Tiro monetina: Testa successo 1

croce insuccesso 0

$$X_i = \begin{cases} 0 & \text{se all'i-esimo lancio ottengo croce} \\ 1 & \text{se all'i-esimo lancio ottengo Testa} \end{cases}$$

Le $X_1 \_\_\_ X_n$ sono i.i.d. $\mathbb{P}_{X_i} = B(p)$

$E[X_i] = p$ $\quad Var[X_i] = p(1-p)$ $\quad \forall i = 1 \_\_ n$

$x_1 \_\_\_ x_n$ $\quad x_i \in \{0,1\}$ $\quad X_i(\omega) = x_i$

$\bar{X}_n := \frac{1}{n} \sum_{i=1}^{n} X_i$ $\quad E[\bar{X}_n] = p$ $\quad Var[\bar{X}_n] = \frac{p(1-p)}{n}$

Chebychev $\quad \mathbb{P}(|\bar{X}_n - p| > t) \leq \frac{p(1-p)}{nt^2} \quad \forall t > 0$

$p \in [0,1]$ $\quad p(1-p) = -p^2 + p - \frac{1}{4} + \frac{1}{4} = \frac{1}{4} - \left(p - \frac{1}{2}\right)^2 \leq \frac{1}{4}$

$$\mathbb{P}(|\bar{X}_n - p| > t) \leq \frac{1}{4nt^2}$$

$$\mathbb{P}(|\bar{X}_n - p| \leq t) \geq 1 - \frac{1}{4nt^2}$$

$x_1 \_\_\_ x_n$ $\quad \bar{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_i$

$t = \frac{5}{100} = \frac{1}{20}$ $\quad \mathbb{P}\left(|\bar{X}_n - p| \leq \frac{1}{20}\right) \geq 1 - \frac{400}{4n} = 1 - \frac{100}{n}$

$1 - \frac{100}{n} \geq \frac{95}{100}$ $\quad \frac{100}{n} \leq \frac{5}{100} = \frac{1}{20}$,

$$n \geq 2000$$

DISTRIBUZIONE GAUSSIANA $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \quad \forall x \in \mathbb{R}$$

Se $\mathbb{P}_X = N(\mu, \sigma^2) \Rightarrow E[X] = \mu \quad Var[X] = \sigma^2$

$\mu = 0 \quad \sigma = 1 \qquad f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad \forall x \in \mathbb{R}$

$N(0,1)$ DISTRIBUZIONE GAUSSIANA STANDARD

$\Phi(t) + \Phi(-t) = 1 \quad \forall t \in \mathbb{R}$ dove Φ è la legge associata a $N(0,1)$

Se $\mathbb{P}_X = N(\mu, \sigma^2) \Rightarrow f(x) > 0 \ \forall x \in \mathbb{R} \Rightarrow F_X$ è una funzione strettamente crescente

$\lim_{t \to -\infty} F(t) = 0$, $\lim_{t \to +\infty} F(t) = 1 \qquad F(t) \in (0,1) \ \forall t \in \mathbb{R}$

$\forall \alpha \in (0,1) \quad \exists!\, x \in \mathbb{R}$ t.c. $F(x) = \alpha$

x si dice quantile relativo ad α

Nel caso particolare $\mathbb{P}_X = N(0,1)$ x si indica con la lettera z_α

$\Phi(z_\alpha) + \Phi(-z_\alpha) = 1 \qquad \Phi(-z_\alpha) = 1 - \alpha \qquad z_{1-\alpha} = -z_\alpha$

X_1 e X_2 v.a. gaussiane indipendenti

$\mathbb{P}_{X_1} = N(\mu_1, \sigma_1^2) \qquad \mathbb{P}_{X_2} = N(\mu_2, \sigma_2^2)$

$\Rightarrow X_1 + X_2$ è gaussiana e $\mathbb{P}_{X_1+X_2} = N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

X gaussiana $\quad \mathbb{P}_X = N(\mu, \sigma^2) \qquad \alpha, \beta \in \mathbb{R} \quad \alpha \neq 0$

