INTERVALLI DI CONFIDENZA, TEST D'IPOTESI

Note Title 21/05/2019

Sia $X_1 \_\_ X_n$ campione statistico e sia θ un parametro che caratterizza la distribuzione del campione.

Siano $L_i = l_i(X_1 \_\_ X_n)$

$L_s = l_s(X_1 \_\_ X_n)$ due statistiche.

(L_i, L_s)

Dico che l'intervallo (L_i, L_s) è un intervallo di confidenza di livello (di fiducia) $1-\alpha$ per il parametro θ se

$$\mathbb{P}(\theta \in (L_i, L_s)) \geq 1-\alpha$$

Dico che la semiretta $(L_i, +\infty)$ è un intervallo di confidenza unilaterale superiore se

$$\mathbb{P}(\theta > L_i) = \mathbb{P}(\theta \in (L_i, +\infty)) \geq 1-\alpha$$

Dico che la semiretta $(-\infty, L_s)$ è un intervallo di confidenza unilaterale inferiore se

$$\mathbb{P}(\theta < L_s) = \mathbb{P}(\theta \in (-\infty, L_s)) \geq 1-\alpha$$

— o —

$X_1 \_\_ X_n$ valore atteso ignoto μ

varianza nota σ^2

$\bar{X}$ valore atteso μ

varianza nota $\frac{\sigma^2}{n}$

Y valore atteso μ

varianza σ^2

$$\mathbb{P}(|Y-\mu| > t) \leq \frac{\sigma^2}{t}$$

$$\mathbb{P}(|\bar{X}-\mu| > t) \leq \frac{\sigma^2}{nt^2} \qquad \mathbb{P}(|\bar{X}-\mu| \leq t) \geq 1-\frac{\sigma^2}{nt^2}$$

$$\mathbb{P}(-t \leq \bar{X}-\mu \leq t) \geq 1-\frac{\sigma^2}{nt^2}$$

$$\mathbb{P}\left(-t \le \mu - \overline{X} \le t\right) \geq 1 - \frac{\sigma^2}{nt^2}$$

$$\frac{\sigma^2}{nt^2} = \alpha \iff t = \frac{\sigma}{\sqrt{n\alpha}}$$

$$\mathbb{P}\left(\overline{X} - \frac{\sigma}{\sqrt{n\alpha}} \le \mu < \overline{X} + \frac{\sigma}{\sqrt{n\alpha}}\right) \geq 1 - \alpha$$

$$(L_i, L_s) = \left(\overline{X} - \frac{\sigma}{\sqrt{n\alpha}}, \overline{X} + \frac{\sigma}{\sqrt{n\alpha}}\right)$$

DISTRIBUZIONE ESPONENZIALE

$X_1 \_\_\_ X_n$ campione statistico con $\mathbb{P}_{X_i} = \exp(\lambda)$
$= f(x)\,dx$

$$f(x) = \begin{cases} 0 & x \le 0 \\ \lambda e^{-\lambda x} & x > 0 \end{cases}$$

$$\lambda e^{-\lambda x} = \frac{\lambda^1}{1} x^{1-1} e^{-\lambda x} = \frac{\lambda^1}{\Gamma(1)} x^{1-1} e^{-\lambda x}$$

$$\exp(\lambda) = \Gamma(1, \lambda)$$

$$S_n = \sum_{i=1}^{n} X_i \qquad \mathbb{P}_{S_n} = \Gamma(n, \lambda) = g(x)\,dx$$

$$g(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{\lambda^n x^{n-1}}{\Gamma(n)} e^{-\lambda x} & x > 0 \end{cases}$$

$$Y = 2\lambda S_n = h(x)\,dx \qquad h(x) = \frac{1}{2\lambda} g\left(\frac{x}{2\lambda}\right)$$

$$h(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{1}{2\cancel{\lambda}} \dfrac{\cancel{\lambda^n}}{\Gamma(n)} \left(\dfrac{x}{2\cancel{\lambda}}\right)^{n-1} e^{-\cancel{\lambda} \cdot \frac{x}{2\cancel{\lambda}}} & x > 0 \end{cases}$$

$$h(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{\left(\frac{1}{2}\right)^n}{\Gamma(n)} x^{n-1} e^{-x/2} & x > 0 \end{cases} \qquad \Gamma\left(n, \frac{1}{2}\right) = \Gamma\left(\frac{2n}{2}, \frac{1}{2}\right)$$

$$\mathbb{P}_Y = \chi^2_{2n}$$

$$1-\alpha = \mathbb{P}\left(\chi^2_{2n,\frac{\alpha}{2}} < Y < \chi^2_{2n,1-\frac{\alpha}{2}}\right) =$$

$$= \mathbb{P}\left(Y < \chi^2_{2n,1-\frac{\alpha}{2}}\right) - \mathbb{P}\left(Y < \chi^2_{2n,\frac{\alpha}{2}}\right)$$

