BINOMIALE NEGATIVA $G(p)$, $G'(p)$

Note Title 09/11/2017

Moneta su cui a ogni lancio esce Testa con probabilità $p \in (0,1)$ ed esce croce con probabilità $1-p \in (0,1)$.
Voglio scrivere come v.a. su un opportuno spazio probabilizzato $(\Omega, \mathcal{E}, \mathbb{P})$

IL NUMERO DI CROCI (INSUCCESSI) CHE OTTENGO PRIMA DI OTTENERE L'n-ESIMA TESTA (SUCCESSO)

$\Omega = \{0, 1\}$ $\qquad$ $0 \to$ insuccesso $\to$ croce

$\qquad$ $1 \to$ successo $\to$ Testa

$\Omega = \{0,1\}^\infty$ $\quad \omega \in \Omega$ $\quad \omega = \{\omega_i\}_{i=1}^\infty$ $\quad \omega_i \in \{0,1\}$

$\bar{z} = (\bar{z}_1, \dots \bar{z}_n) \in \{0,1\}^n$

$E = \{\omega \in \Omega : \omega_1 = \bar{z}_1 \dots \omega_n = \bar{z}_n\}$

CILINDRO FINITO su $\bar{z}$

($\star$) $\mathbb{P}(E) = p^k (1-p)^{n-k}$ se in $\bar{z}$ ci sono k successi (1)
$n-k$ insuccessi (0)

Si può dimostrare che la famiglia data dall'insieme vuoto e da tutti i possibili cilindri finiti è un anello di Ω, che la $\mathbb{P}$ definita in ($\star$) è una funzione σ-additiva su questo anello $\Rightarrow$
Prendo $\mathcal{E} = \sigma$-algebra generata dall'anello dai cilindri finiti e $\mathbb{P}$ = l'unica estensione di $\mathbb{P}$ definita su tale anello a tutto $\mathcal{E}$, che per il Teo. di Carathéodory esiste ed è unica.

Per $\omega \in \Omega$ pongo $X(\omega) = \#$ di insuccessi che ottengo

prima di ottenere l'n-esimo successo se in ω ci sono almeno n successi,
altrimenti pongo $X(\omega) = +\infty$

$\{X = +\infty\}$

$\bar{\omega} = (0, 0, 0, \_\_\_\_\_\_)$

$\mathbb{P}(\{\bar{\omega}\})$

$E_1 = \{\omega \in \Omega : \omega_1 = 0\}$ $\quad E_2 = \{\omega \in \Omega : \omega_1 = \omega_2 = 0\}$

$\dots \quad E_n = \{\omega \in \Omega : \omega_1 = \omega_2 \dots = \omega_n = 0\}$

$\{\bar{\omega}\} = \bigcap_{i=1}^{\infty} E_i \qquad E_i \subseteq E_{i-1}$

$$\mathbb{P}(\{\bar{\omega}\}) = \mathbb{P}\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{i\to\infty} \mathbb{P}(E_i) = \lim_{i\to\infty} (1-p)^i = 0$$

$\mathbb{P}(X = +\infty) = 0$

$\mathbb{P}(X = +\infty) = 0 \qquad X(\Omega) = \{0, 1, 2, \dots n \dots\} \cup \{+\infty\}$

$X(\Omega) = \mathbb{N}_0 \cup \{+\infty\}$

$k \in \mathbb{N}_0 \qquad \mathbb{P}(X = k)$

$\bar{\omega} \in \{X = k\} \qquad \bar{\omega} = (\underbrace{\bar{\omega}_1 \dots \dots \bar{\omega}_{n+k-1}}, \underset{n+k}{1}, \dots)$

n-1 successi
k insuccessi

$$\forall \bar{\omega} \in \{X=k\} \quad \mathbb{P}(\text{cilindro su } \bar{\omega}) = p^n (1-p)^k$$

$$\mathbb{P}(X = k) = \binom{n+k-1}{k} p^n (1-p)^k$$

$$\sum_{k=0}^{\infty} \mathbb{P}(X=k) = 1$$

La distribuzione concentrata sugli interi nonnegativi definita da questa v.a. X si dice