$\Rightarrow \quad Y := \alpha X + \beta$ è ancora gaussiana

$$\mathbb{P}_Y = N(\alpha\mu + \beta, \alpha^2\sigma^2)$$

In particolare, segue che se $\mathbb{P}_X = N(\mu, \sigma^2)$

$\Rightarrow \quad Z := \dfrac{X-\mu}{\sigma}$ è gaussiana standard

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DISTRIBUZIONE $\Gamma(\alpha, \lambda) \qquad \alpha, \lambda > 0$

È la distribuzione A.C. associata alla densità

$$f(x) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$C := \dfrac{\lambda^\alpha}{\Gamma(\alpha)} \qquad \displaystyle\int_{\mathbb{R}} f(x)\,dx = 1$

$$\Gamma : \alpha \in (0, +\infty) \longmapsto \Gamma(\alpha) \in (0, +\infty)$$

$$\Gamma(\alpha) := \int_0^{+\infty} x^{\alpha-1} e^{-x}\,dx$$

$$\Gamma(1) = \int_0^{+\infty} e^{-x}\,dx = -e^{-x}\Big|_{x=0}^{x\to+\infty} = 1$$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^{+\infty} x^{-1/2} e^{-x}\,dx \qquad x = y^2 \qquad dx = 2y\,dy$$

$$= \int_0^{+\infty} y^{-1} e^{-y^2} 2y\,dy = \int_0^{+\infty} 2e^{-y^2}\,dy = \int_{\mathbb{R}} e^{-y^2}\,dy = \sqrt{\pi}$$

$$\Gamma(\alpha+1) = \int_0^{+\infty} x^{\alpha+1-1} e^{-x}\,dx = \int_0^{+\infty} \underbrace{x^\alpha}_{f} \underbrace{e^{-x}}_{g'}\,dx$$

$$= \underbrace{\cancel{-x^\alpha e^{-x}\Big|_{x=0}^{x\to+\infty}}}_{=0} + \int_0^{+\infty} \alpha x^{\alpha-1} e^{-x}\,dx = \alpha \int_0^{+\infty} x^{\alpha-1} e^{-x}\,dx$$

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \quad \forall \alpha > 0$$

$$\Gamma(1) = 1$$

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1$$

$$\Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1$$

$$\Gamma(4) = 3\,\Gamma(3) = 3 \cdot 2 \cdot 1$$

$$\vdots$$

$$\Gamma(n) = (n-1)!$$

$$\Gamma\left(\frac{2n+1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3 \cdot 1}{2 \cdot 2}\sqrt{\pi}$$

$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5 \cdot 3 \cdot 1}{2 \cdot 2 \cdot 2}\sqrt{\pi}$$

$$f(x) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$\alpha = 1 \Rightarrow \quad f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$\Gamma(1, \lambda) = \exp(\lambda)$$

Se $\mathbb{P}_X = \Gamma(\alpha, \lambda) \Rightarrow \mathbb{E}[X] = \dfrac{\alpha}{\lambda} \qquad \mathrm{Var}[X] = \dfrac{\alpha}{\lambda^2}$

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\,dx \qquad \int_{\mathbb{R}} |x| f(x)\,dx = \int_0^{+\infty} \underbrace{x f(x)}_{\geq 0}\,dx$$

$$= \int_0^{+\infty} x f(x)\,dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x \cdot x^{\alpha-1} e^{-\lambda x}\,dx \qquad t = \lambda x \quad x = \frac{t}{\lambda} \quad dx = \frac{1}{\lambda}dt$$

$$= \frac{\cancel{\lambda^\alpha}}{\Gamma(\alpha)} \int_0^{+\infty} \frac{t^\alpha}{\cancel{\lambda^\alpha}} \frac{1}{\lambda} e^{-t}\,dt = \frac{1}{\lambda}\frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^\alpha e^{-t}\,dt$$

$$= \frac{1}{\lambda}\frac{1}{\Gamma(\alpha)}\Gamma(\alpha+1) = \frac{1}{\lambda \cancel{\Gamma(\alpha)}} \alpha \cancel{\Gamma(\alpha)} = \frac{\alpha}{\lambda}$$

$$E[X^2] = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^2 \cdot x^{\alpha-1} e^{-\lambda x} \, dx \quad \ldots$$ per esercizio calcolare $Var[X]$

