STIMATORI DI MASSIMA VEROSIMIGLIANZA

$X_1 \ldots X_n$ campione statistico

Se θ è un parametro che caratterizza la distribuzione del campione e se $Y = f(X_1 \ldots X_n)$ fornisce una stima di θ, allora Y si dice uno stimatore di θ

Supponiamo che la distribuzione del campione ammette densità

CASO DISCRETO

$g(x|\theta)$

$g : x \in \mathbb{R} \to [0, +\infty)$

$g(x) \neq 0$ solo per un insieme discreto di valori di x

$\mathbb{P}(X_i = x) = g(x)$

CASO A.C.

$g(x|\theta)$

$g : x \in \mathbb{R} \to [0, +\infty)$ T.c.

$\mathbb{P}(X_i \in A) = \int_A g(x)\,dx$

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$x_1 \ldots x_n$ $\qquad x_i = X_i(\omega) \quad \forall i = 1 \ldots n$

CASO DISCRETO . $\mathbb{P}(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n) =$

$= \prod_{i=1}^{n} \mathbb{P}(X_i = x_i) = \prod_{i=1}^{n} g(x_i|\theta) =$ densità di $(X_1 \ldots X_n)$ calcolata in $(x_1 \ldots x_n)$

Indico con $f(x|\theta) = \prod_{i=1}^{n} g(x_i|\theta)$ la densità congiunta di $X_1 \ldots X_n$

dove $x = (x_1 \ldots x_n)$

Vedo $f(x_1 \ldots x_n|\theta)$ come funzione di θ e cerco il θ che la massimizza.

CASO A.C.

$$\mathbb{P}\left(|X_1 - x_1| < \delta,\ |X_2 - x_2| < \delta,\ \_\_\_\ |X_n - x_n| < \delta\right) =$$

$$= \prod_{i=1}^{n} \mathbb{P}\left(|X_i - x_i| < \delta\right) \longleftarrow$$

$$\mathbb{P}\left(|X_i - x_i| < \delta\right) = \mathbb{P}\left(x_i - \delta < X_i < x_i + \delta\right) =$$

$$= \int_{x_i - \delta}^{x_i + \delta} g(t|\theta)\,dt \simeq g(x_i|\theta)(2\delta)$$

$$\mathbb{P}\left(|X_1 - x_1| < \delta,\ \text{--}\ \text{--}\ |X_n - x_n| < \delta\right) \simeq \prod_{i=1}^{n} \left(g(x_i|\theta)\, 2\delta\right)$$

$$= (2\delta)^n \prod_{i=1}^{n} g(x_i|\theta) = (2\delta)^n f(x_1 \_\_\_ x_n|\theta)$$

dove $f(x_1 \_\_\_ x_n|\theta)$ è la densità congiunta di $X_1 \_\_\_ X_n$

Il valore di θ che massimizza $f(x|\theta)$ si chiama STIMATORE DI MAX VEROSIMIGLIANZA

$$\hat{\theta} = \operatorname{argmax} f(X_1 \_\_\_ X_n|\theta)$$

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$$f(\underbrace{x_1 \_\_\_ x_n}_{x}|\theta) = \prod_{i=1}^{n} g(x_i|\theta) = f(x|\theta)$$

$$h(x|\theta) = \ln f(x|\theta)$$

$$\ell(x|\theta) = \ln \prod_{i=1}^{n} g(x_i|\theta) = \sum_{i=1}^{n} \ln g(x_i|\theta)$$

DISTRIBUZIONE DI BERNOULLI

Concentrata su $\{0,1\}$ $\quad \mathbb{P}(X=1)=p \quad \mathbb{P}(X=0)=1-p$

$X_1 \_\_\_\_ X_n$ campione statistico $\mathbb{P}_{X_i} := B(p)$

$x_1 \_\_\_\_ x_n$

$$\underbrace{\mathbb{P}(X_i = x_i)}_{=g(x_i|p)} = \begin{cases} p & x_i = 1 \\ 1-p & x_i = 0 \end{cases}$$

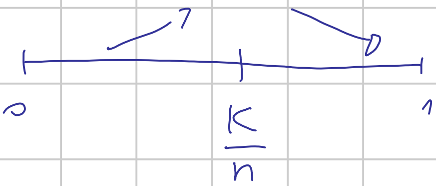
$$K := \sum_{i=1}^{n} x_i$$

$$\ell(x|p) = \sum_{i=1}^{n} \ln g(x_i|p)$$

$$= K \ln p + (n-k)\ln(1-p)$$

$$\frac{\partial \ell(x|p)}{\partial p} = \frac{K}{p} + \frac{(n-k)(-1)}{1-p} = \frac{K(1-\cancel{p}) - p(n-\cancel{k})}{p(1-p)}$$

$$= \frac{K-np}{p(1-p)} \gtrless 0 \qquad K-np \gtrless 0 \qquad p \lessgtr \frac{K}{n}$$



$$\hat{p} = \frac{K}{n} = \frac{\sum_{i=1}^{n} x_i}{n} = \overline{x} \quad -$$

Lo stimatore di massima verosimiglianza è $\overline{X}$

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DISTRIBUZIONE DI POISSON

$$\underbrace{\mathbb{P}(X=k)}_{g(k|\lambda)} = \frac{e^{-\lambda}\lambda^k}{k!} \qquad \forall k = 0,1,2,\ldots$$

