TEST DI SMIRNOV-KOLMOGOROV

Se X è una v.a. con legge F, allora la v.a. $Y = F \circ X$ ha distribuzione $U([0,1])$

$X_1 \dots X_n$ campione statistico con distribuzione $\mathbb{P}_{X_i}$ e legge $F : \mathbb{R} \to [0,1]$ CONTINUA

$\forall i = 1 \dots n \quad \forall t \in \mathbb{R}$

$$Y_i(\omega, t) := \mathbb{1}_{(-\infty, t]}(X_i(\omega)) = \begin{cases} 1 & X_i(\omega) \le t \\ 0 & X_i(\omega) > t \end{cases}$$

$$\mathbb{P}_{Y_i(\cdot, t)} = B(F(t)) \qquad \mathbb{E}[Y_i(\cdot, t)] = F(t)$$

$$Var[Y_i(\cdot, t)] = F(t)(1 - F(t)) \le \frac{1}{4} \quad \forall t \in \mathbb{R}$$

$$G_n(\omega, t) = \frac{1}{n} \sum_{i=1}^{n} Y_i(\omega, t) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{(-\infty, t]}(X_i(\omega))$$

$$D_n(\omega) = \sup_{t \in \mathbb{R}} |G_n(\omega, t) - F(t)|$$

$$\forall t \in \mathbb{R} \quad \mathbb{P}(|G_n(\omega, t) - F(t)| > \varepsilon) \le ⊛$$

$$\mathbb{E}[G_n(\cdot, t)] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[Y_i(\cdot, t)] = F(t)$$

$$(⊛) \le \frac{Var[G_n(\cdot, t)]}{\varepsilon^2}$$

$$Var[G_n(\cdot, t)] = Var\left[\frac{1}{n} \sum_{i=1}^{n} Y_i(\cdot, t)\right] =$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} Var[Y_i(\cdot, t)] \leq \frac{1}{4n^2} n = \frac{1}{4n}$$

$$\Rightarrow \mathbb{P}\left(|G_n(\cdot, t) - F(t)| > \varepsilon \right) \leq \frac{1}{4n\varepsilon^2} \quad \forall t \in \mathbb{R} \ \forall n \in \mathbb{N} \ \forall \varepsilon > 0$$

$$\mathbb{P}\left(|G_n(\cdot, t) - F(t)| \leq \varepsilon \right) \geq \frac{1}{4n\varepsilon^2} \quad \forall t \in \mathbb{R}$$

$$\mathbb{P}\left(D_n \leq \varepsilon \right) \geq \frac{1}{4n\varepsilon^2} \quad \forall n \in \mathbb{N} \ \forall \varepsilon > 0$$

$d > 0$

$$\mathbb{P}(D_n \geq d) = \mathbb{P}\left(\sup_{t \in \mathbb{R}} |G_n(\cdot, t) - F(t)| \geq d \right) =$$

$$= \mathbb{P}\left(\sup_{t \in \mathbb{R}} \left| \frac{1}{n} \#\{i : X_i \leq t\} - F(t) \right| \geq d \right)$$

OSSERVAZIONE ① Se F è strettamente monotona crescente allora $X_i \leq t$ SSE $F(X_i) \leq F(t)$

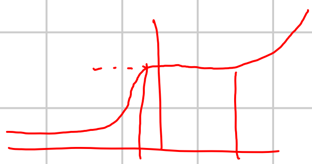
$$\longrightarrow \mathbb{P}\left(\sup_{t \in \mathbb{R}} \left| \frac{1}{n} \#\{i : F \circ X_i \leq F(t)\} - F(t) \right| \geq d \right)$$

