end{cases}$$

$$\begin{cases} x\in D \\ 2x+1 \gtrless 2\sqrt{x^2+x} \end{cases}$$

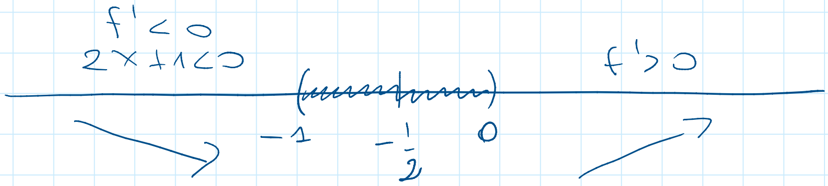
Se $x\in D$ e $2x+1<0 \Rightarrow f'(x)<0$

Se $x\in D$ e $2x+1\geq 0 \Rightarrow$ la disuguaglianza è equivalente a $(2x+1)^2 \gtrless \left(2\sqrt{x^2+x}\right)^2$

$$\cancel{4x^2}+\cancel{4x}+1 \gtrless 4(\cancel{x^2}+\cancel{x})$$

$$1 > 0$$

$\Rightarrow$ Se $x\in D$ e $2x+1\geq 0 \Rightarrow f'(x)>0$



$$f'(x)=\frac{1}{2}(x^2+x)^{-1/2}(2x+1)-1$$

$$f''(x)=\frac{1}{2}\left(\left((x^2+x)^{-1/2}\right)'(2x+1)+(x^2+x)^{-1/2}(2x+1)'\right)$$

$$=\frac{1}{2}\left(-\frac{1}{2}(x^2+x)^{\overset{-3/2}{-\frac{1}{2}-1}}(2x+1)^2+(x^2+x)^{-1/2}\cdot 2\right)$$

$$=\frac{1}{4}(x^2+x)^{-3/2}\left(-(2x+1)^2+2\cdot 2(x^2+x)\right)$$

$$=\frac{1}{4}(x^2+x)^{-3/2}\left(-\cancel{4x^2}-\cancel{4x}-1+\cancel{4x^2}+\cancel{4x}\right)$$

$$=-\frac{1}{4}(x^2+x)^{-3/2}<0 \qquad \cancel{\forall x\in D} \qquad \forall x\,(-\infty,-1)\cup(0,+\infty)$$

$$=\frac{-1}{4}\,\frac{1}{(x^2+x)^{3/2}}$$

$$= \frac{-1}{4} \frac{1}{(x^2+x)^{3/2}}$$

ESERCIZIO $\quad f(x) = \frac{x}{4x+1} e^{-x}$