lunedì 21 ottobre 2019 09:31

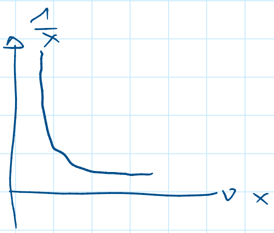
$\sin : x \in \mathbb{R} \mapsto \sin(x) \in \mathbb{R}$
$\cos : x \in \mathbb{R} \mapsto \cos(x) \in \mathbb{R}$

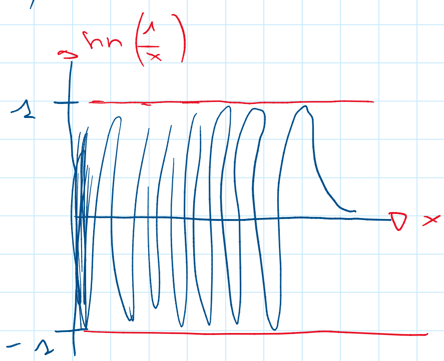
Entrambe periodiche, con periodo 2π
Immagine di entrambe è l'intervallo $[-1, 1]$
Sono continue

$\forall x \in \mathbb{R} \quad \cos^2(x) + \sin^2(x) = 1$

$\nexists \lim_{x \to +\infty} f(x)$ sia per $f(x) = \cos(x)$ che per $f(x) = \sin(x)$

$\nexists \lim_{x \to 0^+} f(x) \qquad f(x) = \sin\left(\frac{1}{x}\right)$





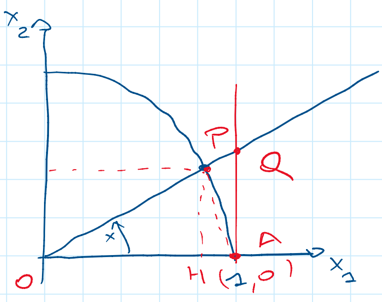
$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

$f(x) = \frac{\sin(x)}{x} \qquad f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$

è una funzione pari

$$f(-x) = \frac{\sin(-x)}{-x} = \frac{-\sin(x)}{-x} = \frac{\sin(x)}{x} = f(x)$$

Considero $\lim_{x \to 0^+} \frac{\sin(x)}{x}$ - Se prendo entrambe sono a posto



$x_1 = \cos(x)$
$x_2 = \sin(x)$

$\frac{\overline{PH}}{\overline{PA}} = \sin(x)$

OPH e OQH sono triangoli simili

$\overline{OH} : \overline{PH} = \overline{OA} : \overline{QA}$
$\cos(x) : \sin(x) = 1 : \overline{QA}$

$\overline{QA} = \frac{\sin(x)}{\cos(x)}$

$\overset{\frown}{PA} = x \qquad \overset{1}{\leq} \overline{QA} = \frac{\sin(x)}{\cos(x)}$

$\overset{\frown}{PA} \geq \overline{PH} \qquad x \geq \sin(x)$

$$\overline{QA} = \frac{\sin(x)}{\cos(x)}$$

$$0 < \sin(x) \leq x \leq \frac{\sin(x)}{\cos(x)} \qquad \text{poiché} \quad x \in \left(0, \frac{\pi}{2}\right)$$

$$0 < \frac{\cos(x)}{\sin(x)} \leq \frac{1}{x} \leq \frac{1}{\sin(x)}$$

$$0 < \cancel{\sin(x)} \frac{\cos(x)}{\cancel{\sin(x)}} \leq \frac{\sin(x)}{x} \leq \frac{1}{\cancel{\sin(x)}} \cancel{\sin(x)}$$

$$\cos(x) \leq \frac{\sin(x)}{x} \leq 1 \qquad \forall x \in \left(0, \frac{\pi}{2}\right)$$

$x \to 0^+$: $\cos(x) \to 1$; $1 \to 1$ ($x \to 0^+$)

