VARIANZA – MARKOV – CHEBICHEV – MODELLI DISCRETI

Note Title 19/10/2017

X v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ $\mathbb{E}[X] = \int_\Omega X(\omega)\mathbb{P}(d\omega) =$

$$= \frac{1}{\mathbb{P}(\Omega)} \int_\Omega X(\omega)\mathbb{P}(d\omega).$$

La varianza è definita:

Sia X v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ t.c. $\mathbb{E}[X]$ esiste finito.

Pongo

$$Var[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$

Se $\mathbb{E}(X)$ esiste finito $\Rightarrow$ $Var[X]$ è sicuramente ben definita e sicuramente $Var[X] \geq 0$.

$$Var[X] = \int_\Omega \left(X(\omega) - \mathbb{E}[X]\right)^2 \mathbb{P}(d\omega)$$

PROPRIETÀ ① $Var[X] \geq 0$; $Var[X] = 0$ SSE X è q.c. costante.

② $Var[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] =$

$$= \mathbb{E}\left[X^2 - 2\mathbb{E}[X]X + (\mathbb{E}[X])^2\right] =$$

$$= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + (\mathbb{E}[X])^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

③ $\alpha, \beta \in \mathbb{R}$ $Y := \alpha X + \beta$

$$\mathbb{E}[Y] = \mathbb{E}[\alpha X + \beta] = \alpha\mathbb{E}[X] + \mathbb{E}[\beta] =$$
$$= \alpha\mathbb{E}[X] + \beta$$

$$Var[\alpha X + \beta] = \mathbb{E}\left[\left((\alpha X + \beta) - \mathbb{E}[\alpha X + \beta]\right)^2\right] =$$

$$= \mathbb{E}\left[\left(\alpha X + \beta - \alpha\mathbb{E}[X] - \beta\right)^2\right] = \mathbb{E}\left[\alpha^2 (X - \mathbb{E}[X])^2\right] =$$

$$= \alpha^2 \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$

cioè $\quad Var[\alpha X + \beta] = \alpha^2 Var[X]$

$$\sigma(X) := \sqrt{Var[X]}$$ SCARTO QUADRATICO MEDIO

$$\sigma(\alpha X + \beta) = \sqrt{\alpha^2 Var[X]} = |\alpha| \sigma(X)$$

— ∘ —

DISUGUAGLIANZA DI MARKOV.

Sia $X : \Omega \to \mathbb{R}$ v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ e sia $I \subset \mathbb{R}$ intervallo t.c. $I \supset X(\Omega)$.

Sia $\varphi : I \to \mathbb{R}$ non negativa strettamente monotona crescente.

Allora

$$\varphi(t)\mathbb{P}(X > t) \leq \mathbb{E}[\varphi \circ X] \qquad \forall t \in I$$

DIM $\quad \varphi(t)\mathbb{P}(X > t) = \qquad \{X > t\} = \{\varphi \circ X > \varphi(t)\}$

$$\varphi(t)\, \mathbb{P}(\varphi \circ X > \varphi(t)) = \varphi(t) \int_\Omega \mathbb{1}_{\{\varphi \circ X > \varphi(t)\}}(\omega)\, \mathbb{P}(d\omega) =$$

$$= \int_\Omega \varphi(t)\, \mathbb{1}_{\{\varphi \circ X > \varphi(t)\}}(\omega)\, \mathbb{P}(d\omega)$$

① $\omega \in \{\varphi \circ X > \varphi(t)\}$

$$\varphi(X(\omega))\mathbb{1}_{\{\varphi \circ X > \varphi(t)\}}(\omega) > \varphi(t)\mathbb{1}_{\{\varphi \circ X > \varphi(t)\}}(\omega)$$

$$\leq \int_\Omega \varphi(X(\omega))\mathbb{1}_{\{\varphi \circ X > \varphi(t)\}}(\omega)\, \mathbb{P}(d\omega) \quad (*)$$

② $\omega \notin \{\varphi \circ X > \varphi(t)\}$

$$\varphi(X(\omega))\mathbb{1}_{\{\varphi \circ X > \varphi(t)\}}(\omega) \geq \varphi(t)\mathbb{1}_{\{\varphi \circ X > \varphi(t)\}}(\omega)$$

$$\varphi(X(\omega))\mathbb{1}_{\{\varphi \circ X > \varphi(t)\}}(\omega) \leq \varphi(X(\omega))$$

