Legge dei Grandi Numeri - Limite Centrale

Note Title 06/11/2017

$$C = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \qquad \sigma_1^2, \sigma_2^2 > 0 \quad \text{e } C \text{ definita positiva}$$

cioè $\det C > 0$, $\operatorname{tr} C > 0$

$$f(x,y) = \frac{1}{2\pi\sqrt{\det C}} \exp\left(-\frac{1}{2}(x,y)\,C^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

Se (X,Y) è una v.a. in $\mathbb{R}^2$ t.c. $\mathbb{P}_{X,Y} = f(x,y)\,dx\,dy$,

allora $\mathbb{P}_X = N(0,\sigma_1^2)$

$\mathbb{P}_Y = N(0,\sigma_2^2)$

$$f(x,y) = \frac{1}{2\pi\sqrt{\det C}} \exp\left(\frac{-x^2}{2\sigma_1^2}\right) \exp\left(\frac{-\left(\sigma_1 y - \frac{\sigma_{12}}{\sigma_1}x\right)^2}{2\det C}\right)$$

$$\operatorname{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[XY] =$$

$$= \int_{\mathbb{R}^2} xy\, f(x,y)\,dx\,dy =$$

$$= \frac{1}{2\pi\sqrt{\det C}} \int_{\mathbb{R}^2} xy \exp\left(\frac{-x^2}{2\sigma_1^2}\right) \exp\left(\frac{-\left(\sigma_1 y - \frac{\sigma_{12}}{\sigma_1}x\right)^2}{2\det C}\right) dx\,dy$$

$$= \frac{1}{2\pi\sqrt{\det C}} \int_{\mathbb{R}} x \exp\left(\frac{-x^2}{2\sigma_1^2}\right) \left(\int_{\mathbb{R}} y \exp\left(\frac{-\left(\sigma_1 y - \frac{\sigma_{12}}{\sigma_1}x\right)^2}{2\det C}\right) dy\right) dx$$

$$t = \frac{\sigma_1 y - \frac{\sigma_{12}}{\sigma_1}x}{\sqrt{2\det C}} \qquad \sigma_1 y - \frac{\sigma_{12}}{\sigma_1}x = t\sqrt{2\det C}$$

$$y = \frac{\sigma_{12}}{\sigma_1^2}x + t\frac{\sqrt{2\det C}}{\sigma_1} \qquad dy = \frac{\sqrt{2\det C}}{\sigma_1}\,dt$$

$$= \frac{1}{2\pi\cancel{\sqrt{\det C}}} \frac{\sqrt{2\cancel{\det C}}}{\sigma_1} \int_{\mathbb{R}} x \exp\left(\frac{-x^2}{2\sigma_1^2}\right) \left(\int_{\mathbb{R}} \left(\frac{\sigma_{12}}{\sigma_1^2}x + t\frac{\sqrt{2\det C}}{\sigma_1}\right) \exp(-t^2)\,dt\right) dx$$

$$= \frac{1}{\sqrt{2}\,\pi\sigma_1} \int_{\mathbb{R}} x \exp\left(\frac{-x^2}{2\sigma_1^2}\right) \left(\frac{\sigma_{12}}{\sigma_1^2}x\sqrt{\pi} + 0\right) dx \;=$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\sigma_{12}}{\sigma_1^3} \int_{\mathbb{R}} x^2 \exp\left(\frac{-x^2}{2\sigma_1^2}\right) dx = \frac{\sigma_{12}}{\sigma_1^2} \int_{\mathbb{R}} \frac{x^2}{\sqrt{2\pi\sigma_1^2}} \exp\left(\frac{-x^2}{2\sigma_1^2}\right) dx$$

$$= \frac{\sigma_{12}}{\sigma_1^2}\, \sigma_1^2 = \sigma_{12}$$

$$f(x|y) = \frac{f(x,y)}{h(y)} = \frac{\dfrac{1}{2\pi\sqrt{\det C}} \exp\left(\dfrac{-(\sigma_2^2x^2 - 2\sigma_{12}xy + \sigma_1^2y^2)}{2\det C}\right)}{\dfrac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(\dfrac{-y^2}{2\sigma_2^2}\right)}$$

$$= \frac{1}{2\pi\sqrt{\det C}} \sqrt{2\pi\sigma_2^2}\, \exp\left(\frac{y^2}{2\sigma_2^2} - \frac{\sigma_2^2x^2 - 2\sigma_{12}xy + \sigma_1^2y^2}{2\det C}\right)$$