$$= \left(1-\frac{\alpha}{2}\right) - \frac{\alpha}{2} = 1-\alpha$$

$$1-\alpha = \mathbb{P}\left(\chi^2_{2n,\frac{\alpha}{2}} < 2\lambda S_n < \chi^2_{2n,1-\frac{\alpha}{2}}\right)$$

$$= \mathbb{P}\left(\chi^2_{2n,\frac{\alpha}{2}} < 2\lambda n \overline{X} < \chi^2_{2n,1-\frac{\alpha}{2}}\right)$$

$$= \mathbb{P}\left(\frac{1}{\chi^2_{2n,1-\frac{\alpha}{2}}} < \frac{1}{2\lambda n\overline{X}} < \frac{1}{\chi^2_{2n,\frac{\alpha}{2}}}\right)$$

$$= \mathbb{P}\left(\frac{2n\overline{X}}{\chi^2_{2n,1-\frac{\alpha}{2}}} < \frac{1}{\lambda} < \frac{2n\overline{X}}{\chi^2_{2n,\frac{\alpha}{2}}}\right)$$

(L_i, L_s) intervallo di confidenza di livello $1-\alpha$ per il valore atteso $\frac{1}{\lambda}$ della distribuzione

CAMPIONI GAUSSIANI

1° CASO Conosco la varianza σ^2 - Voglio un intervallo di confidenza per il valore atteso μ

$X_1 \_\_\_\_ X_n \qquad \mathbb{P}_{X_i} = N(\mu, \sigma^2)$

$\mathbb{P}_{\overline{X}} = N\left(\mu, \frac{\sigma^2}{n}\right) \qquad Z := \dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} \qquad \mathbb{P}_Z = N(0,1)$

$$1-\alpha = \mathbb{P}(|Z| \le t) = \mathbb{P}(-t < Z < t)$$
$$= \mathbb{P}(Z<t) - \mathbb{P}(Z<-t) = \Phi(t) - \Phi(-t)$$
$$= \Phi(t) - (1-\Phi(t)) = 2\Phi(t) - 1$$

$1-\alpha = 2\Phi(t) - 1 \qquad 2\Phi(t) = 2-\alpha,$

$\Phi(t) = 1 - \frac{\alpha}{2} \qquad t = z_{1-\frac{\alpha}{2}}$

$$1-\alpha = \mathbb{P}\left(-z_{1-\frac{\alpha}{2}} < Z < z_{1-\frac{\alpha}{2}}\right) =$$
$$= \mathbb{P}\left(-z_{1-\alpha/2} < \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} < z_{1-\frac{\alpha}{2}}\right) =$$
$$= \mathbb{P}\left(-z_{1-\frac{\alpha}{2}} < \frac{\mu-\overline{X}}{\sigma/\sqrt{n}} < z_{1-\frac{\alpha}{2}}\right)$$
$$= \mathbb{P}\left(\overline{X} - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$$

$$(L_i, L_s) = \left(\overline{X} - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$$

— o —

$1-\alpha = \mathbb{P}(Z<t) \quad \text{sse} \quad t = z_{1-\alpha}$

$$1-\alpha = \mathbb{P}\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} < z_{1-\alpha}\right) =$$

$$= \mathbb{P}\left(\overline{X}-\mu < z_{1-\alpha}\frac{\sigma}{\sqrt{n}}\right)$$

$$= \mathbb{P}\left(\mu > \overline{X} - z_{1-\alpha}\frac{\sigma}{\sqrt{n}}\right)$$

$$(L_i, +\infty) = \left(\overline{X} - z_{1-\alpha}\frac{\sigma}{\sqrt{n}}, +\infty\right)$$

— o —

$$1-\alpha = \mathbb{P}(Z>t) = 1-\mathbb{P}(Z\leq t) = 1-\Phi(t)$$

$$\Phi(t) = \alpha \qquad t = z_\alpha = -z_{1-\alpha}$$

$$1-\alpha = \mathbb{P}\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} > -z_{1-\alpha}\right) = \mathbb{P}\left(\overline{X}-\mu > -z_{1-\alpha}\frac{\sigma}{\sqrt{n}}\right)$$

$$= \mathbb{P}\left(\mu < \overline{X} + z_{1-\alpha}\frac{\sigma}{\sqrt{n}}\right)$$

$$(-\infty, L_s) = \left(-\infty, \overline{X} + z_{1-\alpha}\frac{\sigma}{\sqrt{n}}\right)$$

2° CASO La varianza non è nota.

