Se X e Y sono v.a. indipendenti, entrambe con distribuzione A.C. $\mathbb{P}_X = f(x)\,dx$

$$\mathbb{P}_Y = g(y)\,dy$$

Allora la v.a. bidimensionale (X,Y) ha distribuzione A.C. con densità $h(x,y) = f(x)g(y)$ $\quad \mathbb{P}_{X,Y} = f(x)g(y)\,dx\,dy$

— o —

(X,Y) v.a. bidimensionale con distribuzione A.C. e densità $f(x,y)$ $\quad \mathbb{P}_{X,Y} = f(x,y)\,dx\,dy$

$A \in \mathcal{B}(\mathbb{R})$ $\quad \mathbb{P}_X(A) = \mathbb{P}(X \in A) = \mathbb{P}(X \in A, Y \in \mathbb{R}) =$

$$= \mathbb{P}((X,Y) \in A \times \mathbb{R}) = \int_{A\times\mathbb{R}} f(x,y)\,dx\,dy =$$

$$= \int_A \underbrace{\left(\int_{\mathbb{R}} f(x,y)\,dy\right)}_{g(x)} dx$$

$\forall A \in \mathcal{B}(\mathbb{R})$ $\quad \mathbb{P}_X(A) = \int_A g(x)\,dx$ $\qquad \mathbb{P}_X = g(x)\,dx$

$$g(x) = \int_{\mathbb{R}} f(x,y)\,dy$$

$\mathbb{P}_Y = h(y)\,dy$ $\qquad h(y) = \int_{\mathbb{R}} f(x,y)\,dx$

— o —

X e Y v.a. con distribuzione di Poisson, indipendenti

$\mathbb{P}_X = \mathcal{P}(\lambda)$ $\qquad \mathbb{P}_Y = \mathcal{P}(\mu)$

$\mathbb{P}(X=k) = e^{-\lambda}\dfrac{\lambda^k}{k!}$ $\qquad \mathbb{P}(Y=k) = e^{-\mu}\dfrac{\mu^k}{k!}$ $\qquad \forall k = 0,1,2,\ldots$

$Z := X+Y$

$\forall k \in \mathbb{Z} \quad \mathbb{P}(Z=k) = \sum_{j=-\infty}^{+\infty} \mathbb{P}(X=j)\mathbb{P}(Y=k-j)$

$\mathbb{P}(Z=k) = 0 \quad \forall k = -1, -2, \ldots$

$k \geq 0 \quad \mathbb{P}(Z=k) = \sum_{j=0}^{k} \mathbb{P}(X=j)\mathbb{P}(Y=k-j) \qquad k-j \geq 0 \quad j \leq k$

$= \sum_{j=0}^{k} e^{-\lambda}\frac{\lambda^j}{j!} e^{-\mu}\frac{\mu^{k-j}}{(k-j)!}\frac{k!}{k!} =$

$= e^{-\lambda}e^{-\mu}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^j\mu^{k-j} = \qquad (x+y)^n = \sum\binom{n}{j}x^j y^{n-j}$

$= e^{-(\lambda+\mu)}\frac{1}{k!}(\lambda+\mu)^k \implies \mathbb{P}_Z = \mathcal{P}(\lambda+\mu)$

— o —

X e Y v.a. con distribuzione esponenziale e indipendenti

$\mathbb{P}_X = \exp(\lambda) \qquad \mathbb{P}_Y = \exp(\mu)$

$f(x) = \begin{cases} 0 & x \leq 0 \\ \lambda e^{-\lambda x} & x > 0 \end{cases} \qquad g(x) = \begin{cases} 0 & x \leq 0 \\ \mu e^{-\mu x} & x > 0 \end{cases}$

$\mathbb{P}_Z \qquad Z = X+Y \qquad \mathbb{P}_Z = h(x)\,dx \quad h(x) = \int_{\mathbb{R}} f(y)\,g(x-y)\,dy$

$\begin{cases} y > 0 \\ x-y > 0 \end{cases} \qquad 0 < y < x \qquad x \leq 0 \quad h(x) = 0$

$x > 0 \quad h(x) = \int_0^x \lambda e^{-\lambda y}\mu e^{-\mu(x-y)}\,dy$

$x > 0 \quad h(x) = \lambda\mu\int_0^x e^{-\mu x}e^{(\mu-\lambda)y}\,dy = \lambda\mu e^{-\mu x}\int_0^x e^{(\mu-\lambda)y}\,dy$