② Se F è monotona crescente:

$$X_i \leq t \Rightarrow F(X_i) \leq F(t)$$

$$\begin{cases} F(X_i) \leq F(t) \\ X_i > t \end{cases} \Rightarrow F(X_i) \geq F(t)$$

$$\Rightarrow F(X_i) = F(t)$$



$$A_t := \{\omega \in \Omega : X_i(\omega) > t, \ F(X_i(\omega)) \leq F(t)\} \subset \{\omega \in \Omega : F(X_i) = F(t)\}$$

$$= \{\omega \in \Omega : X_i \in F^{-1}(\{F(t)\})\}$$
$$= \{\omega \in \Omega : X_i \in [a_t, b_t]\}$$

$$\mathbb{P}(A_t) = \mathbb{P}(X_i \in [a_t, b_t]) = \mathbb{P}(X_i \leq b_t) - \mathbb{P}(X_i < a_t)$$
$$= F(b_t) - F(a_t^-) = F(b_t) - F(a_t) =$$
$$= F(t) - F(t) = 0$$

$\Rightarrow$ In ogni caso

U_i : $\mathbb{P}_{U_i} = U([0,1])$

$$\mathbb{P}(D_n \geq d) = \mathbb{P}\left(\sup_{t \in \mathbb{R}} \left| \frac{1}{n} \#\{i : \underbrace{F \circ X_i}_{U_i} \leq F(t)\} - F(t) \right| \geq d\right)$$
$$= \mathbb{P}\left(\sup_{t \in \mathbb{R}} \left| \frac{1}{n} \#\{i : U_i \leq F(t)\} - F(t) \right| \geq d\right) =$$
$$= \mathbb{P}\left(\sup_{y \in (0,1)} \left| \frac{1}{n} \#\{i : U_i \leq y\} - y \right| \geq d\right)$$

$$\lim_{n \to \infty} \mathbb{P}(D_n \sqrt{n} \leq t) = \begin{cases} 0 & t \leq 0 \\ 1 - 2\sum_{j=1}^{\infty} (-1)^{j-1} \exp(-2j^2 t^2) & t > 0 \end{cases}$$

x_1 ——— x_n

$F_0 : \mathbb{R} \to [0,1]$ assegnata $\qquad \lim_{t \to -\infty} F_0(t) = 0$

CONTINUA e MONOTONA CRESCENTE $\qquad \lim_{t \to +\infty} F_0(t) = 1$

H_0) F_0 è la legge del campione

$H_A)$ F_0 non è la legge: $\exists t \in \mathbb{R}$ t.c.

$$\mathbb{P}(X_i \le t) \ne F_0(t)$$

$$H_A: \quad \exists t \in \mathbb{R}: \left| \mathbb{P}(X_i \le t) - F_0(t) \right| > 0$$

$$d_n := \sup_{t \in \mathbb{R}} \left| g_n(x_1, \dots x_n, t) - F_0(t) \right|$$

$$g_n(x_1, \dots x_n, t) = \frac{1}{n} \# \{ i : x_i \le t \}$$

Accetto H_0 sse $\quad d_n < \varepsilon$

Probabilità di commettere errore di 1ª specie

$$\mathbb{P}(D_n \ge \varepsilon \mid F_0(t) = \text{legge del campione}) =$$

$$= \mathbb{P}(D_n \sqrt{n} \ge \varepsilon \sqrt{n} \mid F_0(t) = \text{legge del campione}) =$$

$$= 1 - \mathbb{P}(D_n \sqrt{n} < \varepsilon \sqrt{n} \mid F_0(t) = \text{legge del campione})$$

$$\simeq 1 - \left(1 - 2 \sum_{j=1}^{\infty} (-1)^{j-1} \exp(-2 j^2 \varepsilon^2 n) \right)$$

$$= 2 \sum_{j=1}^{\infty} (-1)^{j-1} \exp(-2 j^2 \varepsilon^2 n) < 2 \exp(-2 \varepsilon^2 n) = \alpha$$

$$a_1 - a_2 + a_3 - a_4 + a_5 \dots \qquad a_j = \exp(-2 j^2 \varepsilon^2 n)$$

$$a_1 - \underbrace{(a_2 - a_3)}_{>0} - \underbrace{(a_4 - a_5)}_{>0} + \dots < a_1 \qquad a_j > a_{j+1} > 0$$

$$\sum (-1)^{j-1} a_j$$

$$2\exp(-2\varepsilon^2 n) = \alpha$$

$$-2\varepsilon^2 n = \ln\frac{\alpha}{2} = -\ln\frac{2}{\alpha}$$

$$\varepsilon^2 = \frac{1}{2n}\ln\frac{2}{\alpha} \qquad \varepsilon = \sqrt{\frac{1}{2n}\ln\left(\frac{2}{\alpha}\right)}$$

$$d_n := \sup_{t\in\mathbb{R}}\left|\frac{1}{n}\#\{i : x_i \le t\} - F_0(t)\right|$$