DISTRIBUZIONE BINOMIALE NEGATIVA DI PARAMETRI n E p e si indica $B(-n,p)$

$$B(-n,p)(\{k\}) = \binom{n+k-1}{k} p^n (1-p)^k \quad k=0,1,2,\dots$$

Si dimostra che se $\mathbb{P}_X = B(-n,p)$, allora

$$E[X] = n\frac{1-p}{p} \qquad Var[X] = \frac{1-p}{p^2}$$

Nel caso particolare $n=1$

$$B(-1,p)(\{k\}) = \binom{1+k-1}{k} p^1 (1-p)^k = p(1-p)^k \quad k=0,1,2,\dots$$

la distribuzione si dice DISTRIBUZIONE GEOMETRICA MODIFICATA e si indica $G'(p)$ $\quad p\in(0,1)$

Sia X v.a. t.c. $\mathbb{P}_X = G'(p)$

$$F_X(t) = \mathbb{P}(X\le t) = \sum_{k\le t} \mathbb{P}(X=k)$$

$$F_X(t) = F_X(\lfloor t\rfloor)$$

$$F_X(\lfloor t\rfloor) = \sum_{k\le \lfloor t\rfloor} \mathbb{P}(X=k)$$

$j\in\mathbb{Z}$ $\quad F_X(j) = 0$ se $j<0$

$j\ge 0$ $\quad F_X(j) = \sum_{k=0}^{j} p(1-p)^k = p\sum_{k=0}^{j} x^k \Big|_{x=1-p}$

$$= p\,\frac{1-x^{j+1}}{1-x}\Big|_{x=1-p} = 1-(1-p)^{j+1}$$

$\mathbb{P}(X\le i+j \mid X\ge i)$ $\quad i,j\in\mathbb{N}_0$

$$= \frac{\mathbb{P}(X\le i+j,\, X\ge i)}{\mathbb{P}(X\ge i)} =$$

$\{X\le i+j,\, X\ge i\} = \{X\le i+j\}\setminus\{X<i\}$

$$= \frac{\mathbb{P}(X \le i+j) - \mathbb{P}(X < i)}{1 - \mathbb{P}(X < i)} = \frac{\mathbb{P}(X \le i+j) - \mathbb{P}(X \le i-1)}{1 - \mathbb{P}(X \le i-1)} =$$

$$= \frac{(\cancel{1} - (1-p)^{i+j+1}) - (\cancel{1} - (1-p)^{i})}{\cancel{1} - (\cancel{1} - (1-p)^{i})} = \frac{(1-p)^{i} - (1-p)^{i+j+1}}{(1-p)^{i}}$$

$$= 1 - (1-p)^{j+1} = \mathbb{P}(X \le j)$$

$$\mathbb{P}(X \le i+j \mid X \ge i) = \mathbb{P}(X \le j) \quad \forall i, j \in \mathbb{N}_0$$

PROPRIETÀ DI MANCANZA DI MEMORIA

Le v.a. con distribuzione geometrica modificata sono le sole v.a. discrete che soddisfano la proprietà di mancanza di memoria su $\mathbb{N}_0$.

DIM Sia X v.a. distribuita su $\mathbb{N}_0$ che soddisfa la proprietà di mancanza di memoria

$$\mathbb{P}(X \le i+j \mid X \ge i) = \mathbb{P}(X \le j) \quad \forall i, j \in \mathbb{N}_0$$

Scelgo $j = 0$

$$\frac{\mathbb{P}(X \le i, X \ge i)}{\mathbb{P}(X \ge i)} = \mathbb{P}(X \le 0) = \mathbb{P}(X = 0)$$

Pongo $p_0 = \mathbb{P}(X = 0) < 1$

$$\mathbb{P}(X = i) = \mathbb{P}(X = 0)\, \mathbb{P}(X \ge i) = p_0 \, \mathbb{P}(X \ge i)$$

