ESEMPI DI DISTRIBUZIONI A.C.

Note Title 23/10/2017

DISTRIBUZIONE UNIFORME SU UN INTERVALLO $[a,b]$.

Si indica $U([a,b])$ ed è la distribuzione A.C. associata alla densità

$$f(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{altrimenti.} \end{cases}$$

$$f(x) \geq 0 \quad \forall x \in \mathbb{R} \ ; \quad \int_{\mathbb{R}} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = 1.$$

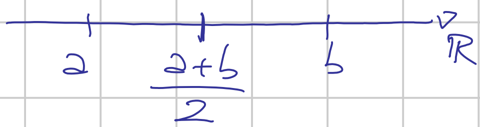
$$U([a,b])(A) = \frac{\mathcal{L}^1(A)}{b-a} \quad \forall A \in \mathcal{B}([a,b])$$

Sia X v.a. t.c. $\mathbb{P}_X = U([a,b])$

$$\mathbb{P}(X \in [a,b]) = 1$$

$$\mathbb{E}[X] = \int_{\mathbb{R}} x\,\mathbb{P}_X(dx) = \int_{\mathbb{R}} x f(x)\,dx = \int_a^b \frac{x}{b-a}\,dx =$$

$$= \frac{1}{2(b-a)} x^2 \Big|_{x=a}^{x=b} = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$



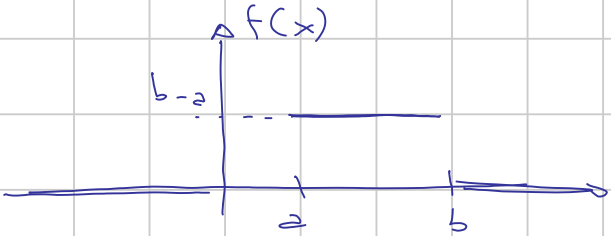
$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2\,\mathbb{P}_X(dx) = \int_{\mathbb{R}} x^2 f(x)\,dx = \int_a^b \frac{x^2}{b-a}\,dx =$$

$$= \frac{1}{3(b-a)} x^3 \Big|_{x=a}^{x=b} = \frac{b^3-a^3}{3(b-a)} = \frac{a^2+ab+b^2}{3}$$

" $a^2+2ab+b^2$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4} =$$

$$= \frac{a^2-2ab+b^2}{12} = \frac{(b-a)^2}{12}$$



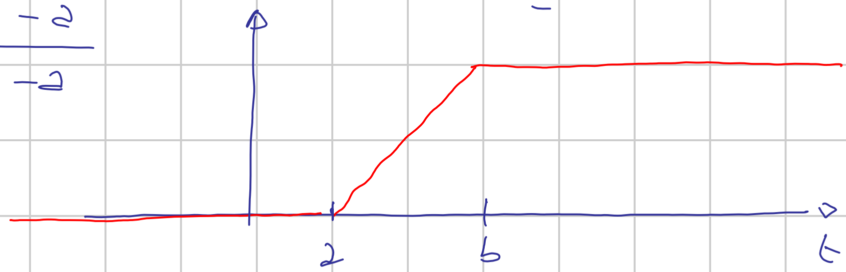
$$F_X(t) = \int_{-\infty}^{t} f(x)\,dx$$

$t \leq a \quad F_X(t) = \int_{-\infty}^{t} 0 \, dx = 0$

$t \in [a, b) \quad F_X(t) = \int_{-\infty}^{t} f(x) \, dx = \int_{-\infty}^{a} 0 \, dx + \int_{a}^{t} \frac{1}{b-a} \, dx =$

$= \int_{a}^{t} \frac{1}{b-a} \, dx = \frac{t-a}{b-a}$

$t \geq b \quad F_X(t) = 1$



DISTRIBUZIONE GAUSSIANA o NORMALE DI PARAMETRI $m \in \mathbb{R}$, $\sigma^2 > 0$.

Si indica col simbolo $N(m, \sigma^2)$ ed è la distribuzione A.C. associata alla densità

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right) \quad \forall x \in \mathbb{R}.$$

$f(x) > 0 \quad \forall x \in \mathbb{R}$

$\int_{\mathbb{R}} f(x) dx = 1$?