Per il Teorema dei due carabinieri:

$$\lim_{x \to 0^+} \frac{\sin(x)}{x} = 1$$

Poiché la funzione $\frac{\sin(x)}{x}$ è pari, $\displaystyle\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

— o —

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2}$$

$$\frac{1 - \cos(x)}{x^2} \cdot \frac{1 + \cos(x)}{1 + \cos(x)} = \frac{1 - \cos^2(x)}{x^2} \cdot \frac{1}{1 + \cos(x)} =$$

$$= \frac{\sin^2(x)}{x^2} \cdot \frac{1}{1 + \cos(x)} = \left(\frac{\sin(x)}{x}\right)^2 \cdot \frac{1}{1 + \cos(x)} \longrightarrow 1^2 \cdot \frac{1}{2} = \frac{1}{2}$$

Quando $x \to 0$ $\frac{\sin(x)}{x} \to 1$

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^n} = \begin{cases} \frac{1}{2} & n = 2 \\ \nexists < \begin{array}{l} \lim_{x \to 0^+} = +\infty \\ \lim_{x \to 0^-} = -\infty \end{array} & n = 3 \\ 0 & n = 1 \\ +\infty & n = 4 \end{cases}$$

$n = 1$

$$\frac{1 - \cos(x)}{x} = \frac{1 - \cos(x)}{x^2} \cdot x \longrightarrow \frac{1}{2} \cdot 0 = 0$$

$n = 3$

$$\frac{1 - \cos(x)}{x^3} = \underbrace{\frac{1 - \cos(x)}{x^2}}_{\to \frac{1}{2}} \cdot \frac{1}{x} \begin{cases} +\infty & x \to 0^+ \\ -\infty & x \to 0^- \end{cases}$$

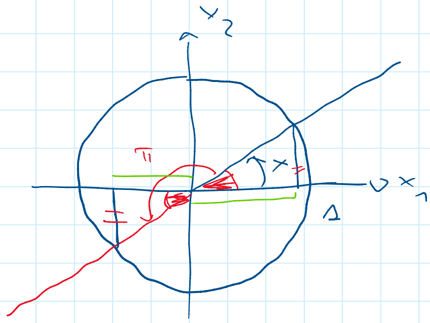
2

$n=4 \quad \frac{1-\cos(x)}{x^4} = \underbrace{\frac{1-\cos(x)}{x^2}}_{\to \frac{1}{2}} \cdot \frac{1}{x^2} \to +\infty$

$tg(x) := \frac{\sin(x)}{\cos(x)} \qquad tg : \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \right\} \to \mathbb{R}$

$tg(x+2\pi) = tg(x) \qquad x \neq \frac{\pi}{2} + k\pi$

$tg(x+\pi) = tg(x) \qquad x \neq \frac{\pi}{2} + k\pi$



$tg(x+\pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)}$

$= \frac{-\sin(x)}{-\cos(x)} = \frac{\sin(x)}{\cos(x)} = tg(x)$

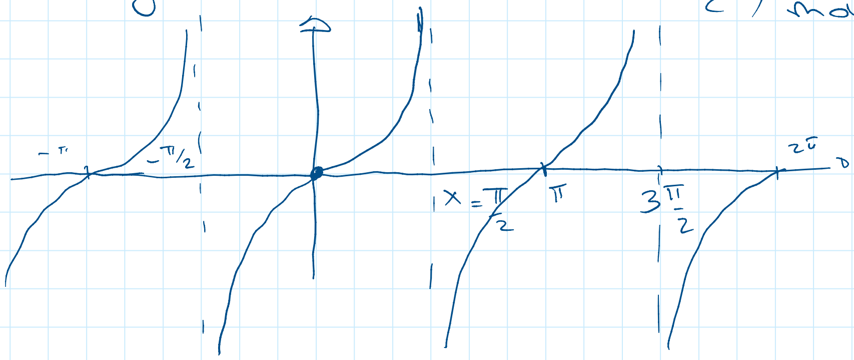
$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \qquad tg(x) = \frac{\sin(x)}{\cos(x)}$ è dispari