$\Rightarrow p_0 \in (0, 1)$

$$\mathbb{P}(X = i) = p_0 \, \mathbb{P}(X \ge i)$$

$$\mathbb{P}(X = i+1) = p_0 \, \mathbb{P}(X \ge i+1)$$

$$\mathbb{P}(X = i) - \mathbb{P}(X = i+1) = p_0 \left(\mathbb{P}(X \ge i) - \mathbb{P}(X \ge i+1) \right)$$

$$\mathbb{P}(X=i)-\mathbb{P}(X=i+1)=p_0\,\mathbb{P}(\{X\geq i\}\setminus\{X\geq i+1\})$$
$$=p_0\,\mathbb{P}(X=i)$$

$$\mathbb{P}(X=i+1)=(1-p_0)\,\mathbb{P}(X=i)$$

$i=0 \qquad \mathbb{P}(X=1)=(1-p_0)p_0$

$i=1 \qquad \mathbb{P}(X=2)=(1-p_0)\,\mathbb{P}(X=1)=(1-p_0)^2 p_0$

$i=2 \qquad \mathbb{P}(X=3)=(1-p_0)\,\mathbb{P}(X=2)=(1-p_0)^3 p_0$

Per induzione dimostro che $\mathbb{P}(X=k)=(1-p_0)^k p_0$

cioè $\quad \mathbb{P}_X=G'(p_0) \qquad p_0\in(0,1)$

— o —

Lancia la solita moneta

$Y(\omega)=\#$ di lanci a cui ottengo il 1° successo

$Y(\omega)=k \iff X(\omega)=k-1 \implies Y=X+1$

$\mathbb{P}(Y=+\infty)=\mathbb{P}(X=+\infty)=0$

$Y(\Omega)=\mathbb{N}^{\cup\{+\infty\}}=\{1,2,3,\_\_\_\_\}\cup\{+\infty\}$

$k\in\mathbb{N} \qquad \mathbb{P}(Y=k)=\mathbb{P}(X=k-1)=p(1-p)^{k-1}$

Chiamo DISTRIBUZIONE GEOMETRICA $G(p)$ la distribuzione concentrata sugli interi positivi t.c.

$$G(p)(\{k\})=p(1-p)^{k-1} \qquad \forall k\geq 1$$

Se Z è una v.a. t.c. $\mathbb{P}_Z=G(p)$, allora

$$\mathbb{E}[Z]=\mathbb{E}[Y]=\mathbb{E}[X+1]=1+\mathbb{E}[X]=1+\frac{1-p}{p}$$
$$=\frac{p+1-p}{p}=\frac{1}{p}$$

$$\mathrm{Var}[Z]=\mathrm{Var}[Y]=\mathrm{Var}[X+1]=\mathrm{Var}[X]=\frac{1-p}{p^2}$$

Esercizio 1 foglio 4

$$\mathbb{P}(X=k) = p(1-p)^k \qquad k=0,1,2,\ldots$$

$$Y(\Omega) = \{0,1\} \Rightarrow \mathbb{P}_Y = Ber(x) \qquad x = ?$$

$$\mathbb{P}(Y=1 \mid X=k) = q^k \qquad k=0,1,2,\ldots$$

$$\{Y=1\} = \bigcup_{k=0}^{\infty} \{Y=1, X=k\}$$

$$\mathbb{P}(Y=1) = \sum_{k=0}^{\infty} \mathbb{P}(Y=1, X=k) = \sum_{k=0}^{\infty} \mathbb{P}(Y=1 \mid X=k)\,\mathbb{P}(X=k)$$

$$= \sum_{k=0}^{\infty} q^k p (1-p)^k = p \sum_{k=0}^{\infty} \left(q(1-p)\right)^k$$

$$= p \sum_{k=0}^{\infty} y^k \Big|_{y=q(1-p)} \qquad \begin{array}{l} q \in [0,1] \\ p \in (0,1) \\ \Rightarrow q(1-p) = (0,1) \end{array}$$