INTEGRALE NOTEVOLE

$$\int_{\mathbb{R}} \exp(-t^2) dt = \sqrt{\pi}$$

$$\int_{\mathbb{R}} f(x) dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right) dx$$

$t = \frac{x-m}{\sigma\sqrt{2}} \qquad x = m + t\underbrace{\sigma\sqrt{2}}_{>0} \qquad dx = \sigma\sqrt{2} \, dt$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} \exp(-t^2) \sigma\sqrt{2} \, dx = \frac{1}{\sqrt{2\pi\sigma^2}} \sigma\sqrt{2}\sqrt{\pi} = 1$$

Se $m=0$ $\sigma=1$, la distribuzione gaussiana di parametri $m=0$ e $\sigma=1$ si dice GAUSSIANA STANDARD

NORMALE STANDARD

e si indica $N(0,1)$

Sia X_0 v.a. t.c. $\mathbb{P}_{X_0} = N(0,1)$

$$\mathbb{E}[X_0] = \int_{\mathbb{R}} x f(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x \exp\left(-\frac{x^2}{2}\right) dx$$

perché $\mathbb{E}[|X_0|] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |x| \exp\left(-\frac{x^2}{2}\right) dx < +\infty$

$\Rightarrow \mathbb{E}[X_0] = 0$

$$\mathrm{Var}[X_0] = \mathbb{E}[X_0^2] = \int_{\mathbb{R}} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (-x) \underbrace{\left(-x \exp\left(-\frac{x^2}{2}\right)\right)}_{\frac{d}{dx}\exp\left(-\frac{x^2}{2}\right)} dx =$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ (-x) \exp\left(-\frac{x^2}{2}\right) \Big|_{x \to -\infty}^{x \to +\infty} + \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2}\right) dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2}\right) dx \qquad t = \frac{x}{\sqrt{2}} \quad x = t\sqrt{2} \quad dx = \sqrt{2}\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-t^2) \sqrt{2}\,dt = \frac{1}{\sqrt{2\pi}} \sqrt{2}\sqrt{\pi} = 1$$

$\mathbb{P}_{X_0} = N(0,1) \Rightarrow \mathbb{E}[X_0] = 0 \quad \mathrm{Var}[X_0] = 1$

Sia X_0 t.c. $\mathbb{P}_{X_0} = N(0,1)$, siano $m \in \mathbb{R}$ e $\sigma > 0$
e considero la v.a. $X = m + \sigma X_0$

$$\mathbb{P}_{X_0} = f_0(x)\,dx \qquad f_0(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$\Rightarrow \mathbb{P}_X = f(x)\,dx \quad \text{con } f(x) = \frac{1}{|\sigma|} f_0\left(\frac{x-m}{\sigma}\right) =$$

$$= \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$

$\Rightarrow \mathbb{P}_X = N(m, \sigma^2)$

Se Y è v.a. t.c. $\mathbb{P}_Y = N(m, \sigma^2) \Rightarrow \mathbb{E}[Y] = \mathbb{E}[m + \sigma X_0]$

dove X_0 è una qualsiasi v.a. t.c. $\mathbb{P}_{X_0} = N(0,1)$

e lo stesso vale per la varianza $Var[Y] = Var[m + \sigma X_0]$

$$\mathbb{E}[Y] = \mathbb{E}[m + \sigma X_0] = \mathbb{E}[m] + \sigma\,\mathbb{E}[X_0] = m + \sigma \cdot 0 = m$$

$$Var[Y] = Var[m + \sigma X_0] = \sigma^2\, Var[X_0] = \sigma^2 \cdot 1 = \sigma^2$$

$$F_Y(t) = \mathbb{P}(Y \leq t) = \mathbb{P}(m + \sigma X_0 \leq t) = \mathbb{P}\left(X_0 \leq \frac{t-m}{\sigma}\right)$$

$$\Rightarrow \quad F_Y(t) = F_{X_0}\left(\frac{t-m}{\sigma}\right) \qquad \forall t \in \mathbb{R}.$$

F_{X_0}, legge associata a $N(0,1)$ si indica Φ

$$\Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

PROPRIETÀ $\quad \Phi(t) + \Phi(-t) = 1 \qquad \forall t \in \mathbb{R}$

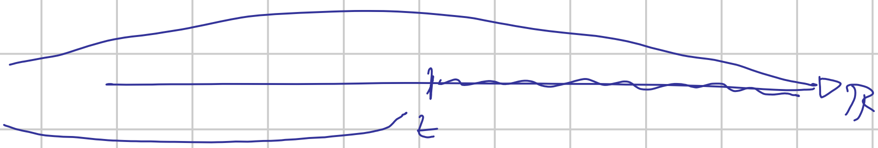
<u>Oss</u> È vera per ogni legge di distribuzione A.C. in cui la densità associata sia una funzione pari.