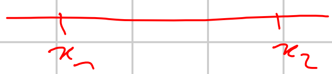
$$\sup_{t\in\mathbb{R}} h(t) = \max\left\{\sup_{t\in A} h(t),\ \sup_{t\in\mathbb{R}\setminus A} h(t)\right\}$$

$$\boxed{x_1 < x_2 < \dots < x_n}$$

$$\max\left\{\begin{array}{ll}
\sup_{t<x_1}\left|\frac{1}{n}\#\{i : x_i \le t\} - F_0(t)\right|, & \sup_{t<x_1}\left|F_0(t)\right| \\
\sup_{t\in[x_1,x_2)}\left|\frac{1}{n}\#\{i : x_i \le t\} - F_0(t)\right|, & \sup_{t\in[x_1,x_2)}\left|\frac{1}{n} - F_0(t)\right| \\
\vdots & \\
\sup_{t\in[x_{n-1},x_n)}\left|\frac{1}{n}\#\{i : x_i \le t\} - F_0(t)\right|, & \sup_{t\in[x_{n-1},x_n)}\left|\frac{n-1}{n} - F_0(t)\right| \\
\sup_{t\ge x_n}\left|\frac{1}{n}\#\{i : x_i \le t\} - F_0(t)\right| & \sup_{t\ge x_n}\left|1 - F_0(t)\right|
\end{array}\right\}$$

$$= \max\left\{\begin{array}{l}
\sup_{t<x_1}\left|F_0(t)\right| \\
\sup_{x_1\le t<x_2}\left|\frac{1}{n} - F_0(t)\right|
\end{array}\right.$$

$$F_0(x_1)$$



⋮

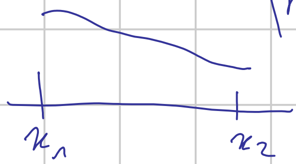
$$\sup_{x_{n-1}\le t< x_n} \left|\frac{n-1}{n} - F_0(t)\right|$$

$$\sup_{t\ge x_n} \left|1 - F_0(t)\right| \Big\} \quad 1 - F_0(x_n)$$

$$\sup_{x_1\le t< x_2} \left|\frac{1}{n} - F_0(t)\right| =$$

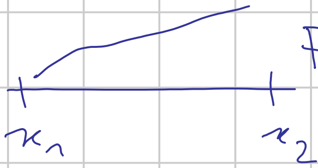
1° CASO $\quad F_0(t) \le \frac{1}{n} \quad \forall t \in [x_1, x_2) \qquad \left|\frac{1}{n} - F_0(x_1)\right|$

$$h(t) = \frac{1}{n} - F_0(t) \ge 0$$



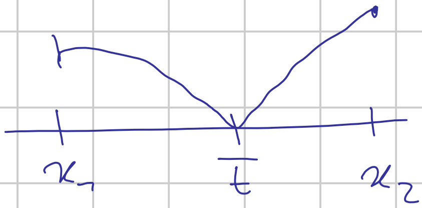
2° CASO $\quad F_0(t) \ge \frac{1}{n} \quad \forall t \in [x_1, x_2) \qquad \left|\frac{1}{n} - F_0(x_2)\right| = F_0(x_2) - \frac{1}{n}$

$$h(t) = F_0(t) - \frac{1}{n} \ge 0$$



3° CASO $\quad F_0(t) - \frac{1}{n}$ si annulla in $[x_1, x_2)$

$$h(t) = \left|\frac{1}{n} - F_0(t)\right|$$



$$\sup_{t\in[x_1,x_2)} \left|\frac{1}{n} - F_0(t)\right| = \max\left\{\left|\frac{1}{n} - F_0(x_1)\right|, \left|\frac{1}{n} - F_0(x_2)\right|\right\}$$