$$\sum_{k=0}^{n} y^k = \frac{1-y^{n+1}}{1-y} \xrightarrow[n\to\infty]{} \frac{1}{1-y}$$

$$\mathbb{P}(Y=1) = p \frac{1}{1-y}\Big|_{y=q(1-p)} = \frac{p}{1-q(1-p)}$$

$$\mathbb{P}(Y=0) = 1 - \mathbb{P}(Y=1) = 1 - \frac{p}{1-q(1-p)} = \frac{1-q(1-p)-p}{1-q(1-p)}$$

$$= \frac{(1-p)(1-q)}{1-q(1-p)}$$

$$Z := XY \qquad \begin{array}{l} X(\Omega) = \{0,1,\ldots\} \\ Y(\Omega) = \{0,1\} \end{array}$$

$$Z(\Omega) = \{0,1,2,\ldots\}$$

$$\mathbb{P}(Z=k) = \mathbb{P}(XY=k)$$

$$k \neq 0 \quad \{XY=k\} = \{Y=1, X=k\}$$

$$\mathbb{P}(Z=k) = \mathbb{P}(Y=1, X=k) = \mathbb{P}(Y=1 \mid X=k)\,\mathbb{P}(X=k)$$
$$= q^k p(1-p)^k = p\,(q(1-p))^k$$

$k=0 \quad \{XY=0\} = \{X=0\} \cup \{Y=0\}$
$$= \{X=0, Y=1\} \cup \{Y=0\}$$

$$\mathbb{P}(Z=0) = \mathbb{P}(Y=1, X=0) + \mathbb{P}(Y=0) =$$
$$= \mathbb{P}(Y=1 \mid X=0)\,\mathbb{P}(X=0) + \mathbb{P}(Y=0)$$
$$= q^0\, p(1-p)^0 + \frac{(1-p)(1-q)}{1-q(1-p)} = p + \frac{(1-p)(1-q)}{1-q(1-p)}$$

$$\mathbb{E}[Z] = \sum_{k=0}^{\infty} k\,\mathbb{P}(Z=k) = \sum_{k=1}^{\infty} k\,\mathbb{P}(Z=k) =$$
$$= \sum_{k=1}^{\infty} k\, p\,(q(1-p))^k = p \sum_{k=1}^{\infty} k z^k \Big|_{z=q(1-p)}$$

$$\sum_{k=1}^{\infty} k z^{k-1} z = z \sum_{k=1}^{\infty} \frac{d}{dz} z^k \qquad z \in (-1,1)$$
$$= z \frac{d}{dz}\left(\sum_{k=1}^{\infty} z^k\right) = z \frac{d}{dz}\left(\frac{1}{1-z} - 1\right) = z \frac{d}{dz}(1-z)^{-1} = z(1-z)^{-2}$$

$$\mathbb{E}[Z] = p z (1-z)^{-2} \Big|_{z=q(1-p)} = \frac{pq(1-p)}{(1-q(1-p))^2}$$

Esercizio 2 foglio 4

$\mathbb{P}_X = \exp(\lambda) \qquad \mathbb{P}_Y = U([0,a]) \qquad a>0 \qquad \lambda>0$

X e Y sono indipendenti

$Z := X+Y \quad T := X-Y$

$$\mathbb{E}[Z] = \mathbb{E}[X] + \mathbb{E}[Y] = \frac{1}{\lambda} + \frac{a}{2}$$

$$\mathbb{E}[T] = \mathbb{E}[X] - \mathbb{E}[Y] = \frac{1}{\lambda} - \frac{a}{2}$$

$\mathbb{P}_X = f(x)\,dx \quad \mathbb{P}_Y = g(y)\,dy$

$$\mathbb{P}_Z = h(z)\,dz \qquad h(x) = \int_{\mathbb{R}} f(y)\, g(x-y)\, dy$$

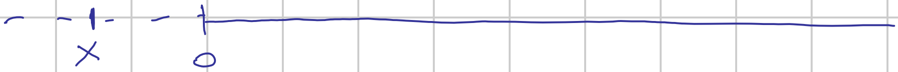
$$f(x) = \begin{cases} 0 & x \le 0 \\ \lambda e^{-\lambda x} & x > 0 \end{cases} \qquad g(x) = \begin{cases} \frac{1}{2} & x \in (0,2) \\ 0 & \text{altrimenti} \end{cases}$$

$$h(x) = \int_0^{+\infty} \lambda e^{-\lambda y} \frac{1}{2} \mathbb{1}_{(0,2)}(x-y)\, dy$$