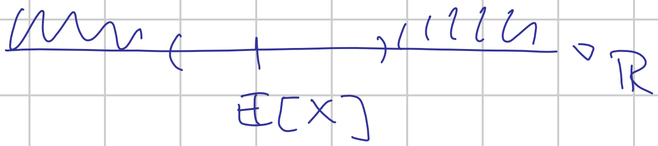
$(*)$
$$\leq \int_\Omega \varphi(X(\omega))\mathbb{P}(d\omega) = \mathbb{E}[\varphi \circ X] \qquad \boxtimes$$

DISUGUAGLIANZA DI CHEBYCHEV

Sia X v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ t.c. $\mathbb{E}[X]$ esiste finito.

Allora

$$\forall \delta > 0 \quad \mathbb{P}\left(|X - \mathbb{E}[X]| > \delta \right) \leq \frac{Var[X]}{\delta^2}$$

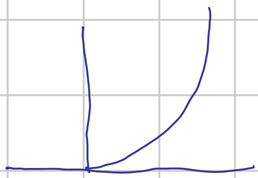


<u>DIM</u> Considero la disug di Markov

$$f(t)\,\mathbb{P}(Y > t) < \mathbb{E}[f \circ Y]$$

con $Y = |X - \mathbb{E}[X]|$ $\qquad I = [0, +\infty)$

$$f : t \in I = [0, +\infty) \longmapsto t^2 \in \mathbb{R}$$



$$t^2\, \mathbb{P}\left(|X - \mathbb{E}[X]| > t \right) < \mathbb{E}[Y^2] = \mathbb{E}\left[|X - \mathbb{E}[X]|^2 \right] = Var[X]$$

Per $t>0$ divido

$$\mathbb{P}\left(|X - \mathbb{E}[X]| > t \right) < \frac{Var[X]}{t^2}.$$

Es. 2 Foglio 2

$\mathbb{P}_X = f(x)dx$ _ $a, b \in \mathbb{R}$ $a \neq 0$ $\qquad Y := aX + b$

$f : x \in \mathbb{R} \mapsto ax + b \in \mathbb{R}$

$Y = f \circ X$

$\psi : \mathbb{R} \to \mathbb{R}$ funzione di Borel non negativa

$$\int_{\mathbb{R}} \psi(y)\, \mathbb{P}_Y(dy) = \int_{\mathbb{R}} \psi(y)\, \mathbb{P}_{f \circ X}(dy) = \int_{\mathbb{R}} \psi(f(x))\, \mathbb{P}_X(dx) =$$

$$= \int_{\mathbb{R}} \psi(ax + b) f(x)\, dx = \int_{-\infty}^{+\infty} \psi(ax + b) f(x)\, dx$$

$y = ax + b \qquad x = \frac{y - b}{a} \qquad dx = \frac{1}{a} dy$

$a>0 \qquad x\to -\infty \quad y\to -\infty$

$\qquad x\to +\infty \quad y\to +\infty$

$$= \int_{-\infty}^{+\infty} \varphi(y)\, f\left(\frac{y-b}{a}\right)\frac{1}{a}\,dy \implies \mathbb{P}_Y = \frac{1}{a} f\left(\frac{y-b}{a}\right) dy$$

$$\int_{\mathbb{R}} \varphi(y)\,\mathbb{P}_Y(dy) = \int_{-\infty}^{+\infty} \varphi(ax+b) f(x)\,dx =$$

$a<0$

$y = ax+b \qquad x = \frac{y-b}{a} \qquad x\to -\infty \quad y\to +\infty$

$\qquad x\to +\infty \quad y\to -\infty$

$$= \int_{+\infty}^{-\infty} \varphi(y)\, f\left(\frac{y-b}{a}\right)\frac{1}{a}\,dy \qquad dx = \frac{1}{a}\,dy$$

$$= \int_{-\infty}^{+\infty} \varphi(y)\, f\left(\frac{y-b}{a}\right)\frac{1}{-a}\,dy \qquad \mathbb{P}_Y = \frac{1}{-a} f\left(\frac{y-b}{a}\right) dy \quad a<0$$

$$\mathbb{P}_Y = \frac{1}{a} f\left(\frac{y-b}{a}\right) dy \quad a>0$$

$$\int_{\mathbb{R}} \varphi(y)\, g(y)\,dy$$

$$\mathbb{P}_Y = \frac{1}{|a|} f\left(\frac{y-b}{a}\right) dy \qquad \forall b\in\mathbb{R},\ \forall a\neq 0$$

$$\mathbb{P}_Y = g(y)\,dy \qquad g(y) = \frac{1}{|a|} f\left(\frac{y-b}{a}\right)$$