$$\Phi(-t) = \int_{-\infty}^{-t} f(x)\,dx \qquad y = -x \qquad x = -y \qquad dx = -1 \cdot dy$$

$$= \int_{+\infty}^{t} f(-y)\,(-1)\,dy = \int_{t}^{+\infty} f(-y)\,dy = \int_{t}^{+\infty} f(y)\,dy =$$

$$= \int_{\mathbb{R}} f(y)\,dy - \int_{-\infty}^{t} f(y)\,dy = 1 - \Phi(t).$$

$$\int_{\mathbb{R}} = \int_{-\infty}^{t} + \int_{t}^{+\infty}$$



DISTRIBUZIONE ESPONENZIALE DI PARAMETRO $\lambda > 0$

Si indica $\exp(\lambda)$ ed è la distribuzione A.C. associata alla densità

$$f(x) = \begin{cases} 0 & x \leq 0, \\ \lambda e^{-\lambda x} & x > 0. \end{cases}$$

Se è una V.a. T.c. $\mathbb{P}_X = \exp(\lambda) \Rightarrow X \geq 0$ p.c.

$f(x) > 0 \quad \forall x > 0$

$$\int_{\mathbb{R}} f(x)dx = \int_0^{+\infty} \lambda e^{-\lambda x} dx = -e^{-\lambda x} \Big|_{x=0}^{x \to +\infty} = 0 - (-1) = 1$$

Se $\mathbb{P}_X = \exp(\lambda)$, voglio calcolare $\mathbb{E}[X]$ e $Var[X]$

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)dx = \int_0^{+\infty} x \lambda e^{-\lambda x} dx = \int_0^{+\infty} (-x) \underbrace{\left(-\lambda e^{-\lambda x}\right)}_{= \frac{d}{dx} e^{-\lambda x}} dx$$

$$= (-x) e^{-\lambda x} \Big|_{x=0}^{x \to +\infty} + \int_0^{+\infty} e^{-\lambda x} dx =$$

$$= -\frac{1}{\lambda} e^{-\lambda x} \Big|_{x=0}^{x \to +\infty} = 0 + \frac{1}{\lambda} \cdot 1 = \frac{1}{\lambda}$$

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)dx = \int_0^{+\infty} x^2 \lambda e^{-\lambda x} dx = \int_0^{+\infty} (-x^2)\left(-\lambda e^{-\lambda x}\right) dx$$

$$= (-x^2) e^{-\lambda x} \Big|_{x=0}^{x \to +\infty} + 2\int_0^{+\infty} x e^{-\lambda x} dx = \frac{2}{\lambda} \underbrace{\int_0^{+\infty} \lambda x e^{-\lambda x} dx}_{\mathbb{E}[X] = \frac{1}{\lambda}}$$

$$\mathbb{E}[X^2] = \frac{2}{\lambda^2} \qquad Var[X] = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}.$$

$$F_X(t) = \int_{-\infty}^{t} f(x)dx$$

$$F_X(t) = 0 \qquad t \leq 0$$

$$t > 0 \qquad F_X(t) = \int_{-\infty}^{t} f(x)dx = \int_0^t \lambda e^{-\lambda x} dx = -e^{-\lambda x} \Big|_{x=0}^{x=t}$$

$$= -e^{-\lambda t} + 1$$

$$F_X(t) = \begin{cases} 0 & t<0, \\ 1-e^{-\lambda t} & t\geq 0 \end{cases}.$$

$t, s \in \mathbb{R} \qquad t, s \geq 0$

$$\mathbb{P}(X \leq t+s \mid X>s) = \mathbb{P}(X\leq t)$$

PROPRIETÀ DI MANCANZA DI MEMORIA

$$\mathbb{P}(X\leq t+s \mid X>s) = \frac{\mathbb{P}(X\leq t+s,\ X>s)}{\mathbb{P}(X>s)} =$$

$$\{X\leq t+s,\ X>s\} = \{X\leq t+s\}\cap\{X>s\} =$$
$$= \{X\leq t+s\}\setminus\{X\leq s\}$$
$$\{X\leq s\}\subseteq\{X\leq t+s\}$$