PROPRIETÀ Siano X e Y v.a. indipendenti

t.c. $\mathbb{P}_X = \Gamma(\alpha, \lambda)$ $\quad \mathbb{P}_Y = \Gamma(\beta, \lambda)$

Allora $\mathbb{P}_{X+Y} = \Gamma(\alpha+\beta, \lambda)$

DIM $\mathbb{P}_X = f(x)\,dx \qquad \mathbb{P}_Y = g(x)\,dx$

$\mathbb{P}_{X+Y} = h(x)\,dx$

$$h(x) = \int_{\mathbb{R}} f(y)\, g(x-y)\,dy = \int_{(0,+\infty)} f(y)\, g(x-y)\,dy$$

$x-y \geq 0 \quad y \leq x$ $$= \int_{(0,x)} f(y)\, g(x-y)\,dy$$

Se $x \leq 0$ $h(x) = 0$, $x > 0$ $h(x) = \int_0^x f(y)\, g(x-y)\,dy$

$$x>0 \quad h(x) = \int_0^x \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\cancel{\lambda y}} \cdot \frac{\lambda^\beta}{\Gamma(\beta)} (x-y)^{\beta-1} e^{-\lambda(x-\cancel{y})}\, dy$$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} e^{-\lambda x} \int_0^x y^{\alpha-1} (x-y)^{\beta-1}\, dy$$

$y = xt \quad dy = x\,dt$

$y=0 \quad t=0$

$y=x \quad t=1$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} e^{-\lambda x} \cdot \cancel{x} \int_0^1 x^{\alpha-1} t^{\alpha-1} x^{\beta-\cancel{1}} (1-t)^{\beta-1}\, dt$$

$$= \underbrace{\frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt}_{C} \; x^{(\alpha+\beta)-1} e^{-\lambda x}$$

$$C = \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha+\beta)}$$

DEF Sia $n \in \mathbb{N}$, la distribuzione $\Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$ si chiama DISTRIBUZIONE DI PEARSON a n-gradi di libertà o anche DISTRIBUZIONE χ^2 a n-gradi di libertà e si indica χ^2_n

$$\chi^2_n := \Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$$

N.B. Se $\mathbb{P}_X = \chi^2_n$ $\quad \mathbb{E}[X] = n \quad \mathrm{Var}[X] = 2n$

Se X e Y sono v.a. indipendenti con

$$\mathbb{P}_X = \chi^2_n, \ \mathbb{P}_Y = \chi^2_k \implies \mathbb{P}_{X+Y} = \chi^2_{n+k} = \Gamma\left(\frac{n+k}{2}, \frac{1}{2}\right)$$

PROPRIETÀ Sia X una v.a. gaussiana standard, allora X^2 è una v.a. con distribuzione di Pearson a 1 grado di libertà

$$\left(\mathbb{P}_X = N(0,1) \implies \mathbb{P}_{X^2} = \chi^2_1 \right)$$

DIM $\mathbb{P}_X = f(x)\,dx \implies \mathbb{P}_{X^2} = g(x)\,dx$

$$g(x) = \begin{cases} \frac{1}{2\sqrt{x}} \left(f(\sqrt{x}) + f(-\sqrt{x}) \right) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \implies f(\sqrt{x}) = f(-\sqrt{x}) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x\right)$$

$$x > 0 \quad g(x) = \frac{1}{\cancel{2}\sqrt{x}} \frac{1}{\sqrt{2\pi}} \cancel{2} \exp\left(-\frac{x}{2}\right) = \frac{1}{\sqrt{2\pi}} x^{-1/2} \exp\left(-\frac{1}{2}x\right)$$

densità associata a $\Gamma\left(\frac{1}{2}, \frac{1}{2}\right) = \chi^2_1$

Conseguenza Sia $X_1 \_\_ X_n$ campione gaussiano con valore atteso μ e varianza σ^2, allora