$$T := \frac{(\overline{X}-\mu)\sqrt{n}}{S}$$

So che se X_1 ___ X_n è un campione gaussiano con $\mathbb{P}_{X_i} = N(\mu, \sigma^2)$

allora $\mathbb{P}_T = t(n-1)$

Indico con $t_{n-1,\alpha}$

$$1-\alpha = \mathbb{P}(|T| \leq t) = \mathbb{P}(-t < T < t) =$$

$$= \mathbb{P}(T < t) - \mathbb{P}(T < -t) = F_T(t) - F_T(-t)$$
$$= F_T(t) - (1 - F_T(t)) = 2F_T(t) - 1$$

$$1 - \alpha = 2F_T(t) - 1 \qquad 2F_T(t) = 2 - \alpha \qquad F_T(t) = 1 - \frac{\alpha}{2}$$

$$t = t_{n-1, 1-\frac{\alpha}{2}}$$

$$1 - \alpha = \mathbb{P}\left(|T| \le t_{n-1, 1-\frac{\alpha}{2}}\right) = \mathbb{P}\left(\left|\frac{(\overline{X} - \mu)\sqrt{n}}{S}\right| \le t_{n-1, 1-\frac{\alpha}{2}}\right)$$

$$= \mathbb{P}\left(|\overline{X} - \mu| \le \frac{S}{\sqrt{n}} t_{n-1, 1-\frac{\alpha}{2}}\right)$$

$$= \mathbb{P}\left(-\frac{S}{\sqrt{n}} t_{n-1, 1-\frac{\alpha}{2}} < \mu - \overline{X} < \frac{S}{\sqrt{n}} t_{n-1, 1-\frac{\alpha}{2}}\right) =$$

$$= \mathbb{P}\left(\overline{X} - \frac{S}{\sqrt{n}} t_{n-1, 1-\frac{\alpha}{2}} < \mu < \overline{X} + \frac{S}{\sqrt{n}} t_{n-1, 1-\frac{\alpha}{2}}\right)$$

$$\left(\overline{X} - \frac{S}{\sqrt{n}} t_{n-1, 1-\frac{\alpha}{2}},\ \overline{X} + \frac{S}{\sqrt{n}} t_{n-1, 1-\frac{\alpha}{2}}\right)$$

$$1 - \alpha = \mathbb{P}(T < t_{n-1, 1-\alpha}) = \mathbb{P}\left(\frac{(\overline{X} - \mu)\sqrt{n}}{S} < t_{n-1, 1-\alpha}\right)$$

$$= \mathbb{P}\left(\overline{X} - \mu < \frac{S}{\sqrt{n}} t_{n-1, 1-\alpha}\right) =$$

$$= \mathbb{P}\left(\mu > \overline{X} - \frac{S}{\sqrt{n}} t_{n-1, 1-\alpha}\right)$$

$$(L_i, +\infty) = \left(\overline{X} - \frac{S}{\sqrt{n}} t_{n-1, 1-\alpha}, +\infty\right)$$

$$1 - \alpha = \mathbb{P}(T > t) = 1 - \mathbb{P}(T < t) \qquad \mathbb{P}(T < t) = \alpha$$

$$t = t_{n-1, \alpha} = -t_{n-1, 1-\alpha}$$

$$1-\alpha = \mathbb{P}\left(T > -t_{n-1,1-\alpha}\right) = \mathbb{P}\left(\frac{(\overline{X}-\mu)\sqrt{n}}{S} > -t_{n-1,1-\alpha}\right)$$

$$= \mathbb{P}\left(\overline{X}-\mu > \frac{-S}{\sqrt{n}}\, t_{n-1,1-\alpha}\right)$$

$$= \mathbb{P}\left(\mu < \overline{X} + \frac{S}{\sqrt{n}}\, t_{n-1,1-\alpha}\right)$$

$$(-\infty, L_s) = \left(-\infty, \overline{X} + \frac{S}{\sqrt{n}}\, t_{n-1,1-\alpha}\right)$$

$$V := \frac{(n-1)S^2}{\sigma^2}$$ so che se $\mathbb{P}_{X_i} = N(\mu, \sigma^2)$ allora $\mathbb{P}_V = \chi^2_{n-1}$

$$1-\alpha = \mathbb{P}\left(\chi^2_{n-1,\frac{\alpha}{2}} < V < \chi^2_{n-1,1-\frac{\alpha}{2}}\right) =$$

$$= \mathbb{P}\left(\chi^2_{n-1,\frac{\alpha}{2}} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{n-1,1-\frac{\alpha}{2}}\right) =$$

$$= \mathbb{P}\left(\frac{1}{\chi^2_{n-1,1-\frac{\alpha}{2}}} < \frac{\sigma^2}{(n-1)S^2} < \frac{1}{\chi^2_{n-1,\frac{\alpha}{2}}}\right)$$

$$= \mathbb{P}\left(\frac{(n-1)S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{n-1,\frac{\alpha}{2}}}\right)$$