$$\mathbb{P}(X\leq t+s,\ X>s) = \mathbb{P}(X\leq t+s) - \mathbb{P}(X\leq s)$$
$$\mathbb{P}(X>s) = 1-\mathbb{P}(X\leq s)$$

$$\mathbb{P}(X\leq t+s\mid X>s) = \frac{\mathbb{P}(X\leq t+s)-\mathbb{P}(X\leq s)}{1-\mathbb{P}(X\leq s)} = \frac{F_X(t+s)-F_X(s)}{1-F_X(s)}$$
$$= \frac{(\cancel{1}-e^{-\lambda(t+s)}) - (\cancel{1}-e^{-\lambda s})}{\cancel{1}-(\cancel{1}-e^{-\lambda s})} = \frac{e^{-\lambda s} - e^{-\lambda t-\lambda s}}{e^{-\lambda s}}$$
$$= 1-e^{-\lambda t} = F_X(t) = \mathbb{P}(X\leq t).$$

Si <u>può dimostrare</u> che la distribuzione esponenziale è l'unica distribuzione concentrata su $[0,+\infty)$ ma non unicamente in 0 t.c. le v.a. aventi tale distribuzione godono della proprietà di mancanza di memoria.

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ES 4 foglio 2

$X := \#\text{Teste} - \#\text{croci}$

$\Omega = \{0,1\}^n = \{\omega = (\omega_1 \ldots \omega_n) : \omega_i \in \{0,1\}\}$

$0 \to$ croce

$1 \to$ Testa

$Y(\omega) =$ v.a. che conta il numero delle Teste uscite

$Y(\omega) = \sum_{i=1}^{n} \omega_i = \#\text{Teste uscite}$

$\Rightarrow \#\text{croci} = n - Y(\omega) = n - \sum_{i=1}^{n} \omega_i$

$$X(\omega) = \sum_{i=1}^{n} \omega_i - \left(n - \sum_{i=1}^{n} \omega_i\right) = 2\sum_{i=1}^{n} \omega_i - n = 2Y(\omega) - n$$

$$X(\Omega) = \{-n, -n+2, -n+4, \ldots \ldots n-2, n\}$$

$k \in \{-n, -n+2, \ldots n-2, n\}$

$\mathbb{P}(X=k)$ $\qquad \{X=k\} = \{Y = ?\}$

$$X = 2Y - n \Rightarrow \{X = k\} = \{2Y - n = k\} = \left\{Y = \frac{n+k}{2}\right\}$$

$$\mathbb{P}(X=k) = \mathbb{P}\left(Y = \frac{n+k}{2}\right) = Bin(n,p)\left(\left\{\frac{n+k}{2}\right\}\right) =$$

$$\begin{cases} \binom{n}{\frac{n+k}{2}} p^{\frac{n+k}{2}} (1-p)^{n-\frac{n+k}{2}} & \frac{n+k}{2} = 0, \ldots n \\ 0 & \text{altrimenti} \end{cases}$$

N.B. $\frac{n+k}{2} = 0, 1, \ldots n-1, n$

$n+k = 0, 2, \ldots 2n-2, 2n$

$k = -n, -n+2, \ldots n-2, n$ $\qquad$ ok

Es 5 foglio 2

$X_1 :=$ # estratto delle 1^a urne $\quad 1, \ldots, n$

$X_2 :=$ # estratto delle 2^a urne $\quad 1, \ldots, n$

$X := \max\{X_1, X_2\} \quad \Rightarrow X(\Omega) = \{1, \ldots, n\}$

$\mathbb{P}_{X_1} = \mathbb{P}_{X_2}$ = distribuzioni uniformi su $\{1, 2, \ldots, n\}$ cioè

$$\mathbb{P}(X_1 = j) = \mathbb{P}(X_2 = j) = \frac{1}{n} \quad \forall j = 1, \ldots, n$$

$\mathbb{P}(X = j) = ?$ $\rightarrow \{X = j\} = \{X_1 = j, X_2 \le j\} \cup \{X_1 < j, X_2 = j\}$

$$\mathbb{P}(X=j) = \mathbb{P}(X_1 = j, X_2 \le j) + \mathbb{P}(X_1 < j, X_2 = j) = (A)$$