ESERCIZIO 8

$$\mathbb{P}_X = f(x)\,dx \qquad f(x) = \begin{cases} c & x\in(0,a) \\ 0 & x\notin(0,a) \end{cases}$$

$$1 = \int_{\mathbb{R}} f(x)\,dx = \int_0^a c\,dx = ca \implies c = \frac{1}{a}$$

$$\text{"}P(X\in\mathbb{R})\text{"} \qquad f(x) = \begin{cases} \frac{1}{a} & x\in(0,a) \\ 0 & x\notin(0,a) \end{cases}$$

$Y := \sqrt{X} \qquad Z := X^2$

$Y = \varphi \circ X \qquad \varphi(x) = \begin{cases} 0 & x < 0 \\ \sqrt{x} & x \geq 0 \end{cases}$ ⟵

$\mathbb{P}(X \leq 0) = \int_{-\infty}^{0} f(x)\,dx = \int_{-\infty}^{0} 0\,dx = 0$

$\psi : \mathbb{R} \to \mathbb{R}$ funzione di Borel nonnegativa

$\int_{\mathbb{R}} \psi(y)\,\mathbb{P}_{\varphi \circ X}(dy) = \int_{\mathbb{R}} \psi(\varphi(x))\,\mathbb{P}_X(dx) = \int_{\mathbb{R}} \psi(\varphi(x)) f(x)\,dx$

$= \int_0^2 \psi(\varphi(x)) \frac{1}{2}\,dx = \int_0^2 \psi(\sqrt{x}) \frac{1}{2}\,dx$

$y = \sqrt{x} \qquad x = y^2 \qquad dx = 2y\,dy$

$x = 0 \qquad y = 0$

$x = 2 \qquad y = \sqrt{2}$

$= \int_0^{\sqrt{2}} \psi(y) \frac{1}{2} 2y\,dy = \int_{\mathbb{R}} \psi(y) g(y)\,dy$

$g(y) = \begin{cases} \frac{2y}{2} & y \in (0, \sqrt{2}), \\ 0 & \text{altrimenti} \end{cases}$

$\mathbb{P}_Y = g(y)\,dy$

$\mathbb{E}[Y] \qquad \mathrm{Var}[Y]$

$\mathbb{E}[Y] = \int_{\Omega} Y(\omega)\,\mathbb{P}(d\omega) = \int_{\mathbb{R}} y\,\mathbb{P}_Y(dy) = \int_{\mathbb{R}} y\,g(y)\,dy =$

$= \int_0^{\sqrt{2}} y \cdot \frac{2y}{2}\,dy = \frac{2}{2} \int_0^{\sqrt{2}} y^2\,dy = \frac{2}{2} \frac{y^3}{3} \Big|_{y=0}^{y=\sqrt{2}} =$

$= \frac{2}{3}\sqrt{2}$

$Y := \sqrt{X}$ $\qquad \varphi(x) = \begin{cases} 0 & x \le 0 \\ \sqrt{x} & x > 0 \end{cases}$ $\qquad f(t) = \begin{cases} \frac{1}{2} & t \in (0,2) \\ 0 & t \notin (0,2) \end{cases}$

$$\mathbb{E}[Y] = \mathbb{E}[\varphi \circ X] =$$

$$= \int_{\mathbb{R}} \varphi(t)\, \mathbb{P}_X(dt) = \int_{\mathbb{R}} \varphi(t) f(t)\, dt$$

$$= \int_0^2 \varphi(t) \frac{1}{2}\, dt = \int_0^2 \frac{1}{2} \sqrt{t}\, dt = \frac{1}{2} \frac{2}{3} t^{3/2} \Big|_{t=0}^{t=2} =$$