$\mu = \lambda \quad h(x) = \lambda^2 x e^{-\lambda x}$

$\mu \neq \lambda \quad h(x) = \lambda\mu e^{-\mu x}\left.\frac{e^{(\mu-\lambda)y}}{\mu-\lambda}\right|_{y=0}^{y=x} = \frac{\lambda\mu}{\mu-\lambda}e^{-\mu x}\left(e^{(\mu-\lambda)x} - 1\right)$

$$= \frac{\lambda\mu}{\mu-\lambda}\left(e^{-\lambda x} - e^{-\mu x}\right)$$

$\mathbb{P}_X = N(0,\sigma^2)$ $\quad$ $\mathbb{P}_Y = N(0,\delta^2)$ $\quad$ X, Y independent.

$\mathbb{P}_X = f(x)\,dx$ $\quad$ $\mathbb{P}_Y = g(x)\,dx$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-x^2}{2\sigma^2}\right) \qquad g(x) = \frac{1}{\sqrt{2\pi\delta^2}}\exp\left(\frac{-x^2}{2\delta^2}\right)$$

$Z := X+Y$ $\quad$ $\mathbb{P}_Z = h(x)\,dx$ $\quad$ $h(x) = \int_{\mathbb{R}} f(y)\,g(x-y)\,dy \quad x\in\mathbb{R}$

$$h(x) = \frac{1}{2\pi\sigma\delta}\int_{\mathbb{R}} \exp\left(\frac{-y^2}{2\sigma^2}\right)\exp\left(\frac{-(x-y)^2}{2\delta^2}\right)dy$$

$$= \frac{1}{2\pi\sigma\delta}\int_{\mathbb{R}} \exp\left(-\frac{1}{2}\left\{\frac{y^2}{\sigma^2} + \frac{x^2-2xy+y^2}{\delta^2}\right\}\right)dy$$

Coeff di y^2 $\quad \frac{1}{\sigma^2}+\frac{1}{\delta^2} = \frac{\delta^2+\sigma^2}{\sigma^2\delta^2}$

coeff di x^2 $\quad \frac{1}{\delta^2}$

Coeff di $-2xy$ $\quad \frac{1}{\delta^2}$

$(ay-bx)^2 + cx^2 =$
$a^2y^2 - 2abxy + b^2x^2 + cx^2$

$$\left(y\frac{\sqrt{\delta^2+\sigma^2}}{\sigma\delta}\right)^2 - 2\,y\frac{\sqrt{\delta^2+\sigma^2}}{\sigma\delta} \times \frac{1}{\delta^{\cancel{2}}}\,\frac{\sigma\cancel{\delta}}{\sqrt{\sigma^2+\delta^2}}$$

$$+\,x^2\frac{\sigma^2}{\delta^2(\sigma^2+\delta^2)} - x^2\frac{\sigma^2}{\delta^2(\sigma^2+\delta^2)} + \frac{x^2}{\delta^2}$$

$$= \left(y\frac{\sqrt{\delta^2+\sigma^2}}{\sigma\delta} - x\frac{\sigma}{\delta\sqrt{\sigma^2+\delta^2}}\right)^2 + \frac{x^2\left(\cancel{\sigma^2}+\cancelto{1}{\delta^2}-\cancel{\sigma^2}\right)}{\cancel{\delta^2}(\sigma^2+\delta^2)} =$$

$$= \left(y\frac{\sqrt{\delta^2+\sigma^2}}{\sigma\delta} - x\frac{\sigma}{\delta\sqrt{\sigma^2+\delta^2}}\right)^2 + \frac{x^2}{\sigma^2+\delta^2}$$

$$h(x) = \frac{1}{2\pi\sigma\delta}\int_{\mathbb{R}}\exp\left(-\frac{1}{2}\left\{\left(y\frac{\sqrt{\delta^2+\sigma^2}}{\sigma\delta} - x\frac{\sigma}{\delta\sqrt{\sigma^2+\delta^2}}\right)^2 + \frac{x^2}{\sigma^2+\delta^2}\right\}\right)dy$$

$$= \frac{1}{2\pi\sigma\delta} \exp\left(\frac{-x^2}{2(\sigma^2+\delta^2)}\right) \int_{\mathbb{R}} \exp\left(-\frac{1}{2}\left(y\frac{\sqrt{\sigma^2+\delta^2}}{\sigma\delta} - x\frac{\sigma}{\delta\sqrt{\sigma^2+\delta^2}}\right)^2\right) dy$$