$tg(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin(x)}{\cos(x)} = -tg(x)$

$\left[0, \frac{\pi}{2}\right) \qquad tg(0) = \frac{\sin(0)}{\cos(0)} = \frac{0}{1} = 0$

$\lim_{x \to \left(\frac{\pi}{2}\right)^-} tg(x) = \lim_{x \to \left(\frac{\pi}{2}\right)^-} \frac{\sin(x)}{\cos(x)} = +\infty \qquad \cos(x) > 0$

$tg(x)$ nell'intervallo $\left[0, \frac{\pi}{2}\right)$ è strettamente monotona crescente



$cotg(x) = \frac{\cos(x)}{\sin(x)} \qquad x \neq k\pi$

$cotg : \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\} \to \mathbb{R}$

$x \neq k\pi \qquad cotg(x+\pi) = \frac{\cos(x+\pi)}{} = \frac{-\cos(x)}{} = cotg(x)$

$x \neq k\pi \qquad \cot g(x+\pi) = \frac{\cos(x+\pi)}{\sin(x+\pi)} = \frac{\cancel{-}\cos(x)}{\cancel{-}\sin(x)} = \cot g(x)$

$\cancel{(0,\pi)} \qquad \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

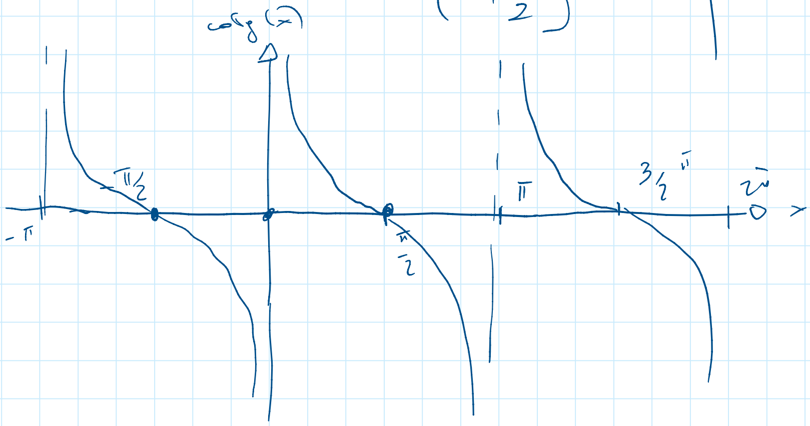
$\cot g(-x) = \frac{\cos(-x)}{\sin(-x)} = \frac{\cos(x)}{-\sin(x)} = -\cot g(x)$

$\left(0, \frac{\pi}{2}\right]$

$\cot g\left(\frac{\pi}{2}\right) = \frac{\cos\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right)} = \frac{0}{1} = 0$

$\lim_{x \to 0^+} \cot g(x) = \lim_{x \to 0^+} \frac{\cos(x)}{\sin(x)} = +\infty$

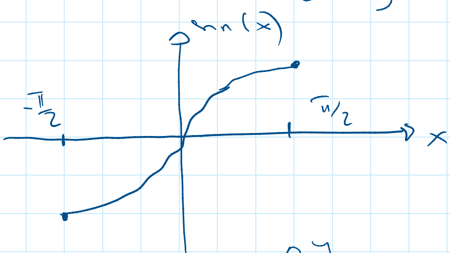
$\cot g(x) = \frac{\cos(x)}{\sin(x)}$ decrescente in $\left(0, \frac{\pi}{2}\right]$