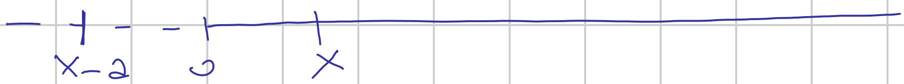
$$\mathbb{1}_{(0,2)}(x-y) \neq 0 \iff 0 < x-y < 2$$

$$-2 < y-x < 0$$

$$x-2 < y < x$$



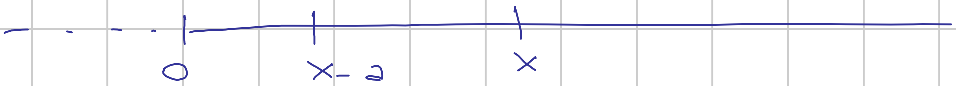
$$h(x) = 0 \qquad x < 0$$



$$\begin{cases} x-2 < 0 \\ x > 0 \end{cases} \qquad \begin{matrix} x < 2 \\ x > 0 \end{matrix}$$

$$h(x) = \int_0^x \frac{\lambda}{2} e^{-\lambda y}\, dy = -\frac{1}{2} e^{-\lambda y} \Big|_{y=0}^{y=x} \qquad x \in (0,2)$$

$$x \in (0,2) \quad h(x) = \frac{1}{2} - \frac{1}{2} e^{-\lambda x} = \frac{1}{2}\left(1 - e^{-\lambda x}\right)$$



$$x - 2 > 0 \qquad x > 2$$

$$h(x) = \int_{x-2}^{x} \frac{\lambda}{2} e^{-\lambda y}\, dy = -\frac{1}{2} e^{-\lambda y} \Big|_{y=x-2}^{y=x} =$$

$$= \frac{1}{2} e^{-\lambda(x-2)} - \frac{1}{2} e^{-\lambda x} = \frac{e^{-\lambda x}}{2}\left(e^{\lambda 2} - 1\right)$$

$$h(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}\left(1 - e^{-\lambda x}\right) & x \in (0,2) \\ \frac{e^{-\lambda x}}{2}\left(e^{\lambda 2} - 1\right) & x > 2 \end{cases}$$

$T = X - Y = X + (-Y)$ $\qquad \mathbb{P}_{-Y} = U([-2,0])$

$-Y = (-1)Y + 0$ $\qquad \mathbb{P}_{-Y} = k(x)dx$

$$k(x) = \frac{1}{|a|} g\left(\frac{x-b}{a}\right) = g(-x) = \begin{cases} \frac{1}{2} & 0 < -x < 2 \\ 0 & \text{altrimenti} \end{cases}$$

$$= \begin{cases} \frac{1}{2} & -2 < x < 0 \\ 0 & \text{altrimenti} \end{cases}$$

$\mathbb{P}_T = \ell(x)dx$ $\qquad \ell(x) = \int_{\mathbb{R}} f(y) k(x-y) dy$

Esercizio 3 foglio 4

$\mathbb{P}_X = \exp(\lambda)$ $\qquad \mathbb{P}_Y = B(p)$, X e Y indipendenti.