$$= \frac{2}{3}\sqrt{2} - 0 = \frac{2}{3}\sqrt{2}$$

$$\mathrm{Var}[Y] := \mathbb{E}\left[(Y - \mathbb{E}[Y])^2\right] = \mathbb{E}[Y^2] - \left(\mathbb{E}[Y]\right)^2.$$

$$= \mathbb{E}[Y^2] - \left(\frac{2}{3}\sqrt{2}\right)^2$$

$$\mathbb{E}[Y^2] = \int_{\mathbb{R}} y^2\, \mathbb{P}_Y(dy) = \int_{\mathbb{R}} y^2 g(y)\, dy$$

$$g(y) = \begin{cases} \frac{2y}{2} & y \in (0, \sqrt{2}) \\ 0 & y \notin (0, \sqrt{2}) \end{cases}$$

$$= \int_0^{\sqrt{2}} y^2 \frac{2y}{2}\, dy = \frac{2}{2} \frac{y^4}{4} \Big|_{y=0}^{y=\sqrt{2}} = \frac{1}{22} y^5 \Big|_{y=0}^{y=\sqrt{2}} = \frac{1}{22} 2^2 = \frac{2}{2}$$

$$\mathrm{Var}[Y] = \frac{2}{2} - \frac{4}{9} 2 = 2\left(\frac{1}{2} - \frac{4}{9}\right) = 2\, \frac{9-8}{18} = \frac{2}{18}$$

$\mathbb{E}[Y^2] =$ $\qquad Y := \sqrt{X}$

$$Y^2 = (\sqrt{X})^2 = X$$

$$\mathbb{E}[Y^2] = \mathbb{E}[X] = \int_{\mathbb{R}} x\, \mathbb{P}_X(dx) = \int_{\mathbb{R}} x f(x)\, dx =$$

$$f(x) = \begin{cases} \frac{1}{2} & x \in (0,2) \\ 0 & x \notin (0,2) \end{cases}$$

$$= \int_0^2 x \cdot \frac{1}{2}\, dx = \frac{1}{2} \frac{x^2}{2} \Big|_{x=0}^{x=2} = \frac{1}{2} \frac{2^2}{2} = \frac{2}{2}$$

$$Z := X^2$$

$$\mathbb{P}_X = f(x)\,dx \quad \Rightarrow \quad \mathbb{P}_{X^2} = g(z)\,dz$$

$$g(z) = \begin{cases} 0 & z \le 0 \\ \frac{1}{2\sqrt{z}}\left(f(\sqrt{z}) + f(-\sqrt{z})\right) & z > 0 \end{cases} \quad !$$

$$f(x) = \begin{cases} \frac{1}{a} & x \in (0, a) \\ 0 & x \notin (0, a) \end{cases}$$

$$-\sqrt{z} \notin (0, a) \quad \Rightarrow \quad f(-\sqrt{z}) = 0 \quad \forall z$$

$$\sqrt{z} \in (0, a) \qquad 0 < \sqrt{z} < a \qquad 0 < z < a^2$$

$$f(\sqrt{z}) = \frac{1}{a}$$

$$z \ge a^2 \qquad g(z) = 0$$

$$z \in (0, a^2) \qquad g(z) = \frac{1}{2\sqrt{z}}\left(\frac{1}{a} + 0\right) = \frac{1}{2a\sqrt{z}}$$

$$Z = X^2 \qquad g(z) = \begin{cases} 0 & z \le 0,\ z \ge a^2 \\ \frac{1}{2a\sqrt{z}} & z \in (0, a^2) \end{cases}$$

$$\mathbb{E}[Z] = \int_{\mathbb{R}} z\,\mathbb{P}_Z(dz) = \int_{\mathbb{R}} z\,g(z)\,dz \quad (1)$$

$$\mathbb{E}[Z] = \mathbb{E}[X^2] = \int_{\mathbb{R}} x^2\,\mathbb{P}_X(dx) = \int_{\mathbb{R}} x^2 f(x)\,dx$$