Calcolo l'esponente:

$$\frac{1}{2\sigma_2^2 \det C}\left(y^2\left(\cancel{\sigma_1^2\sigma_2^2} - \sigma_{12}^2\right) - \sigma_2^4x^2 + 2\sigma_{12}\sigma_2^2xy - \cancel{\sigma_1^2\sigma_2^2y^2}\right)$$

$$= \frac{-1}{2\sigma_2^2 \det C}\left(\sigma_2^4x^2 - 2\sigma_{12}\sigma_2^2xy + \sigma_{12}^2y^2\right) =$$

$$= \frac{-1}{2\sigma_2^2 \det C}\left(\sigma_2^2 x - \sigma_{12}y\right)^2 = \frac{-1}{2\cancel{\sigma_2^2}\det C}\, \sigma_2^{\cancel{4}\,2}\left(x - \frac{\sigma_{12}}{\sigma_2^2}y\right)^2$$

$$f(x|y) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\sigma_2^2}{\det C}}\, \exp\left(\frac{-(\sigma_2^2x - \sigma_{12}y)^2}{2\sigma_2^2\det C}\right) =$$

$$= \frac{1}{\sqrt{2\pi\dfrac{\det C}{\sigma_2^2}}} \exp\left(\frac{-\left(x - \dfrac{\sigma_{12}}{\sigma_2^2}y\right)^2}{2\dfrac{\det C}{\sigma_2^2}}\right)$$

è la densità associata a $N\left(\dfrac{\sigma_{12}}{\sigma_2^2}y, \dfrac{\det C}{\sigma_2^2}\right)$

$$\Rightarrow \mathbb{E}[X|Y=y] = \int_{\mathbb{R}} x f(x|y)\, dx = \frac{\sigma_{12}}{\sigma_2^2}y$$

LEGGE DEBOLE DEI GRANDI NUMERI

Sia $\{X_n\}_{n\in\mathbb{N}}$ successione di v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$.

Supponiamo che

1) Le v.a. X_n siano a due a due scorrelate

   cioè $Cov(X_i, X_j) = 0 \quad i \neq j$

2) $\mathbb{E}[X_i] = E$ finito $\forall i \in \mathbb{N}$

3) $\exists C^2 > 0$ t.c. $Var[X_i] \leq C^2 \quad \forall i \in \mathbb{N}$

Sia $S_n := \sum_{i=1}^{n} X_i$

Allora
$$\int_\Omega \left| \frac{S_n}{n}(\omega) - E \right|^2 \mathbb{P}(d\omega) \leq \frac{C^2}{n} \quad \forall n \in \mathbb{N}$$
$$\mathbb{P}\left(\left| \frac{S_n}{n} - E \right| > \delta \right) \leq \frac{C^2}{n\delta^2} \quad \forall n \in \mathbb{N}$$

DIM $\quad S_n := \sum_{i=1}^{n} X_i$

$$\mathbb{E}[S_n] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} E = nE$$

$$Var[S_n] = Var\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} Var[X_i] + 2\sum_{i<j} \underbrace{\cancel{Cov(X_i, X_j)}}_{=0}$$
$$\leq \sum_{i=1}^{n} C^2 = nC^2$$

$$\mathbb{E}\left[\frac{S_n}{n}\right] = \frac{1}{n}\mathbb{E}[S_n] = \frac{1}{n} nE = E$$

$$Var\left[\frac{S_n}{n}\right] = \frac{1}{n^2} Var[S_n] \leq \frac{1}{n^2} nC^2 = \frac{C^2}{n}$$

$$\int_\Omega \left| \frac{S_n}{n}(\omega) - E \right|^2 \mathbb{P}(d\omega) = \mathbb{E}\left[\left(\frac{S_n}{n} - E\right)^2\right] = Var\left[\frac{S_n}{n}\right] \leq \frac{C^2}{n}$$

$$\mathbb{P}\left(\left| \frac{S_n}{n} - E \right| > \delta \right) \leq \frac{1}{\delta^2} Var\left[\frac{S_n}{n}\right] \leq \frac{C^2}{\delta^2 n}$$