$\mathbb{P}_X = f(x)dx$ $\qquad \mathbb{P}_{X^2} = g(x)dx$ $\qquad g(x) = \begin{cases} \frac{1}{2\sqrt{x}}\left(f(\sqrt{x}) + f(-\sqrt{x})\right) & x > 0 \\ 0 & x \leq 0 \end{cases}$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$g(x)$, per $x > 0$ è $g(x) = \frac{1}{2\sqrt{x}} f(\sqrt{x}) = \frac{1}{2\sqrt{x}} \lambda e^{-\lambda\sqrt{x}}$

$\mathbb{E}[X^2] = \frac{2}{\lambda^2}$, $\qquad \int_{\mathbb{R}} x^2 f(x) dx$ $\qquad \int_{\mathbb{R}} x g(x) dx$

$\mathbb{E}[Z] = \mathbb{E}[X] + \mathbb{E}[Y] = \frac{1}{\lambda} + p$

$$F_Z(t) = \mathbb{P}(X+Y \leq t) = \mathbb{P}(X+Y \leq t, Y=0) + \mathbb{P}(X+Y \leq t, Y=1)$$

$$= \mathbb{P}(X \leq t, Y=0) + \mathbb{P}(X \leq t-1, Y=1)$$

$$= \mathbb{P}(X \leq t)\mathbb{P}(Y=0) + \mathbb{P}(X \leq t-1)\mathbb{P}(Y=1)$$

$$= (1-p)\mathbb{P}(X \leq t) + p\,\mathbb{P}(X \leq t-1)$$

$$\mathbb{P}(X \leq t) = F_X(t) = \begin{cases} 0 & t \leq 0 \\ 1 - e^{-\lambda t} & t > 0 \end{cases}$$

$$F_Z(t) = \begin{cases} 0 & t < 0 \\ (1-p)\left(1 - e^{-\lambda t}\right) & t \in [0,1) \\ (1-p)\left(1-e^{-\lambda t}\right) + p\left(1 - e^{-\lambda(t-1)}\right) & t \geq 1 \end{cases}$$

Esercizio 4 foglio 4

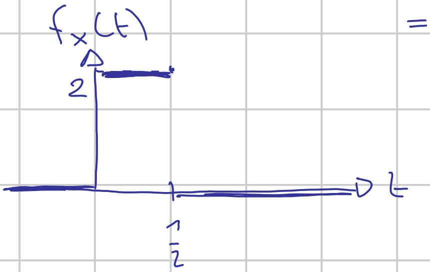
$\mathbb{P}_X = U\left(\left(0, \frac{1}{2}\right)\right)$ $\quad \mathbb{P}_Y = B(p)$ $\quad X$ e Y indipendenti

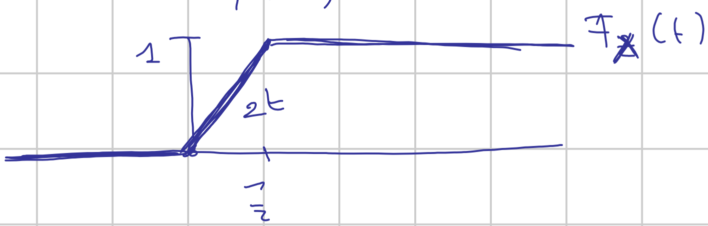
$Z := \max\{X, Y\}$ $\quad$ Legge e suo grafico

$t \in \mathbb{R}$ $\quad F_Z(t) = \mathbb{P}(Z \leq t) = \mathbb{P}\left(\max\{X,Y\} \leq t\right)$