$$d_n = \max\left\{F_0(x_1), \left|\frac{1}{n} - F_0(x_1)\right|, \left|\frac{1}{n} - F_0(x_2)\right|, \left|\frac{2}{n} - F_0(x_2)\right|, \left|\frac{2}{n} - F_0(x_3)\right| \ldots\right.$$

$$\left| \frac{n-1}{n} - F_0(x_{n-1}) \right|, \quad \left| \frac{n-1}{n} - F_0(x_n) \right|,$$
$$\left. 1 - F_0(x_n) \right\}$$

$h_{media} = 170$ cm $\qquad sd = 10$ cm

$$\mathbb{P}_{H_i} = N(170, 100) \qquad Z_i = \frac{H_i - 170}{10}$$

$$\mathbb{P}(H_i > 175) = \mathbb{P}\left(\frac{H_i - 170}{10} > \frac{175 - 170}{10}\right) =$$

$$= \mathbb{P}\left(Z_i > \frac{1}{2}\right) = 1 - \Phi(0.5) \simeq 1 - 0.69146$$

$$\simeq 1 - 0.69 = 0.31.$$

H_1 ——— H_{10} $\qquad \bar{H}_{10} \qquad \mathbb{P}(\bar{H}_{10} > 175)$

$$\mathbb{P}_{H_i} = N(170, 100) \qquad \mathbb{P}_{\bar{H}_{10}} = N\left(170, \frac{100}{10}\right) =$$

$$= N(170, 10)$$

$$Z = \frac{\bar{H}_{10} - 170}{\sqrt{10}} \qquad \mathbb{P}_Z = N(0,1)$$

$$\mathbb{P}(\bar{H}_{10} > 175) = \mathbb{P}\left(\frac{\bar{H}_{10} - 170}{\sqrt{10}} > \frac{175 - 170}{\sqrt{10}}\right)$$

$$= \mathbb{P}\left(Z > \frac{5}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{1}{2}\sqrt{10}\right) \simeq \mathbb{P}(Z > 1.58)$$

$$= 1 - \Phi(1.58) \simeq 1 - 0.94295 \simeq 1 - 0.94 = 0.06$$

H_1 - - - - - H_{100} $\qquad \mathbb{P}_{H_i} = N(170, 100)$

$$\mathbb{P}(\bar{H}_{100} > 175) \qquad \mathbb{P}_{\bar{H}_{100}} = N\left(170, \frac{100}{100}\right) =$$

$$Z = \overline{H}_{100} - 170 \qquad \mathbb{P}_Z = N(0,1) = N(170, 1)$$

$$\mathbb{P}(\overline{H}_{100} > 175) = \mathbb{P}(\overline{H}_{100} - 170 > 175 - 170)$$

$$= \mathbb{P}(Z > 5) = 1 - \Phi(5) \sim 0.0000003$$

$$\Phi(4.42) = 1 \qquad \underline{\Phi(5) = 0.999997}$$

— o —

$$\mathbb{P}_T = \exp(\lambda) \qquad \mathbb{P}(\lambda \in [0.24, 0.25]) = 0.99$$

$$\mathbb{P}\left(T > \frac{1}{2}\right) \qquad f(x) = \begin{cases} 0 & x \leq 0 \\ \lambda e^{-\lambda x} & x > 0 \end{cases}$$

$$F(t) = \begin{cases} 0 & t \leq 0 \\ 1 - e^{-\lambda t} & t > 0 \end{cases}$$

$$1 - F\left(\frac{1}{2}\right) = 1 - \left(1 - e^{-\lambda \frac{1}{2}}\right) = e^{-\lambda/2}$$

