TEST D'IPOTESI PER CAMPIONI GAUSSIANI

Note Title 29/05/2018

$X_1 \dots X_n$ campione gaussiano di valore atteso ignoto e varianza σ^2 nota

$H_0: \quad \mu = \mu_0 \qquad\qquad H_A: \mu \neq \mu_0$

$\mathbb{P}_{X_i} = N(\mu_0, \sigma^2) \quad \text{sse} \quad \mathbb{E}[\bar{X}] = \mu_0$

Accetto H_0 sse $|\bar{x} - \mu_0| < \varepsilon$

$$\alpha = \mathbb{P}\left(|\bar{X} - \mu_0| > \varepsilon \;\middle|\; \mu = \mu_0 \right) = (*)$$

Se $\mu = \mu_0 \Rightarrow \mathbb{P}_{\bar{X}} = N\left(\mu_0, \frac{\sigma^2}{n}\right)$

$$(*) = \mathbb{P}\left(\frac{|\bar{X} - \mu_0|}{\sigma/\sqrt{n}} > \frac{\varepsilon}{\sigma/\sqrt{n}} \;\middle|\; \mu = \mu_0 \right)$$

$$= \mathbb{P}\left(|Z| > \frac{\varepsilon\sqrt{n}}{\sigma} \right) \quad \text{con} \quad \mathbb{P}_Z = N(0,1)$$

$$= \mathbb{P}\left(Z > \frac{\varepsilon\sqrt{n}}{\sigma} \right) + \mathbb{P}\left(Z < \frac{-\varepsilon\sqrt{n}}{\sigma} \right)$$

$$= 1 - \Phi\left(\frac{\varepsilon\sqrt{n}}{\sigma} \right) + \underbrace{\Phi\left(\frac{-\varepsilon\sqrt{n}}{\sigma} \right)}_{1 - \Phi\left(\frac{\varepsilon\sqrt{n}}{\sigma}\right)}$$

$$\alpha = 2\left(1 - \Phi\left(\frac{\varepsilon\sqrt{n}}{\sigma} \right) \right)$$

$$\Leftrightarrow \Phi\left(\frac{\varepsilon\sqrt{n}}{\sigma} \right) = 1 - \frac{\alpha}{2} \Leftrightarrow \frac{\varepsilon\sqrt{n}}{\sigma} = z_{1-\frac{\alpha}{2}}$$

$$\varepsilon = \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}$$

Accetto H_0 sse $|\bar{x} - \mu_0| < \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}$

$$\beta(\mu) = \mathbb{P}\left(|\bar{X} - \mu_0| < \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}} \;\middle|\; E[\bar{X}] = \mu \neq \mu_0 \right)$$

$$= \mathbb{P}\left(\mu_0 - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}} < \bar{X} < \mu_0 + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}} \;\middle|\; E[\bar{X}] = \mu \neq \mu_0 \right)$$

$$\Downarrow \quad \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \text{ è } N(0,1)$$

$$= \mathbb{P}\left(\frac{\mu_0 - \mu - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}}{\sigma/\sqrt{n}} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{\mu_0 - \mu + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}}{\sigma/\sqrt{n}} \;\middle|\; E[\bar{X}] = \mu \neq \mu_0 \right)$$

$$= \mathbb{P}\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} - z_{1-\frac{\alpha}{2}} < Z < \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + z_{1-\frac{\alpha}{2}} \right) \quad \text{con } \mathbb{P}_Z = N(0,1)$$

$$= \mathbb{P}\left(Z < \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + z_{1-\frac{\alpha}{2}} \right) - \mathbb{P}\left(Z < \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} - z_{1-\frac{\alpha}{2}} \right)$$

$$= \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + z_{1-\frac{\alpha}{2}} \right) - \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + z_{\frac{\alpha}{2}} \right)$$

$$\beta(\mu) = \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + z_{1-\frac{\alpha}{2}} \right) - \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + z_{\frac{\alpha}{2}} \right)$$

1 $\mu > \mu_0 \Rightarrow \mu_0 - \mu < 0 \qquad \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} < 0$

$$\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + z_{\frac{\alpha}{2}} < z_{\frac{\alpha}{2}} \qquad \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + z_{\frac{\alpha}{2}} \right) < \Phi\left(z_{\frac{\alpha}{2}}\right) = \frac{\alpha}{2}$$

$$\mu > \mu_0 \Rightarrow \beta(\mu) \simeq \Phi\Bigg(\underbrace{\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + z_{1-\frac{\alpha}{2}}}_{\leq z_{1-\frac{\alpha}{2}}} \Bigg) \leq 1 - \frac{\alpha}{2}$$