$$\int_{\mathbb{R}} e^{-t^2} dt = \sqrt{\pi} \qquad t = \frac{1}{\sqrt{2}}\left(y\frac{\sqrt{\sigma^2+\delta^2}}{\sigma\delta} - x\frac{\sigma}{\delta\sqrt{\sigma^2+\delta^2}}\right)$$

$$dt = \frac{\sqrt{\sigma^2+\delta^2}}{\sigma\delta\sqrt{2}} dy$$

$$h(x) = \frac{1}{\sqrt{2}\cancel{2}\pi\cancel{\sigma}\cancel{\delta}} \exp\left(\frac{-x^2}{2(\sigma^2+\delta^2)}\right) \int_{\mathbb{R}} \exp(-t^2) \frac{\cancel{\sigma}\cancel{\delta}\cancel{\sqrt{2}}}{\sqrt{\sigma^2+\delta^2}} dt$$

$$= \frac{1}{\sqrt{2(\sigma^2+\delta^2)\pi}} \exp\left(\frac{-x^2}{2(\sigma^2+\delta^2)}\right) \qquad \forall x \in \mathbb{R}$$

$$\Rightarrow \mathbb{P}_{X+Y} = N(0, \sigma^2+\delta^2)$$

$$\sigma^2 = \delta^2 \qquad \mathbb{P}_{X+Y} = N(0, 2\sigma^2)$$

X_1, X_2, X_3 v.a. i.i.d $\quad \mathbb{P}_{X_i} = N(0,\sigma^2)$

$$\mathbb{P}_{X_1+X_2} = N(0, 2\sigma^2) \qquad \mathbb{P}_{\underbrace{(X_1+X_2)}_{N(0,2\sigma^2)} + \underbrace{X_3}_{N(0,\sigma^2)}} = N(0, 3\sigma^2)$$

$X_1 \dots X_n$ v.a. i.i.d $\quad \mathbb{P}_{X_i} = N(0,\sigma^2)$

$$S_n := \sum_{i=1}^{n} X_i \quad \Rightarrow \mathbb{P}_{S_n} = N(0, n\sigma^2)$$

$$\tilde{X}_n := \frac{S_n}{n} \qquad \mathbb{E}[\tilde{X}_n] = \mathbb{E}\left[\frac{S_n}{n}\right] = \frac{1}{n}\mathbb{E}[S_n] = 0$$

$$Var[\tilde{X}_n] = Var\left[\frac{S_n}{n}\right] = \frac{1}{n^2} Var[S_n] = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

$$\Rightarrow \mathbb{P}_{\tilde{X}_n} = N\left(0, \frac{\sigma^2}{n}\right)$$

$\mathbb{P}_X = \exp(\lambda)$ $\quad \mathbb{P}_Y = \exp(\mu)$ $\quad X, Y$ indipendenti.

$Z := \min(X,Y)$

$Z = \varphi \circ (X,Y)$ $\quad \varphi : (x,y) \in \mathbb{R}^2 \mapsto \min(x,y) \in \mathbb{R}$

$\psi : \mathbb{R} \to \mathbb{R}$ funzione di Borel nonnegativa

$$\int_{\mathbb{R}} \psi(t)\, \mathbb{P}_Z(dt) = \int_{\mathbb{R}} \psi(t)\, \mathbb{P}_{\varphi\circ(X,Y)}(dt) = \int_{\mathbb{R}^2} \psi(\varphi(x,y))\, \mathbb{P}_{X,Y}(dx\,dy)$$

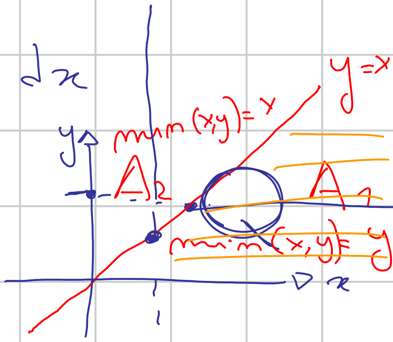
$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \psi(\min(x,y))\, \mathbb{P}_Y(dy) \right) \mathbb{P}_X(dx)$$

$$\mathbb{P}_X = f(x)\,dx \qquad f(x) = \begin{cases} 0 & x \leq 0 \\ \lambda e^{-\lambda x} & x > 0 \end{cases}$$

$$\mathbb{P}_Y = g(y)\,dy \qquad g(y) = \begin{cases} 0 & y \leq 0 \\ \mu e^{-\mu y} & y > 0 \end{cases}$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \psi(\min(x,y))\, g(y)\, dy \right) f(x)\, dx$$

$$\int_0^{+\infty} \left(\int_0^{+\infty} \psi(\min(x,y))\, \mu e^{-\mu y}\, dy \right) \lambda e^{-\lambda x}\, dx$$



$$= \iint_Q \psi(\min(x,y))\, \mu\lambda e^{-\lambda x} e^{-\mu y}\, dx\,dy$$

$$= \iint_{A_1} \psi(y)\, \mu\lambda e^{-\lambda x} e^{-\mu y}\, dx\,dy + \iint_{A_2} \psi(x)\, \mu\lambda e^{-\lambda x} e^{-\mu y}\, dx\,dy$$