$$\lambda \in [0.24, 0.25]$$

$$\Updownarrow$$

$$0.24 \leq \lambda \leq 0.25$$

$$0.12 \leq \frac{\lambda}{2} \leq 0.125$$

$$-0.125 \leq -\frac{\lambda}{2} \leq -0.12$$

$$0.882 \simeq e^{-0.125} \leq e^{-\lambda/2} \leq e^{-0.12} \simeq 0.887$$

$$\mathbb{P}\left(e^{-\lambda/2} \in (0.882, 0.887)\right) = 0.99$$

$$\mathbb{P}\left(T > \frac{1}{2}\right) = e^{-\lambda/2}$$

— ~ —

s_1 $\qquad$ $s_2^2 = 5 s_1^2$

c_1 $\qquad$ $c_2 = \frac{1}{8} c_1$

$n_2 = 8 n_1$

\# esperimenti n_1 $\qquad$ \# esperimenti n_2

$$\left(\bar{x}_1 - \frac{s_1}{\sqrt{n_1}} t_{n_1-1, 1-\frac{\alpha}{2}}, \bar{x}_1 + \frac{s_1}{\sqrt{n_1}} t_{n_1-1, 1-\frac{\alpha}{2}}\right)$$

$$\left(\bar{x}_2 - \frac{s_2}{\sqrt{n_2}} t_{n_2-1}, \bar{x}_2 + \frac{s_2}{\sqrt{n_2}} t_{n_2-1, 1-\frac{\alpha}{2}}\right)$$

Scelgo il 1° metodo sse

$$\frac{s_1}{\sqrt{n_1}} t_{n_1-1, 1-\frac{\alpha}{2}} < \frac{s_2}{\sqrt{n_2}} t_{n_2-1, 1-\frac{\alpha}{2}}$$

$$\frac{\cancel{s_1}}{\cancel{\sqrt{n_1}}} t_{n_1-1, 1-\frac{\alpha}{2}} < \frac{\sqrt{5}\cancel{s_1}}{2\sqrt{2\cancel{n_1}}} t_{8n_1-1, 1-\frac{\alpha}{2}}$$

$$t_{n_1-1, 1-\frac{\alpha}{2}} < \frac{\sqrt{5}}{2\sqrt{2}} t_{8n_1-1, 1-\frac{\alpha}{2}}$$

$n = 250$ $\qquad 1-\alpha = 0.9$

$\bar{x} = 65 \qquad s^2 = 300$

$$\left(\bar{x} - \frac{s}{\sqrt{n}} t_{n-1, 1-\frac{\alpha}{2}} , \bar{x} + \frac{s}{\sqrt{n}} t_{n-1, 1-\frac{\alpha}{2}} \right)$$

$s = \sqrt{300} = 10\sqrt{3}$ $\qquad t_{249, 0.95} = 1.6449$ "TAVOLE"

1.650996 "R"

$$\left(65 - \frac{10\sqrt{3}}{5\sqrt{10}} t_{249, 0.95} , 65 + \frac{10\sqrt{3}}{5\sqrt{10}} t_{249, 0.95} \right)$$

$$\simeq (63.185, 66.815)$$

$$\left(\frac{(n-1)s^2}{\chi^2_{n-1, 1-\frac{\alpha}{2}}} , \frac{(n-1)s^2}{\chi^2_{n-1, \frac{\alpha}{2}}} \right)$$

$$\left(\frac{249 \cdot 300}{\underbrace{\chi^2_{249, 0.95}}_{\simeq 286.8078}} , \frac{249 \cdot 300}{\underbrace{\chi^2_{249, 0.05}}_{\simeq 213.4653}} \right) \simeq (260.45, 349.90)$$

— o —

$\mathbb{P}_X = N(1.4, \sigma^2)$ $\qquad \sigma^2$ ignoto

1.549	1.389	1.225	1.082	1.682
1.165	1.301	1.213	1.364	1.448

$\bar{x} = 1.341865$ $\qquad 1-\alpha = 0.95$

$$\left(\bar{x} - \frac{s}{\sqrt{n}} t_{n-1,1-\frac{\alpha}{2}} , \quad \bar{x} + \frac{s}{\sqrt{n}} t_{n-1,1-\frac{\alpha}{2}}\right)$$

$n = 10$ $\qquad n-1 = 9$ $\qquad \alpha = 0.05$ $\qquad \frac{\alpha}{2} = 0.025$