$x_1 \_\_\_\_ x_n$ $\qquad g(x_i|\lambda) = \dfrac{e^{-\lambda}\lambda^{x_i}}{x_i!}$

$$l(x|\lambda) = \sum_{i=1}^{n} \ln g(x_i|\lambda) = \sum_{i=1}^{n} \ln\left(\frac{e^{-\lambda}\lambda^{x_i}}{x_i!}\right) =$$

$$= \sum_{i=1}^{n} \left(-\lambda + x_i \ln\lambda - \ln x_i!\right) =$$

$$= -n\lambda + \ln\lambda \underbrace{\left(\sum_{i=1}^{n} x_i\right)}_{n\bar{x}} - \sum_{i=1}^{n} \ln(x_i!)$$

$$= n(-\lambda + \bar{x}\ln\lambda) - \sum_{i=1}^{n} \ln(x_i!)$$

$$\frac{d\, l(x|\lambda)}{d\lambda} = n\left(-1 + \frac{\bar{x}}{\lambda}\right) = \frac{n}{\lambda}(\bar{x} - \lambda) \gtrless 0$$

$$\lambda - \bar{x} \lesseqgtr 0 \qquad \lambda \lesseqgtr \bar{x} \,,$$

$$=0 \qquad \hat{\lambda} = \bar{X}$$

DISTRIBUZIONE GAUSSIANA $N(\mu, \sigma^2)$

$$g(x_i|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x_i-\mu)^2}{2\sigma^2}\right)$$

$$l(x|\mu,\sigma^2) = \sum_{i=1}^{n} \ln g(x_i|\mu,\sigma^2) =$$

$$= \sum_{i=1}^{n} \ln\left((2\pi\sigma^2)^{-1/2} \cdot \exp\left(\frac{-(x_i-\mu)^2}{2\sigma^2}\right)\right)$$

$$= \sum_{i=1}^{n} \left(-\frac{1}{2}\ln(2\pi\sigma^2) + \frac{-(x_i-\mu)^2}{2\sigma^2}\right) =$$

$$= \sum_{i=1}^{n} \left(-\frac{1}{2}\ln(2\pi) - \frac{1}{\cancel{2}}\cancel{2}\ln(\sigma) - \frac{(x_i-\mu)^2}{2\sigma^2}\right)$$

$$= -\frac{n}{2}\ln(2\pi) - n\ln(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial f_h}{\partial \mu} = +\frac{1}{2\sigma^2} \sum_{i=1}^{n} 2(x_i-\mu) = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) =$$

$$= \frac{1}{\sigma^2}\left(\sum_{i=1}^{n} x_i - n\mu\right) = \frac{n}{\sigma^2}(\bar{x}-\mu)$$

$$\frac{\partial f_h}{\partial \sigma} = \frac{-n}{\sigma} - \frac{1}{2}\,\frac{-2}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2 = \frac{-n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$= \frac{1}{\sigma^3}\left(-n\sigma^2 + \sum_{i=1}^{n}(x_i-\mu)^2\right)$$

$$\begin{cases} \bar{x}-\mu = 0 \\ -n\sigma^2 + \sum_{i=1}^{n}(x_i-\mu)^2 = 0 \end{cases} \qquad \begin{cases} \mu = \bar{x} \\ \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 \end{cases}$$

$$\begin{cases} \mu = \bar{x} \\ \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \frac{n-1}{n}\,\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \end{cases}$$

$$= \frac{n-1}{n} S_x^2$$

DISTRIBUZIONE UNIFORME SU UN INTERVALLO $[a,b]$

$$g(x|a,b) = \begin{cases} 0 & x \notin [a,b] \\ \frac{1}{b-a} & x \in [a,b] \end{cases}$$

$$f(x_1 \ldots x_n | a,b) = \prod_{i=1}^{n} g(x_i | a,b) =$$

$$\begin{cases} \frac{1}{(b-a)^n} & \text{se } a \le x_i \le b \quad \forall i = 1 \ldots n \\ 0 & \text{altrimenti} \end{cases}$$

$$a = \min\{x_1 \_\_ x_n\}$$
$$b = \max\{x_1 \_\_ x_n\}$$

$$\hat{A} = \min\{X_1 \_\_ X_n\}$$
$$\hat{B} = \max\{X_1 \_\_ X_n\}$$

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$X_1 \_\_ X_n$ campione statistico la cui distribuzione ha forma nota ma dipende da un parametro θ