$A_1 : \ y \in (0, +\infty) \quad x \in (y, +\infty) \qquad A_2 : x \in (0, +\infty) \quad y \in (x, +\infty)$

$$= \int_0^{+\infty} \left(\int_y^{+\infty} \psi(y)\, \lambda\mu e^{-\lambda x} e^{-\mu y}\, dx \right) dy + \int_0^{+\infty} \left(\int_x^{+\infty} \psi(x)\, \lambda\mu e^{-\lambda x} e^{-\mu y}\, dy \right) dx$$

$$= \int_0^{+\infty} \psi(y)\, \mu e^{-\mu y} \left(\int_y^{+\infty} \lambda e^{-\lambda x}\, dx \right) dy + \int_0^{+\infty} \psi(x)\, \lambda e^{-\lambda x} \left(\int_x^{+\infty} \mu e^{-\mu y}\, dy \right) dx$$

$$= \int_0^{+\infty} f(y)\, \mu e^{-\mu y} \left(-e^{-\lambda x}\right)\Big|_{x=y}^{x\to+\infty} dy + \int_0^{+\infty} f(x)\, \lambda e^{-\lambda x} \left(-e^{-\mu y}\right)\Big|_{y=x}^{y\to+\infty} dx$$

$$= \int_0^{+\infty} f(y)\, \mu e^{-\mu y} \left(0 - \left(-e^{-\lambda y}\right)\right) dy + \int_0^{+\infty} f(x)\, \lambda e^{-\lambda x} \left(0 - \left(-e^{-\mu x}\right)\right) dx$$

$$= \int_0^{+\infty} f(x)\, \mu e^{-\mu x} e^{-\lambda x}\, dx + \int_0^{+\infty} f(x)\, \lambda e^{-\lambda x} e^{-\mu x}\, dx$$

$$= \int_0^{+\infty} f(x)\, (\mu+\lambda)\, e^{-(\mu+\lambda)x}\, dx = \int_{\mathbb{R}} f(x)\, g(x)\, dx$$

$$g(x) = \begin{cases} 0 & x \le 0 \\ (\mu+\lambda)\, e^{-(\mu+\lambda)x} & x > 0 \end{cases}$$

$\mathbb{P}_{\min(X,Y)} = g(x)\,dx \qquad \mathbb{P}_{\min(X,Y)} = \exp(\lambda+\mu)$

— o —

DISTRIBUZIONI GAMMA

FUNZIONE GAMMA DI EULERO

$\alpha > 0 \qquad \int_0^{+\infty} x^{\alpha-1} e^{-x}\, dx \qquad \alpha > 0 \quad \alpha - 1 > -1$

$\int_0^1 x^p\, dx$

$$\int_\varepsilon^1 x^p\, dx = \begin{cases} \log(x)\Big|_{x=\varepsilon}^{x=1} & p = -1 \\ \dfrac{1}{p+1}\, x^{p+1}\Big|_{x=\varepsilon}^{x=1} & p \ne -1 \end{cases}$$

$$= \begin{cases} 0 - \log(\varepsilon) & p = -1 \\ \dfrac{1}{p+1}\left(1 - \varepsilon^{p+1}\right) & p \ne -1 \end{cases} \xrightarrow{\varepsilon \to 0^+} \begin{cases} +\infty & p = -1 \\ \dfrac{1}{p+1} & p > -1 \\ +\infty & p < -1 \end{cases}$$

$\forall \alpha > 0 \quad \int_0^{+\infty} x^{\alpha-1} e^{-x}\, dx$ è un integrale convergente

Considero la funzione $\Gamma : \alpha \in (0, +\infty) \longmapsto \int_0^{+\infty} x^{\alpha-1} e^{-x}\, dx$