(1) $$\mathbb{E}[Z] = \int_{\mathbb{R}} z\,g(z)\,dz = \int_0^{a^2} z\,\frac{1}{2a}\,\frac{1}{\sqrt{z}}\,dz =$$

$$= \frac{1}{2a}\int_0^{a^2} z^{1/2}\,dz = \frac{1}{2a}\,\frac{2}{3}\,z^{3/2}\Big|_{z=0}^{z=a^2} =$$

$$= \frac{1}{3a}\,a^3 = \frac{a^2}{3}$$

$$\mathrm{Var}[Z] = \mathbb{E}[(Z-\mathbb{E}[Z])^2] = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2$$
$$= \mathbb{E}[Z^2] - \left(\frac{a^2}{3}\right)^2$$

$$\mathbb{E}[Z^2] = \int_{\mathbb{R}} z^2 \, \mathbb{P}_Z dz = \int_{\mathbb{R}} z^2 g(z) \, dz \quad (*)$$

$$\mathbb{E}[Z^2] = \mathbb{E}[X^4] = \int_{\mathbb{R}} x^4 \, \mathbb{P}_X(dx) = \int_{\mathbb{R}} x^4 f(x) \, dx$$

$$\mathbb{E}[Z^2] = \int_0^{a^2} z^2 \frac{1}{2a\sqrt{z}} \, dz = \frac{1}{2a} \int_0^{a^2} z^{3/2} \, dz =$$
$$= \frac{1}{2a} \frac{2}{5} z^{5/2} \Big|_{z=0}^{z=a^2} = \frac{1}{5a} a^5 = \frac{a^4}{5}$$

$$\mathrm{Var}[Z] = \frac{a^4}{5} - \left(\frac{a^2}{3}\right)^2 = \frac{a^4}{5} - \frac{a^4}{9} = \frac{a^4(9-5)}{45} =$$
$$= \frac{4}{45} a^4$$

ESEMPI DI DISTRIBUZIONI DISCRETE

DISTRIBUZIONE DI DIRAC

È la distribuzione discreta concentrata in un solo punto x_0

$$\delta_{x_0}(A) = \begin{cases} 1 & x_0 \in A \\ 0 & x_0 \notin A \end{cases}$$

$$\{t_j\}_{j\in J} = \{x_0\} \qquad p_0 = \mathbb{P}(X = x_0) = 1$$

$$\mathbb{P}_X(A) = \sum_{j \in J} t_j p_j$$

$$(\Omega, \mathcal{E}, \mathbb{P}) \qquad X : \omega \in \Omega \longmapsto x_0 \in \mathbb{R}$$

$$\mathbb{E}[X] = x_0 \mathbb{P}(X = x_0) = x_0 \cdot 1 = x_0$$

$$\mathrm{Var}[X] = 0$$

DISTRIBUZIONE DI BERNOULLI DI PARAMETRO $p \in [0,1]$

$\{t_j\}_{j \in J} = \{0, 1\}$ $\quad B(p)$

$p_0 = \mathbb{P}_X(\{0\}) = B(p)(\{0\}) = 1-p$

$p_1 = \mathbb{P}_X(\{1\}) = B(p)(\{1\}) = p$

Sia X v.a. t.c. $\mathbb{P}_X = B(p)$

$\mathbb{E}[X] = 0 \cdot \mathbb{P}(X=0) + 1\,\mathbb{P}(X=1) = 0(1-p) + 1 \cdot p = p$

$Var[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathbb{E}[X^2] - p^2$

$= 0\,\mathbb{P}(X^2=0) + 1 \cdot \mathbb{P}(X^2=1) - p^2$

$= \mathbb{P}(X=1) - p^2 = p - p^2 = p(1-p)$

— o —

Esce Testa con probabilità p

esce Croce con probabilità $1-p$

$T \longleftrightarrow 1$

$C \longleftrightarrow 0$

$\Omega = \{0, 1\} \qquad \mathcal{E} = \mathcal{P}(\Omega)$

$\mathbb{P}(\{0\}) = 1-p$

$\mathbb{P}(\{1\}) = p$

$X : \omega \in \Omega \mapsto X(\omega) \in \mathbb{R}$

$X(\omega) = \omega$

$\mathbb{P}(X=0) = \mathbb{P}(\{\omega = 0\}) = \mathbb{P}(\{0\}) = 1-p$

$\mathbb{P}(X=1) = \mathbb{P}(\{\omega = 1\}) = \mathbb{P}(\{1\}) = p$

— o —

$\Omega = \{1, \ldots, 6\} \qquad \mathcal{E} = \mathcal{P}(\Omega) \qquad \mathbb{P}(\{\omega\}) = \frac{1}{6}$