$= \mathbb{P}(X \leq t, Y \leq t) = \mathbb{P}(X \leq t)\,\mathbb{P}(Y \leq t)$

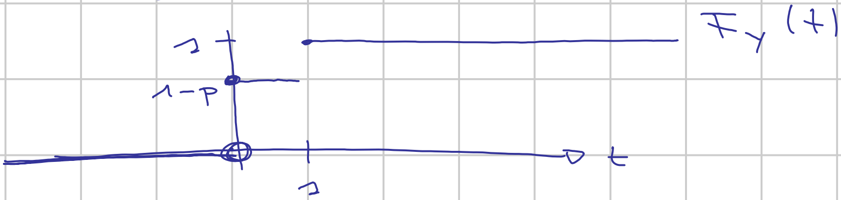
$= F_X(t) F_Y(t)$



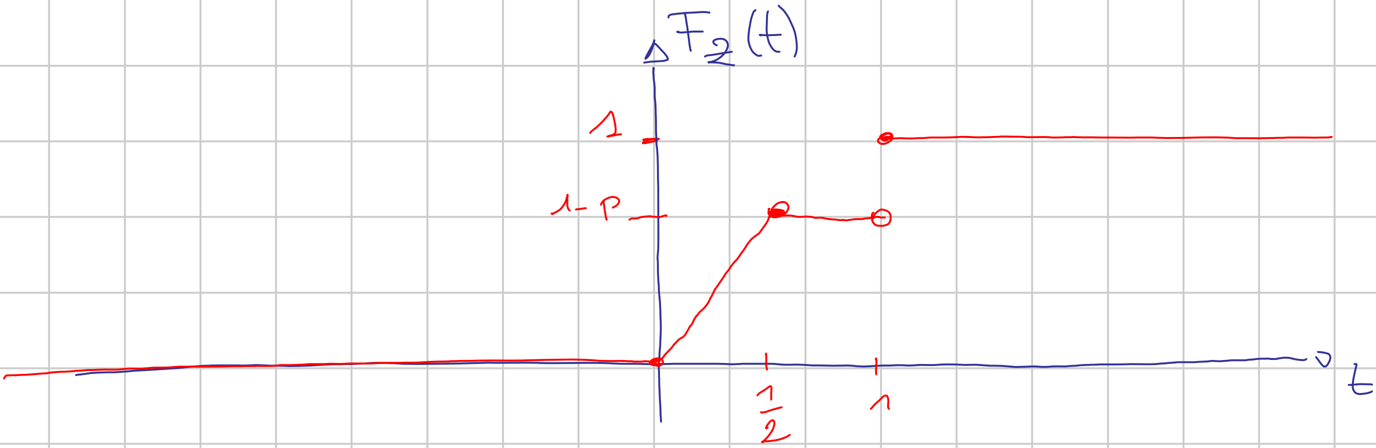


$\mathbb{P}(Y = 1) = p$

$\mathbb{P}(Y = 0) = 1 - p$



$$F_Z(t) = \begin{cases} 0 & t < 0 \\ 2(1-p)t & 0 \leq t < \frac{1}{2} \\ (1-p) & \frac{1}{2} \leq t < 1 \\ 1 & t \geq 1 \end{cases}$$



$Z = \max(X, Y)$ $\quad Z = \varphi \circ (X, Y)$

$\varphi : (x, y) \in \mathbb{R}^2 \longmapsto \max(x, y) \in \mathbb{R}$

ψ funzione di Borel nonnegativa

$$\int_{\mathbb{R}} \psi(t)\, \mathbb{P}_Z(dt) = \int_{\mathbb{R}^2} \psi(\varphi(x,y))\, \mathbb{P}_{X,Y}(dx\,dy) =$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \psi(\max(x,y))\, \mathbb{P}_Y(dy) \right) \mathbb{P}_X(dx)$$

$$= \int_{\mathbb{R}} \left(\psi(\max(x,0)) \overbrace{\mathbb{P}(Y=0)}^{1-p} + \psi(\max(x,1)) \overbrace{\mathbb{P}(Y=1)}^{p} \right) \mathbb{P}_X(dx)$$

$$\mathbb{P}_X = f(x)\,dx \qquad f(x) = \begin{cases} 2 & x \in (0, \frac{1}{2}) \\ 0 & \text{altrimenti} \end{cases}$$

$$= 2\int_0^{1/2} \left((1-p)\psi(\max(x,0)) + p\,\psi(\max(x,1)) \right) dx$$

$$= 2\int_0^{1/2} \left((1-p)\psi(x) + p\,\psi(1) \right) dx =$$

$$= \int_0^{1/2} \left(2(1-p)\psi(x) + 2p\,\psi(1) \right) dx$$

$$= \int_{\mathbb{R}} 2(1-p)\mathbb{1}_{(0,\frac{1}{2})}(x)\, \psi(x)\,dx + \cancel{\tfrac{1}{2}}\cancel{2}\,p\,\psi(1)$$