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2 \text{ ha distribuzione } \chi^2_n$$

TEOREMA Sia $X_1 \_\_ X_n$ campione statistico gaussiano $\mathbb{P}_{X_i} = N(\mu, \sigma^2)$

Allora $\overline{X}$ e S^2 sono v.a. indipendenti

Sia $Z_i := \frac{X_i - \mu}{\sigma}$, $\forall i = 1 \_\_ n$ la standardizzazione del campione

Allora $\overline{Z} = \frac{\overline{X} - \mu}{\sigma}$, le v.a. $\overline{Z}$ e $\sum_{i=1}^{n} (Z_i - \overline{Z})^2$ sono indipendenti

e $\sum_{i=1}^{n} (Z_i - \overline{Z})^2$ ha distribuzione χ^2_{n-1}

DIM $n=2$ $\quad \mathbb{P}_{X_1+X_2} = N(2\mu, 2\sigma^2)$

$$\mathbb{P}_{\overline{X}} = N\left(\mu, \frac{\sigma^2}{2}\right)$$

$$S^2 = (X_1 - \overline{X})^2 + (X_2 - \overline{X})^2 = \left(X_1 - \frac{X_1 + X_2}{2} \right)^2 + \left(X_2 - \frac{X_1 + X_2}{2} \right)^2$$

$$\Rightarrow S^2 = \frac{1}{2} (X_1 - X_2)^2$$

sono indipendenti sse

$$\overline{X} = \frac{1}{2} (X_1 + X_2)$$

$U := X_1 + X_2$ e $V := X_1 - X_2$

sono indipendenti

$\mathbb{P}_U = N(2\mu, 2\sigma^2) \qquad \mathbb{P}_V = N(0, 2\sigma^2)$

$\mathbb{P}_{U,V} \qquad (U, V) = \varphi \circ (X_1, X_2)$

$\varphi : (x,y) \in \mathbb{R}^2 \longmapsto (x+y,\ x-y) \in \mathbb{R}^2$

$\psi : \mathbb{R}^2 \to (0,+\infty)$ di Borel

$$\int_{\mathbb{R}^2} \psi(u,v)\, \mathbb{P}_{U,V}(du\,dv) = \int_{\mathbb{R}^2} \psi \circ \varphi(x,y)\, \mathbb{P}_{X_1,X_2}(dx\,dy)$$

$$= \int_{\mathbb{R}^2} \psi(x+y, x-y) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(y-\mu)^2}{2\sigma^2}\right) dx\,dy$$

$u = x+y \qquad x = \frac{u+v}{2}$

$v = x-y \qquad y = \frac{u-v}{2} \qquad J = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \qquad |\det J| = \frac{1}{2}$

$$= \int_{\mathbb{R}^2} \psi(u,v) \frac{1}{2\pi\sigma^2\, 2} \exp\left(-\frac{(u-2\mu)^2}{2(\sigma\sqrt{2})^2}\right) \exp\left(\frac{-v^2}{2(\sigma\sqrt{2})^2}\right) du\,dv$$

$$= \int_{\mathbb{R}^2} \psi(u,v) \big(\text{densità di } U\big)(u) \big(\text{densità di } V\big)(v)\, du\,dv$$

$n \geq 3 \qquad \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad \bar{X}_{n-1} = \frac{1}{n-1} \sum_{i=1}^{n-1} X_i$

$$\bar{X}_n - \bar{X}_{n-1} = \frac{1}{n}\left(X_n - \bar{X}_{n-1}\right)$$

$$S_n^2 = \frac{1}{n-1}\left\{(n-2) S_{n-1}^2 + \left(1 - \frac{1}{n}\right)\left(X_n - \bar{X}_{n-1}\right)^2\right\}$$

Hp di induzione: S_{n-1}^2, $\bar{X}_{n-1}$ sono indipendenti