$\{X_1 \in A\}, \{X_2 \in B\}$ sono eventi indipendenti

$$(A) = \mathbb{P}(X_1 = j)\,\mathbb{P}(X_2 \le j) + \mathbb{P}(X_1 < j)\,\mathbb{P}(X_2 = j) = (B)$$

$$\{X_2 \le j\} = \bigcup_{k=1}^{j} \{X_2 = k\} \qquad \mathbb{P}(X_2 \le j) = \sum_{k=1}^{j} \mathbb{P}(X_2 = k) = \sum_{k=1}^{j} \frac{1}{n} = \frac{j}{n}$$

$$\{X_1 < j\} = \bigcup_{k=1}^{j-1} \{X_1 = k\} \qquad \mathbb{P}(X_1 < j) = \sum_{k=1}^{j-1} \mathbb{P}(X_1 = k) = \sum_{k=1}^{j-1} \frac{1}{n} = \frac{j-1}{n}$$

$$(B) = \frac{1}{n} \cdot \frac{j}{n} + \frac{j-1}{n} \cdot \frac{1}{n} = \frac{2j-1}{n^2}$$

$$\mathbb{P}(X = j) = \frac{2j-1}{n^2} \qquad \forall j = 1, \ldots, n$$

$$\mathbb{E}[X] = \sum_{j=1}^{n} j\,\mathbb{P}(X = j) = \sum_{j=1}^{n} \frac{j(2j-1)}{n^2} =$$

$$= \frac{2}{n^2} \sum_{j=1}^{n} j^2 - \frac{1}{n^2} \sum_{j=1}^{n} j$$

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

$$\mathbb{E}[X] = \frac{2}{n^{\cancel{2}}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{6} - \frac{1}{n^{\cancel{2}}} \frac{\cancel{n}(n+1)}{2} =$$

$$= \frac{n+1}{6n}\left(4n+2-3\right) = \frac{(n+1)(4n-1)}{6n}$$

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$$X(\Omega) = \{1, \dots, n\} \qquad X = \max\{X_1, X_2\}$$

$$\{X=j\} = \{X \le j\} \setminus \{X \le j-1\}$$

$$\mathbb{P}(X=j) = \mathbb{P}(X \le j) - \mathbb{P}(X \le j-1)$$

$$= \mathbb{P}(\max\{X_1, X_2\} \le j) - \mathbb{P}(\max\{X_1, X_2\} \le j-1)$$

$$= \mathbb{P}(X_1 \le j, X_2 \le j) - \mathbb{P}(X_1 \le j-1, X_2 \le j-1)$$

$$= \mathbb{P}(X_1 \le j)\mathbb{P}(X_2 \le j) - \mathbb{P}(X_1 \le j-1)\mathbb{P}(X_2 \le j-1)$$

$$= \frac{j}{n} \cdot \frac{j}{n} - \frac{j-1}{n}\frac{j-1}{n} = \frac{j^2 - (j^2 - 2j + 1)}{n^2} = \frac{2j-1}{n^2}$$

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DISTRIBUZIONE CONGIUNTA.

Siano X e Y v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ e supponiamo che siano entrambe discrete

$$X(\Omega) = \{x_i\}_{i \in I} \qquad Y(\Omega) = \{y_j\}_{j \in J}$$

$$(X,Y) : \omega \in \Omega \mapsto (X(\omega), Y(\omega)) \in \mathbb{R}^2$$

$$(X,Y)(\Omega) \subseteq X(\Omega) \times Y(\Omega) = \{(x_i, y_j) : i \in I, j \in J\}$$

$\Rightarrow$ (X,Y) è una funzione vettoriale la cui immagine è un insieme finito o numerabile

$$p_{ij} := \mathbb{P}(X = x_i, Y = y_j) \qquad (i,j) \in I \times J$$

$$p_i := \mathbb{P}(X = x_i) \qquad q_j := \mathbb{P}(Y = y_j)$$

PROP $\forall i \in I \qquad p_i = \sum_{j \in J} p_{ij}$

$\forall j \in J \qquad q_j = \sum_{i \in I} p_{ij}$

DIM $p_i = \mathbb{P}(X = x_i)$ $\qquad \{X = x_i\} = \bigcup_{j \in J} \{X = x_i, Y = y_j\}$

$$p_i = \mathbb{P}(X = x_i) = \mathbb{P}\left(\bigcup_{j \in J} \{X = x_i, Y = y_j\}\right) =$$

$$= \sum_{j \in J} \underbrace{\mathbb{P}(X = x_i, Y = y_j)}_{p_{ij}} = \sum_{j \in J} p_{ij}$$

DENSITÀ DISCRETA CONGIUNTA DI X E Y NEL PUNTO (x_i, y_j)