La funzione Γ si chiama Γ di EULERO

$$\Gamma(1) = \int_0^{+\infty} x^0 e^{-x}\,dx = \int_0^{+\infty} e^{-x}\,dx = -e^{-x}\Big|_{x=0}^{x\to+\infty} = 1$$

$$\Gamma(\alpha+1) = \int_0^{+\infty} x^{\alpha+1-1} e^{-x}\,dx = \int_0^{+\infty} \underset{g^\alpha}{x^\alpha}\, \underset{g'}{e^{-x}}\,dx =$$

$$x^\alpha(-e^{-x})\Big|_{x=0}^{x\to+\infty} + \int_0^{+\infty} \alpha x^{\alpha-1} e^{-x}\,dx = = \alpha \int_0^{+\infty} x^{\alpha-1} e^{-x}\,dx .$$

$$\boxed{\Gamma(\alpha+1) = \alpha\Gamma(\alpha)}$$

$\Gamma(1) = 1$

$\Gamma(2) = 1\cdot\Gamma(1) = 1\cdot 1 = 1$

$\Gamma(3) = 2\Gamma(2) = 2\cdot 1$

$\Gamma(4) = 3\Gamma(3) = 3\cdot 2\cdot 1$

$\Gamma(5) = 4\Gamma(4) = 4\cdot 3\cdot 2\cdot 1$

$$\Gamma(n) = (n-1)!$$

Per α, λ parametri reali positivi chiamiamo DISTRIBUZIONE GAMMA DI PARAMETRI α E λ, la distribuzione $\Gamma(\alpha,\lambda)$ a.c. associata alla densità

$$f(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & x > 0 \end{cases}$$

$f(x) \ge 0$ sì

$$\int_{\mathbb{R}} f(x)\,dx = \int_0^{+\infty} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^{\alpha-1} e^{-\lambda x}\,dx$$

$t = \lambda x \qquad x = \dfrac{t}{\lambda} \qquad dx = \dfrac{1}{\lambda}\,dt$

$$\frac{\cancel{\lambda^\alpha}}{\Gamma(\alpha)} \int_0^{+\infty} \frac{t^{\alpha-1}}{\cancel{\lambda^{\alpha-1}}} e^{-t} \frac{1}{\cancel{\lambda}}\,dt = \frac{1}{\Gamma(\alpha)}\Gamma(\alpha) = 1$$

$\mathbb{P}_X = \Gamma(\alpha, \lambda)$ $\qquad X \geq 0$ q.c.

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_0^{+\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^\alpha e^{-\lambda x}\,dx \qquad t = \lambda x, \quad x = \frac{t}{\lambda}$$

$$= \frac{\cancel{\lambda^\alpha}}{\Gamma(\alpha)} \int_0^{+\infty} \frac{t^\alpha}{\cancel{\lambda^\alpha}} e^{-t} \frac{1}{\lambda}\,dt = \frac{1}{\lambda \Gamma(\alpha)} \int_0^{+\infty} t^{\overset{\alpha+1-1}{\alpha =}} e^{-t}\,dt =$$

$$= \frac{1}{\lambda \Gamma(\alpha)} \Gamma(\alpha+1) = \frac{\alpha \Gamma(\alpha)}{\lambda \Gamma(\alpha)} = \frac{\alpha}{\lambda}$$

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)\,dx = \int_0^{+\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha+1} e^{-\lambda x}\,dx \qquad t = \lambda x, \quad x = \frac{t}{\lambda}$$

$$= \frac{\cancel{\lambda^\alpha}}{\Gamma(\alpha)} \int_0^{+\infty} \frac{t^{\alpha+1}}{\lambda^{\cancel{\alpha}+1}} e^{-t} \frac{1}{\lambda}\,dt = \frac{1}{\lambda^2 \Gamma(\alpha)} \int_0^{+\infty} t^{\alpha+1} e^{-t}\,dt$$

$$\alpha + 1 = (\alpha+2) - 1$$

$$= \frac{1}{\lambda^2 \Gamma(\alpha)} \Gamma(\alpha+2) = \frac{1}{\lambda^2 \Gamma(\alpha)} (\alpha+1) \Gamma(\alpha+1) =$$

$$= \frac{\alpha+1}{\lambda^2 \Gamma(\alpha)} \alpha \Gamma(\alpha) = \frac{(\alpha+1)\alpha}{\lambda^2}$$

$$\mathrm{Var}[X] = \frac{(\alpha+1)\alpha}{\lambda^2} - \left(\frac{\alpha}{\lambda}\right)^2 = \frac{\alpha}{\lambda^2}(\alpha + 1 - \alpha) = \frac{\alpha}{\lambda^2}$$