N.B. Con le ipotesi fatte vale la LEGGE FORTE DEI

GRANDI NUMERI

$$\lim_{n\to\infty} \frac{S_n}{n}(\omega) = E \qquad \mathbb{P}\text{-q.c.}$$

$\mathbb{R}^N \qquad Q = [0,1]^N \qquad f \in C^0(Q) \qquad \int_{\mathbb{R}^N} f(x)\,dx$

$(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N), \mu)$ distribuzione di probabilità

$f: \mathbb{R}^N \to \mathbb{R}$ limitata $\quad |f(x)| \le M \quad \forall x \in \mathbb{R}^N$

$\overline{f} = \int_{\mathbb{R}^N} f(x)\mu(dx) \qquad \int_{\mathbb{R}^N} |f(x)|\mu(dx) \le M\mu(\mathbb{R}^N) = M \cdot 1 = M$

Sia $\{X_n\}_{n\in\mathbb{N}}$ una successione di v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

$\mathbb{P}_{X_n} = \mu$ e le X_n a due a due scorrelate

$Y_n = f(X_n)$

$$\mathbb{E}[|Y_n|] = \int_\Omega |f(X_n(\omega))|\,\mathbb{P}(d\omega) = \int_{\mathbb{R}^N} |f(x)|\mu(dx) \le M\mu(\mathbb{R}^N) = \mu \cdot 1 = \mu$$

$$\mathbb{E}[Y_n] = \int_\Omega f(X_n(\omega))\,\mathbb{P}(d\omega) = \int_{\mathbb{R}^N} f(x)\mu(dx) = \overline{f}$$

$$\mathrm{Var}[Y_n] = \int_\Omega (\overset{f(X_n)}{Y_n} - \overline{f})^2(\omega)\,\mathbb{P}(d\omega) =$$

$$= \int_{\mathbb{R}^N} (f(x) - \overline{f})^2 \mu(dx) \le 4M^2 \int_{\mathbb{R}^N} \mu(dx) = 4M^2 \cdot 1 = 4M^2$$

$$|f(x) - \overline{f}| \le |f(x)| + |\overline{f}| \le M + M = 2M$$

Applico la legge debole dei grandi numeri alla successione Y_n, $\quad S_n = \sum_{i=1}^n Y_i = \sum_{i=1}^n f(X_i)$

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^n f(X_i) - \overline{f}\right| > \delta\right) \le \frac{4M^2}{\delta^2 n} \qquad \forall n \in \mathbb{N} \quad \forall \delta > 0$$

Supponiamo di voler valutare $\overline{f}$ con un errore del 5‰

$\delta = 5‰ = \frac{5}{1000} = 5 \cdot 10^{-3}$ e con probabilità di stare sotto la soglia di errore ~~pari~~ superiore al 95 %

$$\mathbb{P}\left(\left| \frac{1}{n} \sum_{i=1}^{n} f(X_i) - \bar{f} \right| > 5 \cdot 10^{-3} \right) < 5\% = 5 \cdot 10^{-2}$$

Mi basterà avere $\frac{4M^2}{\delta^2 n} < 5 \cdot 10^{-2}$

$$\frac{4M^2}{(5 \cdot 10^{-3})^2 n} < 5 \cdot 10^{-2} \qquad n > \frac{4M^2}{25 \cdot 10^{-6}} \cdot \frac{1}{5 \cdot 10^{-2}}$$

$$n > \frac{4}{125} M^2 10^8$$

$$x_n = X_n(\omega)$$

$$\frac{1}{n} \sum_{i=1}^{n} f(x_i)$$

TEOREMA CENTRALE DEL LIMITE

Sia $\{X_n\}_{n \in \mathbb{N}}$ successione di v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ che siano i.i.d, $\mathbb{E}[X_n] = E$ finito, $Var[X_n] = \sigma^2 > 0$ finita