$1 - \frac{\alpha}{2} = 0.975$

$t_{9,0.975} \simeq 2.2622$

$sd \cong 0.184$

$$\left(1.342 - \frac{0.184}{\sqrt{10}} 2.2622 , \; 1.342 + \frac{0.184}{\sqrt{10}} 2.2622\right)$$

$$\simeq (1.210, \; 1.474)$$

— o —

$\alpha = 0.05$

$H_0 \quad \mu = 1.4$ $\qquad H_A \quad \mu \neq 1.4$

$$\left|\bar{x} - 1.4\right| < \frac{s}{\sqrt{n}} t_{n-1,1-\frac{\alpha}{2}}$$

$$0.068 = \left|1.342 - 1.4\right| < \frac{0.1836}{\sqrt{10}} 2.2622 \simeq 0.132$$

$H_0 \quad \mu \leq 1.4$ $\qquad H_A \quad \mu > 1.4$

Accetto H_0 sse $\bar{x} < 1.4 + \frac{s}{\sqrt{n}} t_{n-1,1-\alpha}$ $\qquad t_{9,0.95} \simeq 1.8331$

SS=

$$1.342 < 1.4 + \frac{0.184}{\sqrt{10}} \quad 1.8331$$

878	933	824	795	954 ...
769	771	888	802	802

$n = 10$

$\bar{x} = 841.6 \quad sd = 66.94807 \quad N(\mu_x, \sigma_x^2)$

943	1012	1001	861	1209
883	867	788	890	917

$\bar{y} = 937,1 \quad sd_y = 116.4965$

$N(\mu_y, \sigma_y^2)$

$1-\alpha = 0.95$

$$\bar{x} - \bar{y} - L < \mu_x - \mu_y < \bar{x} - \bar{y} + L$$

$$L = \frac{t_{n+k-2, 1-\frac{\alpha}{2}} \sqrt{(n-1)s_x^2 + (k-1)s_y^2}}{\sqrt{n+k-2}} \sqrt{\frac{1}{n} + \frac{1}{k}}$$

$k = n$

$$\frac{t_{2n-2, 1-\frac{\alpha}{2}} \sqrt{\cancel{(n-1)}(s_x^2 + s_y^2)}}{\sqrt{\cancel{2}\cancel{(n-1)}}} \quad \frac{\cancel{\sqrt{2}}}{\sqrt{n}}$$

$k = n = 10$

$$\simeq \frac{t_{18, 0.975}^{\simeq 2.1} \sqrt{(66.95)^2 + (116.5)^2}}{\sqrt{10}} \simeq 89.27$$

$$\bar{x} - \bar{y} = 841.6 - 937.1 = -95.5$$

$$\mu_x - \mu_y = \left(-95{,}5 - 89.27 \,,\, \underbrace{-95.5 + 89.27}\right)$$

$$Y_1 \ldots Y_{100} \qquad \{0, 1, 2\} \qquad \binom{2}{k}\cancel{\left(\frac{1}{2}\right)^k}\left(\frac{1}{2}\right)^{2-\cancel{k}}$$

$$\boxed{\alpha = 0.05} \qquad B\left(2, \frac{1}{2}\right) \qquad p_0^0 = p_2^0 = \frac{1}{4} \quad p_1^0 = \frac{1}{2}$$

$$\#_0 = 30 \qquad \#_1 = 45 \qquad \#_2 = 25$$

$$\sum_{j=0}^{2} \frac{(\#_j - n p_j^0)^2}{n p_j^0} < \chi^2_{\underbrace{k-1}_{2},\, 0.95} \simeq 5.9915$$

$$\frac{\left(30 - 100\,\frac{1}{4}\right)^2}{100 \cdot \frac{1}{4}} + \frac{\left(45 - 100 \cdot \frac{1}{2}\right)^2}{100 \cdot \frac{1}{2}} + \frac{\left(25 - 100 \cdot \frac{1}{4}\right)^2}{100 \cdot \frac{1}{4}}$$

$$= \frac{25}{25} + \frac{25}{50} + 0 = 1 + \frac{1}{2} = 1.5$$