② $\mu < \mu_0$

$$\beta(\mu) = \Phi\left(\frac{\mu_0-\mu}{\sigma/\sqrt{n}} + z_{1-\frac{\alpha}{2}}\right) - \Phi\left(\frac{\mu_0-\mu}{\sigma/\sqrt{n}} + z_{\frac{\alpha}{2}}\right)$$

$$= \cancel{1} - \Phi\left(\frac{\mu-\mu_0}{\sigma/\sqrt{n}} - z_{1-\frac{\alpha}{2}}\right) - \left(\cancel{1} - \Phi\left(\frac{\mu-\mu_0}{\sigma/\sqrt{n}} - z_{\frac{\alpha}{2}}\right)\right)$$

$$= \Phi\left(\frac{\mu-\mu_0}{\sigma/\sqrt{n}} - z_{\frac{\alpha}{2}}\right) - \Phi\left(\frac{\mu-\mu_0}{\sigma/\sqrt{n}} - z_{1-\frac{\alpha}{2}}\right)$$

$$= \Phi\left(\frac{\mu-\mu_0}{\sigma/\sqrt{n}} + z_{1-\frac{\alpha}{2}}\right) - \Phi\left(\frac{\mu-\mu_0}{\sigma/\sqrt{n}} + z_{\frac{\alpha}{2}}\right)$$

$$\mu < \mu_0 \Rightarrow \frac{\mu-\mu_0}{\sigma/\sqrt{n}} + z_{\frac{\alpha}{2}} < z_{\frac{\alpha}{2}}$$

$$\Phi\left(\frac{\mu-\mu_0}{\sigma/\sqrt{n}} + z_{\frac{\alpha}{2}}\right) < \frac{\alpha}{2}$$

$$\mu < \mu_0 \qquad \beta(\mu) \simeq \Phi\left(\underbrace{\frac{\mu-\mu_0}{\sigma/\sqrt{n}} + z_{1-\frac{\alpha}{2}}}_{< z_{1-\frac{\alpha}{2}}}\right) < 1 - \frac{\alpha}{2}$$

— o —

H_0 $\mu = \mu_0$ $\qquad$ $H_A: \mu > \mu_0$

Accetto H_0 se $\bar{x} < \mu_0 + \varepsilon$

$$\alpha = P(\bar{X} > \mu_0 + \varepsilon \mid \mu = \mu_0) = P\left(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > \frac{\mu_0+\varepsilon-\mu_0}{\sigma/\sqrt{n}} \,\middle|\, \mu=\mu_0\right)$$

$$= P\left(Z > \frac{\varepsilon\sqrt{n}}{\sigma}\right) = 1 - \Phi\left(\frac{\varepsilon\sqrt{n}}{\sigma}\right)$$

$$\iff \frac{\varepsilon\sqrt{n}}{\sigma} = z_{1-\alpha} \iff \varepsilon = \frac{\sigma}{\sqrt{n}} z_{1-\alpha}$$

Accetto H_0 sse $\bar{x} < \mu_0 + \frac{\sigma}{\sqrt{n}} z_{1-\alpha}$

— o —

$H_0: \mu \leq \mu_0$ $\quad$ $H_A: \mu > \mu_0$

Accetto H_0 sse $\bar{x} < \mu_0 + \varepsilon$

$$\cancel{\alpha =} \mathbb{P}\left(\bar{X} > \mu_0 + \varepsilon \mid \mathbb{E}[\bar{X}] = \mu \leq \mu_0 \right) =$$

$$= \mathbb{P}\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} > \frac{\mu_0 + \varepsilon - \mu}{\sigma/\sqrt{n}} \mid \mathbb{E}[\bar{X}] = \mu \leq \mu_0 \right)$$

$$= \mathbb{P}\left(Z > \frac{\mu_0 - \mu + \varepsilon}{\sigma/\sqrt{n}} \right) = 1 - \Phi\left(\frac{\mu_0 - \mu + \varepsilon}{\sigma/\sqrt{n}} \right)$$