Sia $T = t(X_1 \_\_ X_n)$ stimatore del parametro θ

$$b_\theta(T) = E[T] - \theta$$ bias dello stimatore

$$r(T,\theta) = E[(T-\theta)^2] = E\Big[\big(\underbrace{T - E[T]} + \underbrace{E[T] - \theta}_{b_\theta(T)}\big)^2\Big]$$

$$= E\Big[(T - E[T])^2 + 2b_\theta(T)(T - E[T]) + b_\theta^2(T)\Big]$$

$$= Var[T] + 2b_\theta(T)\,\cancel{E[T - E[T]]}^{=0} + b_\theta^2(T)$$

$$r(T,\theta) = Var[T] + b_\theta^2(T)$$

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$X_1 \_\_ X_n$ campione statistico $P_{X_i} = U([0,\theta])$

$$E[X_i] = \frac{\theta}{2} \qquad Var[X_i] = \frac{\theta^2}{12}$$

$$E[\bar{X}] = E[X_i] = \frac{\theta}{2} \qquad Var[\bar{X}] = \frac{1}{n} Var[X_i] = \frac{\theta^2}{12n}$$

$\boxed{T = 2\overline{X}}$ $\quad \mathbb{E}[T] = \theta \quad Var[T] = 4\, Var[\overline{X}] = \frac{\theta^2}{3n}$

$\hat{T} := \max(X_1 \_\_\_ X_n)$

$F_{\hat{T}}(t) = \mathbb{P}(\hat{T} \leq t) = \mathbb{P}(X_1 \leq t, X_2 \leq t \ldots X_n \leq t)$

$= \prod_{i=1}^{n} \mathbb{P}(X_i \leq t) = \prod_{i=1}^{n} F_{X_i}(t) = \prod_{i=1}^{n} F_X(t) = (F_X(t))^n$

$$F_X(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{\theta} & 0 \leq t \leq \theta \\ 1 & t > \theta \end{cases}$$

$$F_{\hat{T}}(t) = \begin{cases} 0 & t < 0 \\ \left(\frac{t}{\theta}\right)^n & 0 \leq t \leq \theta \\ 1 & t > \theta \end{cases}$$

$\mathbb{P}_{\hat{T}} = g(t)\,dt$ con $g(t) = \begin{cases} \frac{n t^{n-1}}{\theta^n} & t \in [0, \theta] \\ 0 & \text{altrimenti} \end{cases}$

$$\mathbb{E}[\hat{T}] = \int_{\mathbb{R}} t\, g(t)\,dt = \int_0^\theta \frac{n}{\theta^n} t^n\,dt = \frac{n}{\theta^n} \frac{t^{n+1}}{n+1}\Big|_{t=0}^{t=\theta}$$

$\mathbb{E}[\hat{T}] = \frac{n}{n+1}\theta \qquad b_\theta(\hat{T}) = \frac{n}{n+1}\theta - \theta = \frac{-\theta}{n+1}$

$$\mathbb{E}[\hat{T}^2] = \int_{\mathbb{R}} t^2 g(t)\,dt = \int_0^\theta \frac{n}{\theta^n} t^{n+1}\,dt =$$

$$= \frac{n}{\theta^n} \frac{t^{n+2}}{n+2} \Big|_{t=0}^{t=\theta} = \frac{n}{n+2} \theta^2$$

$$\mathrm{Var}[\hat{T}] = \mathbb{E}[\hat{T}^2] - (\mathbb{E}[\hat{T}])^2 = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2$$

$$= \frac{n\theta^2}{(n+2)(n+1)^2}$$

$$r(\hat{T}, \theta) = \mathrm{Var}[\hat{T}] + b_\theta^2(\hat{T}) = \frac{n\theta^2}{(n+2)(n+1)^2} + \frac{\theta^2}{(n+1)^2}$$

$$= \frac{2\theta^2}{(n+1)(n+2)}$$

$$T_c = c\hat{T} \qquad c \in \mathbb{R}$$

$$\mathbb{E}[T_c] = c\,\mathbb{E}[\hat{T}] = \frac{cn\theta}{n+1}$$

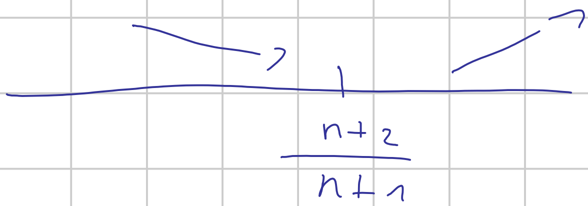
$$\mathrm{Var}[T_c] = c^2\,\mathrm{Var}[\hat{T}] = \frac{nc^2\theta^2}{(n+1)(n+2)}$$

$$b_\theta(T_c) = \mathbb{E}[T_c] - \theta = \frac{cn\theta}{n+1} - \theta = \frac{\theta(nc-n-1)}{n+1}$$

$$r(T_c, \theta) = \mathrm{Var}[T_c] + b_\theta^2(T_c) = \frac{c^2 n\theta^2}{(n+1)(n+2)} + \frac{\theta^2(nc-n-1)^2}{(n+1)^2}$$

$$\frac{dr}{dc}(T_c, \theta) = \frac{2n\theta^2}{(n+2)(n+1)}\left[c(n+1) - (n+2)\right] \gtrless 0$$

$$c \gtrless \frac{n+2}{n+1}$$



$$\bar{c} := \frac{n+2}{n+1}$$

$$r\left(T_{\overline{c}}, \theta\right) = \frac{\theta^2}{(n+1)^2}$$