$X : \omega \in \Omega \mapsto X(\omega) \in \mathbb{R}$

$X(\omega) = \mathbb{1}_{\{1\}}(\omega)$

DISTRIBUZIONE BINOMIALE DI PARAMETRI $n \in \mathbb{N}$ E $p \in [0,1]$

È la distribuzione discreta $B(n,p)$ concentrata

$\{0, 1, \dots n\}$ t.c.

$$B(n,p)(\{k\}) = \mathbb{P}_X(\{k\}) = \binom{n}{k} p^k (1-p)^{n-k} \quad \forall k = 0, 1, \dots, n$$

$$\sum_{k=0}^{n} \mathbb{P}_X(\{k\}) \overset{?}{=} 1$$

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \overset{?}{=} 1$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \quad \Leftarrow \text{ si dimostra per induzione}$$

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1^n = 1$$

$T \leftrightarrow 1$

$C \leftrightarrow 0$

$$\Omega = \{(\omega_1 \dots \omega_n) : \omega_i \in \{0,1\}\} = \{0,1\}^n$$

$$\mathcal{E} = \mathcal{P}(\Omega)$$

$$\mathbb{P}(\{(\omega_1 \dots \omega_n)\})$$

$n = 2$ $\quad \{(\bar\omega_1, \bar\omega_2)\} = \{\omega : \omega_1 = \bar\omega_1\} \cap \{\omega : \omega_2 = \bar\omega_2\}$

$$(\bar\omega_1, \bar\omega_2) = (1, 1)$$

$$\mathbb{P}(\{(\bar\omega_1, \bar\omega_2)\}) = \mathbb{P}(\{\omega_1 = \bar\omega_1\}) \, \mathbb{P}(\{\omega_2 = \bar\omega_2\})$$
$$= p \cdot p = p^2$$

$$(\bar\omega_1, \bar\omega_2, \dots \bar\omega_n) \in \Omega$$

$$\mathbb{P}(\{(\bar\omega_1 \dots \bar\omega_n)\})$$

$$\{(\bar\omega_1 \_ \bar\omega_n)\} = \bigcap_{i=1}^{n} \{\omega : \omega_i = \bar\omega_i\}$$

$$\mathbb{P}(\{(\bar\omega_1 \_ \bar\omega_n)\}) = \prod_{i=1}^{n} \mathbb{P}(\{\omega : \omega_i = \bar\omega_i\})$$

$$= p^k (1-p)^{n-k}$$

$k :=$ # componenti di $\bar\omega = (\bar\omega_1 \_ \bar\omega_n)$ che valgono 1

$$X : \omega = (\omega_1 \_ \omega_n) \in \Omega \longmapsto \sum_{i=1}^{n} \omega_i \in \mathbb{R}$$

$$X(\Omega) = \{0, 1, 2, \_ n\}$$

$k \in \{0, 1, \_ n\}$ $\{X = k\} = \{\omega \in \Omega : \omega$ ha k componenti che valgono 1, $n-k$ componenti che valgono $0\}$

$\omega \in \{X = k\}$

$$\mathbb{P}(\{\omega\}) = p^k (1-p)^{n-k}$$

$$\{X = k\} = \bigcup_{\omega \in \{X=k\}} \{\omega\}$$

$$\mathbb{P}(X = k) = \sum_{\omega \in \{X=k\}} \mathbb{P}(\{\omega\}) = \sum_{\omega \in \{X=k\}} p^k (1-p)^{n-k} =$$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mathbb{E}[X] = \sum_{k=0}^{n} k \, \mathbb{P}(X = k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$X_i : \omega = (\omega_1 \_ \omega_n) \longmapsto \omega_i \in \mathbb{R}$$

? $X = \sum_{i=1}^{n} X_i$? $X(\omega) = \sum_{i=1}^{n} \omega_i = \sum_{i=1}^{n} X_i(\omega)$