$\mu \leq \mu_0$ $\qquad$ $\frac{\mu_0 - \mu + \varepsilon}{\sigma/\sqrt{n}} \geq \varepsilon / \sigma/\sqrt{n}$

$$\mathbb{P}\left(\bar{X} > \mu_0 + \varepsilon \mid \mathbb{E}[\bar{X}] = \mu < \mu_0 \right) \leq 1 - \Phi\left(\frac{\varepsilon \sqrt{n}}{\sigma} \right) = \alpha$$

Scelgo $\frac{\varepsilon \sqrt{n}}{\sigma} = z_{1-\alpha}$

Accetto H_0 $\mu \leq \mu_0$ sse $\bar{x} < \mu_0 + \frac{\sigma}{\sqrt{n}} z_{1-\alpha}$

— o —

In modo del tutto analogo

$H_0: \mu = \mu_0$ $\quad$ $H_A: \mu < \mu_0$

Accetto H_0 sse $\bar{x} > \mu_0 - \varepsilon$

Si trova $\varepsilon = \frac{\sigma}{\sqrt{n}} z_{1-\alpha}$

$H_0 \quad \mu \geq \mu_0 \qquad\qquad H_A \quad \mu < \mu_0$

Accetto H_0 sse $\bar{x} > \mu_0 - \varepsilon$

Si trova $\varepsilon = \frac{\sigma}{\sqrt{n}} z_{1-\alpha}$

— o —

CAMPIONE GAUSSIANO DI VALORE ATTESO E VARIANZA IGNOTI.

$H_0: \quad \mu = \mu_0 \qquad\qquad H_A: \mu \neq \mu_0$

H_0 è vera sse $\mathbb{E}[\bar{X}] = \mu_0$

$$\mathbb{E}\left[\frac{(\bar{X}-\mu_0)\sqrt{n}}{S}\right] = \mathbb{E}\left[\frac{1}{S}\right]\sqrt{n}\,\mathbb{E}\left[\bar{X}-\mu_0\right] = 0$$

$\mathbb{E}\left[\frac{(\bar{X}-\mu_0)\sqrt{n}}{S}\right] = 0$ sse H_0 è vera

Accetto H_0 sse $\left|\frac{(\bar{x}-\mu_0)\sqrt{n}}{S}\right| < \varepsilon$

$$\alpha = \mathbb{P}\left(\frac{|\bar{X}-\mu_0|\sqrt{n}}{S} > \varepsilon \,\middle|\, \mu = \mu_0\right) = \qquad \mathbb{P}_T = t(n-1)$$

$$= \mathbb{P}(|T| > \varepsilon) = \mathbb{P}(T > \varepsilon) + \mathbb{P}(T < -\varepsilon)$$

$$= 1 - F_T(\varepsilon) + F_T(-\varepsilon) = 2 - 2F_T(\varepsilon)$$

$$F_T(\varepsilon) = 1 - \frac{\alpha}{2} \iff \varepsilon = t_{n-1,\, 1-\frac{\alpha}{2}}$$

Accetto H_0 sse $\frac{|\bar{x}-\mu_0|\sqrt{n}}{S} < t_{n-1,1-\frac{\alpha}{2}}$

ovvero sse

$$\mu_0 - \frac{s}{\sqrt{n}} t_{n-1, 1-\frac{\alpha}{2}} < \bar{x} < \mu_0 + \frac{s}{\sqrt{n}} t_{n-1, 1-\frac{\alpha}{2}}$$

— o —

$H_0 \quad \mu = \mu_0$ $\qquad\qquad H_A : \mu > \mu_0$

$$\frac{(\bar{x} - \mu_0)\sqrt{n}}{s} < \varepsilon$$

$$\alpha = \mathbb{P}\left(\frac{(\bar{X} - \mu_0)\sqrt{n}}{S} > \varepsilon \;\middle|\; \mathbb{E}[\bar{X}] = \mu_0 \right)$$

$= \mathbb{P}(T > \varepsilon)$ $\qquad$ con $\qquad \mathbb{P}_T = t(n-1)$

$= 1 - F_T(\varepsilon)$

$\Rightarrow \quad F_T(\varepsilon) = 1 - \alpha$ $\qquad$ cioè $\quad \varepsilon = t_{n-1, 1-\alpha}$