Per $n \geq 1$, $S_n := \sum_{k=1}^{n} X_k$

Allora $\forall t \in \mathbb{R}$ $\mathbb{P}\left(\frac{S_n - nE}{\sigma \sqrt{n}} \leq t \right) \to \Phi(t)$ per $n \to \infty$

Inoltre

$$\lim_{n \to \infty} \sup_{t \in \mathbb{R}} \left| \underbrace{\mathbb{P}\left(\frac{S_n - nE}{\sigma \sqrt{n}} \leq t \right)}_{\text{legge di } \frac{S_n - nE}{\sigma\sqrt{n}}} - \Phi(t) \right| = 0$$

o, equivalentemente

$$\lim_{n \to \infty} \sup_{t \in \mathbb{R}} \left| \mathbb{P}(S_n \leq t) - \Phi\left(\frac{t - nE}{\sigma \sqrt{n}} \right) \right| = 0$$

TEOREMA DI BERRY - ESSEEN

Sia $\{X_n\}_{n\in\mathbb{N}}$ successione di v.a. i.i.d

$E[X_n]=0$

$Var[X_n]=\sigma^2$

$E[|X_n|^3]=\gamma<+\infty$

Sia $S_n:=\sum_{k=1}^{n}X_k$

Allora $\left|\mathbb{P}\left(\frac{S_n}{\sigma\sqrt{n}}\leq t\right)-\Phi(t)\right|\leq\frac{C}{\sqrt{n}}\quad\forall t\in\mathbb{R}$

$C=\frac{0.8\,\gamma}{\sigma^3}$

ES 1 FOGLIO 5

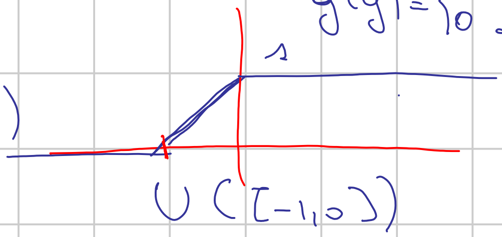
$h(y)=\begin{cases}1 & y\in(0,1)\\ 0 & \text{altrimenti}\end{cases}$

$\mathbb{P}_X=B(p)$

$\mathbb{P}(Y\leq t\mid X=1)=\begin{cases}0 & t<0\\ t & t\in[0,1)\\ 1 & t\geq 1\end{cases}$

$g(y)=\begin{cases}1 & y\in(-1,0)\\ 0 & \text{altrimenti}\end{cases}$

$\mathbb{P}(Y\leq t\mid X=0)=\begin{cases}0 & t<-1\\ t+1 & t\in[-1,0)\\ 1 & t\geq 0\end{cases}$



$\mathbb{P}(Y\leq t)=\mathbb{P}(Y\leq t\mid X=1)\mathbb{P}(X=1)+\mathbb{P}(Y\leq t\mid X=0)\mathbb{P}(X=0)$

$t<-1\qquad \mathbb{P}(Y\leq t)=0$

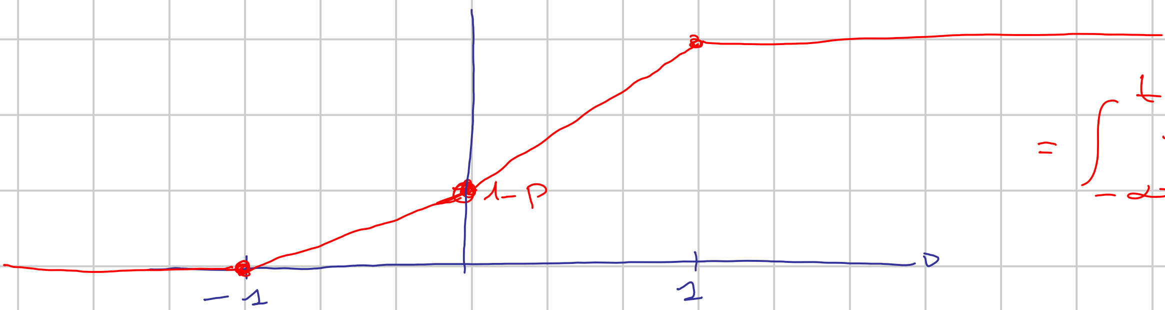
$t\in[-1,0)\qquad \mathbb{P}(Y\leq t)=(t+1)(1-p)$