$\forall \omega \in \Omega$

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i]$$

$$\mathbb{P}(X_i = 1) = p \iff \mathbb{P}(X_i = 0) = 1 - p$$

$$\mathbb{P}_{X_i} = \mathcal{B}(p) \qquad \mathbb{E}[X_i] = p$$

$$\mathbb{E}[X] = \sum_{i=1}^{n} p = np.$$

NON È VERA IN GENERALE

Si dimostra che $\mathrm{Var}[X] = \sum_{i=1}^{n} \mathrm{Var}[X_i] = np(1-p)$

DISTRIBUZIONE DI POISSON DI PARAMETRO $\lambda > 0$

È la distribuzione concentrata sugli interi non negativi

Si indica $\mathcal{P}(\lambda)$

$$\mathbb{P}(X = k) = \mathcal{P}(\lambda)(\{k\}) = e^{-\lambda} \frac{\lambda^k}{k!} \qquad \forall k = 0, 1, 2, \ldots$$

$$\sum_{k=0}^{\infty} \mathcal{P}(\lambda)(\{k\}) \stackrel{?}{=} 1$$

$$\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda} = 1$$

Se X è una v.a. t.c. $\mathbb{P}_X = \mathcal{P}(\lambda)$

$\mathbb{E}[X]$?

$\mathrm{Var}[X]$?

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k\, \mathbb{P}(X = k) = \sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1} k e^{-\lambda} \frac{\lambda^k}{k!} =$$

$$= \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{(k-1)!} = e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^{j+1}}{j!} = e^{-\lambda} \lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} =$$

$k - 1 = j \qquad k = j + 1$

$$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda} = \lambda$$

$$\mathbb{E}[X^2] = \sum_{k=0}^{\infty} k^2 \mathbb{P}(X=k) = \sum_{k=0}^{\infty} k^2 e^{-\lambda} \frac{\lambda^k}{k!} =$$

$$= \sum_{k=1}^{\infty} k^2 e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{k \lambda^k}{(k-1)!} =$$

$$= e^{-\lambda} \sum_{k=1}^{\infty} \frac{(k-1+1)\lambda^k}{(k-1)!} =$$

$$= e^{-\lambda} \left\{ \sum_{k=1}^{\infty} \frac{(k-1)\lambda^k}{(k-1)!} + \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \right\} =$$

$$= e^{-\lambda} \left\{ \sum_{k=2}^{\infty} \frac{(k-1)\lambda^k}{(k-1)!} + \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \right\}$$

$$= e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^k}{(k-2)!} + \underbrace{e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}}_{\lambda} =$$

$k-2=j$

$$= e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^{j+2}}{j!} + \lambda =$$

$$= e^{-\lambda} \lambda^2 \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} + \lambda = \cancel{e^{-\lambda}} \cdot \lambda^2 \cdot \cancel{e^{\lambda}} + \lambda = \lambda^2 + \lambda$$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

— o —

Sia $\{p_n\}_{n\in\mathbb{N}} \subset [0,1]$ $\exists \lim_{n\to\infty} n p_n = \lambda > 0$

Per $k \in \mathbb{N}$ $\quad B(n, p_n)(\{k\}) = \begin{cases} 0 & k > n \\ \binom{n}{k} p_n^k (1-p_n)^{n-k} & 0 \le k \le n \end{cases}$

Per $k \in \mathbb{N}$ fisso

$$\lim_{n\to\infty} B(n, p_n)(\{k\}) = \lim_{n\to\infty} \binom{n}{k} p_n^k (1-p_n)^{n-k}$$

$$\frac{n!}{k!(n-k)!} p_n^k (1-p_n)^{n-k} =$$

$$= \frac{1}{k!} \frac{\overbrace{n(n-1)\cdots(n-k+1)}^{k \text{ fattori}:\ n^k + P_{n-1}(k)}}{n^k} n^k p_n^k (1-p_n)^{n-k}$$

$$= \frac{1}{k!} \frac{n^k + P_{n-1}(k)}{n^k} (np_n)^k \left[\left(1-p_n\right)^{\frac{1}{p_n}}\right]^{p_n(n-k)}$$

$$p_n(n-k) = np_n - kp_n \to 0$$

$$\downarrow$$

$$\lambda$$

$$\lim_{x \to 0} (1-x)^{\frac{1}{x}} = e^{-1}$$

$$\longrightarrow \frac{1}{k!} \cdot 1 \cdot \lambda^k \left[e^{-1}\right]^{\lambda} = e^{-\lambda} \frac{\lambda^k}{k!} = P(\lambda)(\{k\})$$