Accetto H_0 sse $\dfrac{(\bar{x} - \mu_0)\sqrt{n}}{s} < t_{n-1, 1-\alpha}$

cioè sse

$$\bar{x} < \mu_0 + \frac{s}{\sqrt{n}} t_{n-1, 1-\alpha}$$

— o —

$H_0 : \mu \le \mu_0$ $\qquad\qquad H_A : \mu > \mu_0$

Accetto H_0 sse $\dfrac{(\bar{x} - \mu_0)\sqrt{n}}{s} \le \varepsilon$

$$\mathbb{P}\left(\frac{(\bar{X} - \mu_0)\sqrt{n}}{S} > \varepsilon \;\middle|\; \mathbb{E}[\bar{X}] = \mu \le \mu_0 \right)$$

Se $\mathbb{E}[\bar{X}] = \mu \Rightarrow \dfrac{(\bar{X} - \mu)\sqrt{n}}{S}$ ha distribuzione $t(n-1)$

$$\mu \le \mu_0 \qquad \frac{(\bar{X} - \mu)\sqrt{n}}{S} \ge \frac{(\bar{X} - \mu_0)\sqrt{n}}{S}$$

$$\left\{ \frac{(\bar{X} - \mu_0)\sqrt{n}}{S} > \varepsilon \right\} \subseteq \left\{ \frac{(\bar{X} - \mu)\sqrt{n}}{S} > \varepsilon \right\}$$

$$\Rightarrow \mathbb{P}\left(\frac{(\bar{X}-\mu_0)\sqrt{n}}{S} > \varepsilon \;\middle|\; \mathbb{E}[\bar{X}] = \mu \leq \mu_0 \right) \leq$$

$$\mathbb{P}\left(\frac{(\bar{X}-\mu)\sqrt{n}}{S} > \varepsilon \;\middle|\; \mathbb{E}[\bar{X}] = \mu \leq \mu_0 \right)$$

$$= \mathbb{P}(T > \varepsilon) \qquad \text{con} \qquad \mathbb{P}_T = t(n-1)$$

Se cerco ε t.c. $\mathbb{P}(T > \varepsilon) = \alpha$ $\Rightarrow$
la prob di commettere errore di 1ª specie è $\leq \alpha$

$$1 - F_T(\varepsilon) = \alpha \qquad \varepsilon = t_{n-1,\,1-\alpha}$$

Accetto H_0 sse $\dfrac{(\bar{x}-\mu_0)\sqrt{n}}{S} < t_{n-1,\,1-\alpha}$

— o —

$H_0: \mu = \mu_0$ $\qquad$ $H_A: \mu < \mu_0$

Accetto H_0 sse $\dfrac{(\bar{x}-\mu_0)\sqrt{n}}{S} > \mu_0 - \varepsilon$

Trovo $\varepsilon = t_{n-1,\,1-\alpha}$

— o —

$H_0: \mu \geq \mu_0$ $\qquad$ $H_A: \mu < \mu_0$

Accetto H_0 sse $\bar{x} > \mu_0 - \varepsilon$
Trovo $\varepsilon = t_{n-1,\,1-\alpha}$

TEST D'IPOTESI SULLA VARIANZA DI CAMPIONI GAUSSIANI

$X_1 \_\_ X_n$ campione gaussiano con valore atteso μ e varianza σ^2 entrambi ignoti

$H_0: \sigma^2 = \sigma_0^2$ $\qquad H_A: \sigma^2 \neq \sigma_0^2$

H_0 è vera sse $\mathbb{E}[S^2] = \sigma_0^2$ sse

$$\mathbb{E}\left[\frac{S^2}{\sigma_0^2}\right] = 1$$

Accetto H_0 sse $1 - \varepsilon_1 < \frac{S^2}{\sigma_0^2} < 1 + \varepsilon_2$

cioè

Accetto H_0 sse $(n-1)(1-\varepsilon_1) < \frac{(n-1)S^2}{\sigma_0^2} < (n-1)(1+\varepsilon_2)$

$$\alpha = \mathbb{P}\left(\frac{(n-1)S^2}{\sigma_0^2} > (n-1)(1+\varepsilon_2) \,\middle|\, \sigma^2 = \sigma_0^2\right) +$$

$$+ \mathbb{P}\left(\frac{(n-1)S^2}{\sigma_0^2} < (n-1)(1-\varepsilon_1) \,\middle|\, \sigma^2 = \sigma_0^2\right)$$