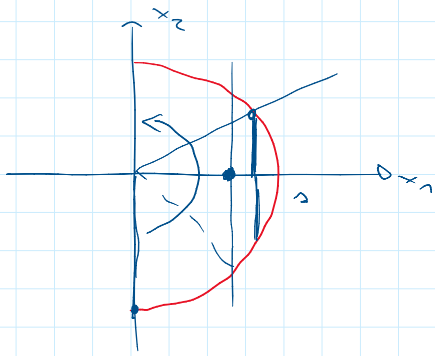


FUNZIONI TRIGONOMETRICHE INVERSE

$f(x) = \sin(x)$

$f(x) = \sin\big|_{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$





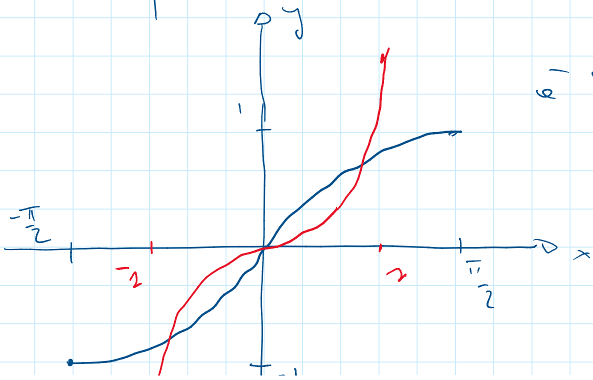
$\sin\big|_{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]} : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$

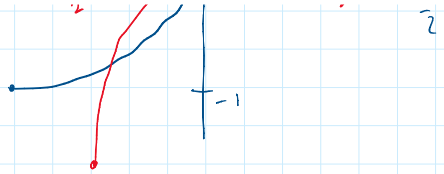
è biunivoca invertibile.

L'inversa si chiama

ARCO IL CUI SENO

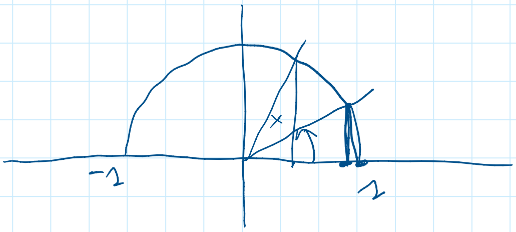
asin arcsin





asin   arcsin

$$\text{asin} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



$x \in [0, \pi]$

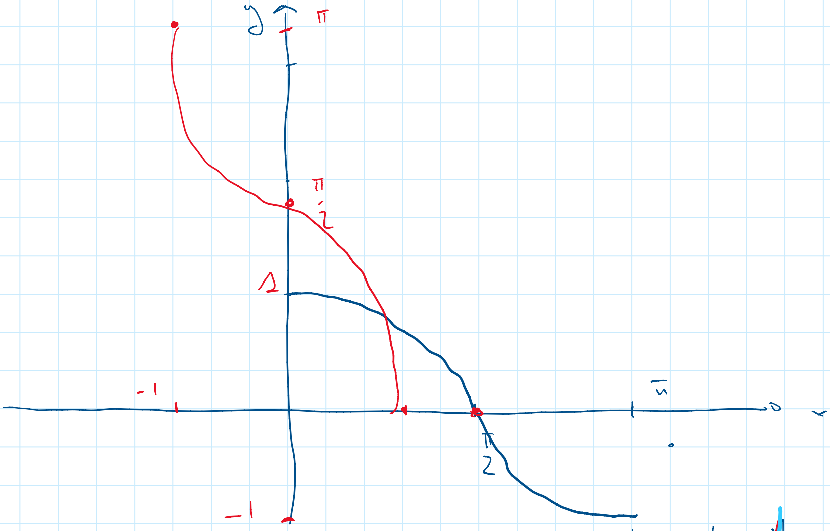
$$\cos\big|_{[0,\pi]} : [0,\pi] \to [-1, 1]$$

strett monotona decrescente
$\Rightarrow$ iniettiva.
Suriettiva.