$t\in[0,1)\qquad \mathbb{P}(Y\leq t)=tp+1\cdot(1-p)$

$t\geq 1\qquad \mathbb{P}(Y\leq t)=1\cdot p+1\cdot(1-p)=1$

$F_Y(t)=\mathbb{P}(Y\leq t)$

$=\int_{-\infty}^{t}f(y)\,dy$



$$f(y) = \begin{cases} 0 & y < -1 \ \vee \ y > 1 \\ 1-p & -1 \le y < 0 \\ p & 0 \le y < 1 \end{cases}$$

$$\mathbb{E}[Y] = \int_{\mathbb{R}} y f(y)\, dy = \int_{-1}^{0} y(1-p)\, dy + \int_0^1 y p\, dy$$

$$= \frac{1-p}{2} y^2 \Big|_{y=-1}^{y=0} + \frac{p}{2} y^2 \Big|_{y=0}^{y=1} = \frac{-(1-p)}{2} + \frac{p}{2} = \frac{2p-1}{2}$$

$$\mathbb{E}[Y^2] = \int_{\mathbb{R}} y^2 f(y)\, dy = \int_{-1}^{0} y^2(1-p) + \int_0^1 y^2 p\, dy =$$

$$= \frac{1-p}{3} y^3 \Big|_{y=-1}^{y=0} + \frac{p}{3} y^3 \Big|_{y=0}^{y=1} = \frac{1-p}{3} + \frac{p}{3} = \frac{1}{3}$$

$$\mathrm{Var}[Y] = \frac{1}{3} - \left(\frac{2p-1}{2}\right)^2 = \frac{1}{3} - \frac{4p^2 - 4p + 1}{4} = \frac{1}{12} + p - p^2$$

$$\mathbb{E}[Y | X = t] = \varphi(t)$$

$$\varphi(0) \quad \varphi(1)$$

$$\mathbb{E}[Y | X = 0] = \frac{1}{\mathbb{P}(X=0)} \int_{\Omega} Y(\omega) \mathbb{1}_{\{X=0\}}(\omega)\, \mathbb{P}(d\omega) =$$

$$= \int_{\Omega} Y(\omega)\, \mathbb{P}_{\{X=0\}}(d\omega) = \int_{\mathbb{R}} y g(y)\, dy \qquad g(y) = \begin{cases} +1 & y \in (-1, 0) \\ 0 & \text{altrimenti} \end{cases}$$

$$= \int_{-1}^{0} y \cdot 1\, dy = \frac{y^2}{2} \Big|_{y=-1}^{y=0} = -\frac{1}{2}$$

$$\mathbb{E}[Y | X = 1] = \frac{1}{\mathbb{P}(X=1)} \int_{\Omega} Y(\omega) \mathbb{1}_{\{X=1\}}(\omega)\, \mathbb{P}(d\omega)$$

$$= \int_{\Omega} Y(\omega)\, \mathbb{P}_{\{X=1\}}(d\omega) = \int_{\mathbb{R}} y h(y)\, dy = \int_0^1 y \cdot 1\, dy = \frac{y^2}{2} \Big|_{y=0}^{y=1}$$

$$= \frac{1}{2}$$

$$\mathbb{E}[Y|X](\omega) = \mathbb{E}[Y|X=0]\,\mathbb{1}_{\{X=0\}}(\omega) +$$

$$+\ \mathbb{E}[Y|X=1]\,\mathbb{1}_{\{X=1\}}(\omega)$$

$$= -\frac{1}{2}\,\mathbb{1}_{\{X=0\}}(\omega) + \frac{1}{2}\,\mathbb{1}_{\{X=1\}}(\omega) .$$