$V = \frac{(n-1)S^2}{\sigma_0^2}$ ha distribuzione $\mathbb{P}_V = \chi^2_{n-1}$ se $\sigma^2 = \sigma_0^2$

$$\alpha = \mathbb{P}\left(V > (n-1)(1+\varepsilon_2)\right) + \mathbb{P}\left(V < (n-1)(1-\varepsilon_1)\right)$$

con $\mathbb{P}_V = \chi^2_{n-1}$

$$\begin{cases} \mathbb{P}\left(V > (n-1)(1+\varepsilon_2)\right) = \frac{\alpha}{2} \\ \mathbb{P}\left(V < (n-1)(1-\varepsilon_1)\right) = \frac{\alpha}{2} \end{cases}$$

$$F_V\left((n-1)(1+\varepsilon_2)\right) = 1 - \frac{\alpha}{2}, \quad F_V\left((n-1)(1-\varepsilon_1)\right) = \frac{\alpha}{2}$$

$$(n-1)(1+\varepsilon_2) = \chi^2_{n-1,\,1-\frac{\alpha}{2}}$$

$$(n-1)(1-\varepsilon_1) = \chi^2_{n-1,\,\frac{\alpha}{2}}$$

Accetto H_0 SSE $\chi^2_{n-1,\frac{\alpha}{2}} < \frac{(n-1)s^2}{\sigma_0^2} \leq \chi^2_{n-1,1-\frac{\alpha}{2}}$

— o —

$H_0: \ \sigma^2 = \sigma_0^2 \qquad H_A: \ \sigma^2 > \sigma_0^2$

Accetto H_0 se $\frac{s^2}{\sigma_0^2} < 1+\varepsilon$

$$\alpha = \mathbb{P}\left(\frac{S^2}{\sigma_0^2} \geq 1+\varepsilon \ \middle|\ \sigma^2 = \sigma_0^2\right) =$$

$$= \mathbb{P}\left(\frac{(n-1)S^2}{\sigma_0^2} \geq (n-1)(1+\varepsilon) \ \middle|\ \sigma^2 = \sigma_0^2\right)$$

$$= \mathbb{P}\big(V > (n-1)(1+\varepsilon)\big) \quad \text{con } \mathbb{P}_V = \chi^2_{n-1}$$

$$\alpha = 1 - F_V\big((n-1)(1+\varepsilon)\big)$$

$$\Longleftrightarrow (n-1)(1+\varepsilon) = \chi^2_{n-1,1-\alpha}$$

Accetto H_0 SSE $\frac{(n-1)s^2}{\sigma_0^2} < \chi^2_{n-1,1-\alpha}$

— o —

$H_0: \ \sigma^2 \leq \sigma_0^2 \qquad H_A: \ \sigma^2 > \sigma_0^2$

Accetto H_0 SSE $\frac{s^2}{\sigma_0^2} \leq 1+\varepsilon$

$$\mathbb{P}\left(\frac{S^2}{\sigma_0^2} > 1+\varepsilon \;\middle|\; Var[X_i] = \sigma^2 \leq \sigma_0^2\right) \leq \star$$

$\sigma^2 \leq \sigma_0^2$ $\qquad \frac{(n-1)S^2}{\sigma^2} \geq \frac{(n-1)S^2}{\sigma_0^2}$

$$\star \leq \mathbb{P}\left(\frac{(n-1)S^2}{\sigma^2} > (n-1)(1+\varepsilon) \;\middle|\; Var[X_i] = \sigma^2 \leq \sigma_0^2\right)$$

$$= \mathbb{P}(V > (n-1)(1+\varepsilon)) = \alpha \quad \text{con} \quad \mathbb{P}_V = \chi^2_{n-1}$$

$$1 - F_V((n-1)(1+\varepsilon)) = \alpha \quad \text{cioè} \quad (n-1)(1+\varepsilon) = \chi^2_{n-1,1-\alpha}$$

Accetto H_0 sse $\frac{S^2(n-1)}{\sigma_0^2} < \chi^2_{n-1,1-\alpha}$

— o —

H_0: $\sigma^2 = \sigma_0^2$ $\qquad H_A$: $\sigma^2 < \sigma_0^2$

Accetto H_0 sse $\frac{S^2}{\sigma_0^2} \geq 1 - \varepsilon$

$$(n-1)(1-\varepsilon) = \chi^2_{n-1,\alpha}$$

Accetto H_0 sse $\frac{(n-1)S^2}{\sigma_0^2} \geq \chi^2_{n-1,\alpha}$

— o —

Analogo se Testo H_0: $\sigma^2 \geq \sigma_0^2$ $\qquad H_A$: $\sigma^2 < \sigma_0^2$