La funzione inversa è una funzione $[-1,1] \to [0,\pi]$
si chiama ARCO IL CUI COSENO

acos   arccos

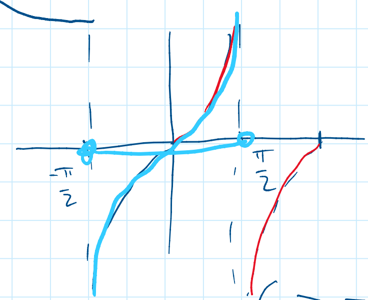
$$\text{acos} : [-1,1] \to [0,\pi]$$



$(\pi, -1)$

$\left(\frac{\pi}{2}, 0\right)$

$tg(x)$ $x \neq \frac{\pi}{2} + k\pi$



$$f(x) = tg\big|_{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}(x)$$

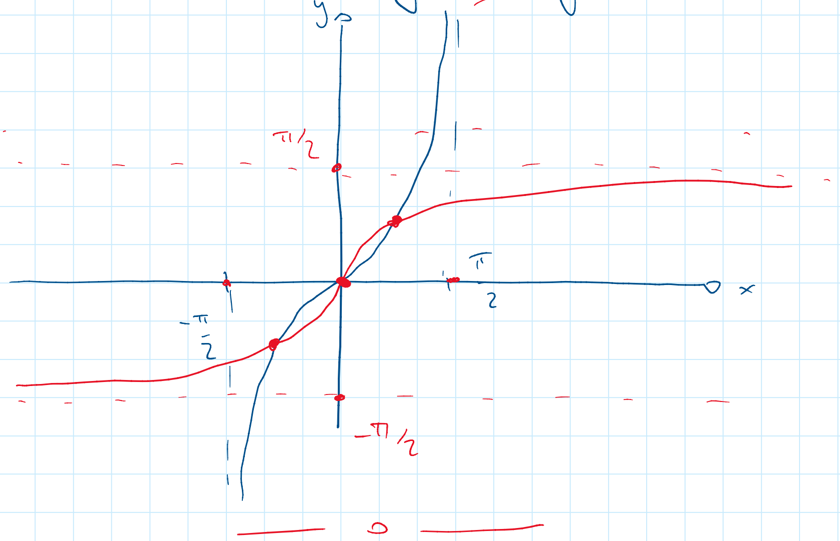
$$tg\big|_{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)} : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$$

$tg\big|_{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$ è strett. monotona crescente e quindi iniettiva.
E' suriettiva su tutto $\mathbb{R}$

$\Rightarrow$ E' invertibile.

La funzione inversa è definita su tutto $\mathbb{R}$ a valori in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Si chiama ARCO LA CUI TANGENTE

atan arctan atg ~~arctg~~



$p \in \mathbb{R}$ $\quad p > 0 \qquad p^n \qquad x \in \mathbb{Q} \qquad p^x$

$x \in \mathbb{R} \qquad x > 0 \qquad x = a_0, a_1 a_2 a_3 \ldots \ldots \qquad a_0 = \lfloor x \rfloor$

$x_0 = a_0$

$x_1 = a_0, a_1 = a_0 + \frac{a_1}{10}$

$x_2 = a_0, a_1 a_2$

$\vdots$

$x_n = a_0, a_1 \ldots a_n$

$$a_0 = x_0 \le x_1 \le x_2 \le \ldots \le x_{n-1} \le x_n \le a_0 + 1 \Leftarrow$$

$p > 1 \qquad p^{a_0} = p^{x_0} \le p^{x_1} \le \ldots \le p^{x_{n-1}} \le p^{x_n} \le p^{a_0+1}$