$$(L_i, L_s) = \left(\frac{(n-1)S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \frac{(n-1)S^2}{\chi^2_{n-1,\frac{\alpha}{2}}}\right)$$

$$1-\alpha = \mathbb{P}\left(V < \chi^2_{n-1,1-\alpha}\right) = \mathbb{P}\left(\frac{(n-1)S^2}{\sigma^2} < \chi^2_{n-1,1-\alpha}\right)$$

$$= \mathbb{P}\left(\frac{\sigma^2}{(n-1)S^2} > \frac{1}{\chi^2_{n-1,1-\alpha}}\right)$$

$$= \mathbb{P}\left(\sigma^2 > \frac{(n-1)S^2}{\chi^2_{n-1,\,1-\alpha}}\right)$$

$$(L_i, +\infty) = \left(\frac{(n-1)S^2}{\chi^2_{n-1,\,1-\alpha}}, +\infty\right)$$

$$1-\alpha = \mathbb{P}(V > t) = 1 - \mathbb{P}(V < t) \qquad \mathbb{P}(V < t) = \alpha$$

$$\Rightarrow \quad t = \chi^2_{n-1,\alpha}$$

$$1-\alpha = \mathbb{P}(V > \chi^2_{n-1,\alpha}) = \mathbb{P}\left(\frac{(n-1)S^2}{\sigma^2} > \chi^2_{n-1,\alpha}\right)$$

$$= \mathbb{P}\left(\sigma^2 < \frac{(n-1)S^2}{\chi^2_{n-1,\alpha}}\right)$$

$$(0, L_s) = \left(0, \frac{(n-1)S^2}{\chi^2_{n-1,\alpha}}\right)$$

———— o ————

$$x_1 \text{———} x_n \qquad x_n \in \{0, 1\}$$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \left|\bar{x} - \frac{1}{2}\right| < \varepsilon$$

$$\left|\bar{x} - \frac{1}{2}\right| \geq \varepsilon$$

$$\mathbb{P}\left(\left|\bar{X} - \frac{1}{2}\right| \geq \varepsilon \;\middle|\; p = \frac{1}{2}\right) =$$

$$= \mathbb{P}\left(\bar{X} \geq \frac{1}{2} + \varepsilon \;\middle|\; p = \frac{1}{2}\right) + \mathbb{P}\left(\bar{X} \leq \frac{1}{2} - \varepsilon \;\middle|\; p = \frac{1}{2}\right) \quad (A)$$

$$X_1 \text{——} X_n \qquad \mathbb{P}_{X_i} = B\left(\frac{1}{2}\right)$$

$\Rightarrow \quad S_n := \sum_{i=1}^{n} X_i \qquad \mathbb{P}_{S_n} = Bin(n, \frac{1}{2})$

$$A = \mathbb{P}\left(\frac{S_n}{n} \geq \frac{1}{2} + \varepsilon \,\middle|\, p = \frac{1}{2} \right) + \mathbb{P}\left(\frac{S_n}{n} \leq \frac{1}{2} - \varepsilon \,\middle|\, p = \frac{1}{2} \right)$$

$$= \mathbb{P}\left(S_n \geq \frac{n}{2} + \varepsilon n \,\middle|\, p = \frac{1}{2} \right) + \mathbb{P}\left(S_n \leq \frac{n}{2} - \varepsilon n \,\middle|\, p = \frac{1}{2} \right)$$

Indico con F la legge $B(n, \frac{1}{2})$

$$= 1 - F\left(\left(\frac{n}{2} + \varepsilon n \right)^{-} \right) + F\left(\frac{n}{2} - \varepsilon n \right)$$

$X_1 \ldots X_n$ campione statistico

H_0 IPOTESI NULLA

$\mathcal{F} \qquad H_0 \quad F \in \mathcal{F}_0 \qquad H_A \quad F \notin \mathcal{F}_0$
$F \in \mathcal{F} \setminus \mathcal{F}_0$

$\Theta \qquad H_0 \quad \theta \in \Theta_0 \qquad H_A \quad \theta \in \Theta \setminus \Theta_0$

$f(X_1 \ldots X_n)$ statistica del campione

$f : \mathbb{R}^n \to G \subset \mathbb{R}$

Fisso $A \subset G$ e se $f(x_1 \ldots x_n) \in A$ accetto H_0

se $f(x_1 \ldots x_n) \notin A$ rifiuto H_0

l'insieme A si dice REGIONE DI ACCETTAZIONE

l'insieme $C := G \setminus A$ si dice REGIONE CRITICA (o di rifiuto)

$$\alpha := \mathbb{P}\left(\varphi(X_1 \_\_ X_n) \notin \mathcal{A} \mid H_0 \right)$$

$$\beta := \mathbb{P}\left(\varphi(X_1 \_\_\_ X_n) \in \mathcal{A} \mid H_A \right)$$