$$\mathbb{P}_Z = (1-p)\,U\left((0,\tfrac{1}{2})\right) + p\,\delta_1$$

Esercizio 5 foglio 4

$X(\Omega) = Y(\Omega) = \mathbb{N}_0$

$$\mathbb{P}(Y=i \mid X+Y=k) = \begin{cases} \binom{k}{i} p^i (1-p)^{k-i} & i=0, \_\, k \\ 0 & i > k \end{cases}$$

$$\mathbb{P}_{X+Y} = \mathrm{Poiss}(\lambda)$$

$\mathbb{P}(Y=i) \qquad i \in \mathbb{N}_0 \qquad \{Y=i\} = \bigcup_{k=0}^{\infty} \{Y=i, X+Y=k\}$

$$\mathbb{P}(Y=i) = \sum_{k=0}^{\infty} \mathbb{P}(Y=i, X+Y=k) =$$

$$= \sum_{k=0}^{\infty} \mathbb{P}(Y=i \mid X+Y=k)\, \underbrace{\mathbb{P}(X+Y=k)}_{e^{-\lambda}\frac{\lambda^k}{k!}} \quad \forall k \geq 0$$

$$= \sum_{k=i}^{+\infty} \binom{k}{i} p^i (1-p)^{k-i}\, e^{-\lambda} \frac{\lambda^k}{k!} =$$

$$= \sum_{k=i}^{+\infty} \frac{\cancel{k!}}{i!(k-i)!} p^i (1-p)^{k-i}\, e^{-\lambda} \frac{\lambda^{k-i}}{\cancel{k!}} \cdot \lambda^i$$

$$= \frac{p^i}{i!} e^{-\lambda} \lambda^i \sum_{k=i}^{+\infty} \frac{1}{(k-i)!} (\lambda(1-p))^{k-i} \qquad j = k-i$$

$$= \frac{(p\lambda)^i e^{-\lambda}}{i!} \sum_{j=0}^{\infty} \frac{(\lambda(1-p))^j}{j!} = \frac{(p\lambda)^i \cancel{e^{-\lambda}}}{i!} e^{\cancel{\lambda}-\lambda p}$$

$$\Rightarrow \mathbb{P}_Y = \mathrm{Poiss}(\lambda p)$$

$$\mathbb{P}(X=j) \qquad \{X=j\} = \bigcup_{k=0}^{\infty} \{X=j, X+Y=k\}$$

$$= \bigcup_{k=0}^{\infty} \{Y=k-j, X+Y=k\}$$

$$\mathbb{P}(X=j) = \sum_{k=0}^{\infty} \mathbb{P}(Y=k-j \mid X+Y=k)\, \mathbb{P}(X+Y=k)$$

$$0 \leq k-j \leq k \qquad k-j \geq 0$$

Pouvio $\qquad k \geq j$

$$\sum_{k=j}^{+\infty} \binom{k}{k-j} p^{k-j} (1-p)^{k-(k-j)}\, e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \sum_{k=j}^{\infty} \frac{\cancel{k!}}{j!(k-j)!} p^{k-j} (1-p)^j\, e^{-\lambda} \frac{\lambda^{k-j}}{\cancel{k!}} \cdot \lambda^j$$

$$= \frac{1}{j!} e^{-\lambda} \lambda^j (1-p)^j \sum_{k=j}^{\infty} \frac{1}{(k-j)!} (p\lambda)^{k-j} \qquad i = k-j$$

$$= \frac{1}{j!} e^{-\lambda} (\lambda(1-p))^j \sum_{i=0}^{\infty} \frac{(p\lambda)^i}{i!} = \frac{1}{j!} e^{-\lambda} (\lambda(1-p))^j\, e^{\lambda p}$$

$$= \frac{1}{j!} e^{-\lambda(1-p)} \lambda(1-p)^j \Rightarrow \mathbb{P}_X = \mathrm{Poiss}(\lambda(1-p))$$