Definisco $\qquad p^x := \sup\{p^{x_n} : n \in \mathbb{N}\}$

$p \in (0,1) \qquad p^{a_0+1} \le p^{x_n} \le p^{x_{n-1}} \le \ldots \le p^{x_1} \le p^{x_0} = p^{a_0}$

$$p^x := \inf\{p^{x_n} : n \in \mathbb{N}\}$$

$p = 1 \qquad p^x = 1.$

Si può dimostrare che rimangono vere le regole di calcolo

$$p^x \cdot p^y = p^{x+y}$$
$$(p^x)^y = p^{xy}$$

$x \in \mathbb{R}$, $x < 0$ $\qquad p^x := \dfrac{1}{p^{-x}}$

$$\lim_{x \to x_0} f(x)^{g(x)} \qquad \lim_{x \to x_0} f(x) = L$$
$f(x) > 0$
$$\lim_{x \to x_0} g(x) = M$$

Se L ed M sono entrambi finiti $\Rightarrow$ ok L^M

$f(x) \to L \in \mathbb{R} \qquad g(x) \to +\infty$

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e \simeq 2.89\ldots$$ NUMERO DI NEPERO

$r \in \mathbb{R} \qquad \lim_{x \to +\infty} \left(1 + \frac{r}{x}\right)^x$

1° CASO $\quad r > 0 \qquad \left(1 + \frac{r}{x}\right)^x = \left(1 + \frac{r}{x}\right)^{\frac{x}{r} \cdot r} =$

$$= \left(\left(1 + \frac{r}{x}\right)^{\frac{x}{r}}\right)^r$$

$y := \frac{x}{r} \qquad x \to +\infty$ SSE $y \to +\infty$

$$\left(\left(1 + \frac{r}{x}\right)^{\frac{x}{r}}\right)^r = \Bigg(\underbrace{\left(1 + \frac{1}{y}\right)^y}_{y \to +\infty \to e}\Bigg)^r \to e^r$$

$r = -1$

$$\lim_{x \to +\infty} \left(1 - \frac{1}{x}\right)^x = e^{-1} = e^r$$

$1 - \frac{1}{x} = \frac{x-1}{\ldots} \qquad y = x - 1 \qquad x = y + 1$

$$1-\frac{1}{x} = \frac{x-1}{x} \qquad y = x-1 \qquad x = y+1$$

$$\Downarrow$$

$$x \to +\infty \text{ SSE } y \to +\infty$$

$$= \frac{y}{y+1} = \frac{1}{\frac{y+1}{y}} = \frac{1}{1+\frac{1}{y}}$$

$$\left(1-\frac{1}{x}\right)^x = \left(\frac{1}{1+\frac{1}{y}}\right)^{y+1} =$$

$$\frac{1}{\left(1+\frac{1}{y}\right)^{y+1}} = \frac{1}{\underbrace{\left(1+\frac{1}{y}\right)^y}_{\to e} \cdot \underbrace{\left(1+\frac{1}{y}\right)}_{\to 1}} \to \frac{1}{e} = e^{-1}$$

$$r<0 \qquad \lim_{x\to+\infty}\left(1+\frac{r}{x}\right)^x$$

$$\left(1+\frac{r}{x}\right)^x \qquad s=-r, \quad s>0$$

$$= \left(1-\frac{s}{x}\right)^x \qquad y = \frac{x}{s} \qquad y\to+\infty \text{ SSE } x\to+\infty$$

$$= \left(1-\frac{s}{x}\right)^{\frac{x}{s}\cdot s} = \Big[\underbrace{\left(1-\frac{1}{y}\right)^y}_{\to e^{-1}}\Big]^s \to \left(e^{-1}\right)^s = e^{-s} = e^{r}$$

$$\forall r\in\mathbb{R}, \ \cancel{r\neq 0} \qquad \lim_{x\to+\infty}\left(1+\frac{r}{x}\right)^x = e^r$$

NB. $r=0$ $\quad 1+\frac{r}{x} = 1 \quad \forall x \quad \left(1+\frac{r}{x}\right)^x = 1^x = 1 \quad \forall x$

$$1 = e^0$$