Calcolo delle Probabilità –2016-2017

Quarto Appello – 7 luglio 2017

Domanda 1) (9 punti) Si ha un'urna $b \geq 2$ palline bianche e $r \geq 2$ palline rosse. Si estraggono (senza reimbussolamento) 2 palline.
Per ogni pallina bianca estratta si lancia una moneta su cui esce testa con probabilità $\frac{1}{3}$.
Per ogni pallina rossa estratta si lancia una moneta su cui esce testa con probabilità $\frac{2}{3}$.
Calcolare la probabilità di ottenere 2 teste nel lancio delle monete.
Sapendo di aver ottenuto 2 teste, calcolare la probabilità di aver estratto solo palline bianche.

$\dfrac{b(b-1)+4br+4r(r-1)}{9(b+r)(b+r-1)}$, $\dfrac{b(b-1)}{b(b-1)+4br+4r(r-1)}$

Svolgimento TT = estraggo 2 teste

$$\mathbb{P}(TT) = \mathbb{P}(TT|BB)\mathbb{P}(BB) + \mathbb{P}(TT|BR)\mathbb{P}(BR) + \mathbb{P}(TT|RR)\mathbb{P}(RR)$$

$$\mathbb{P}(TT|BB) = \binom{2}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^0 = \left(\frac{1}{3}\right)^2 \qquad \mathbb{P}(BB) = \frac{\binom{b}{2}\binom{r}{0}}{\binom{b+r}{2}} = \frac{b(b-1)}{(b+r)(b+r-1)}$$

$$\mathbb{P}(TT|BR) = \frac{1}{3}\cdot\frac{2}{3} \qquad \mathbb{P}(BR) = \frac{\binom{b}{1}\binom{r}{1}}{\binom{b+r}{2}} = \frac{2br}{(b+r)(b+r-1)}$$

$$\mathbb{P}(TT|RR) = \binom{2}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^0 = \left(\frac{2}{3}\right)^2 \qquad \mathbb{P}(RR) = \frac{\binom{b}{0}\binom{r}{2}}{\binom{b+r}{2}} = \frac{r(r-1)}{(b+r)(b+r-1)}$$

$$\Rightarrow \mathbb{P}(TT) = \left(\frac{1}{3}\right)^2\frac{b(b-1)}{(b+r)(b+r-1)} + \frac{1}{3}\frac{2}{3}2\frac{br}{(b+r)(b+r-1)} + \left(\frac{2}{3}\right)^2\frac{r(r-1)}{(b+r)(b+r-1)}$$

$$= \frac{1}{9(b+r)(b+r-1)}\left(b(b-1)+4br+4r(r-1)\right)$$

$$\mathbb{P}(BB|TT) = \frac{\mathbb{P}(TT|BB)\mathbb{P}(BB)}{\mathbb{P}(TT)} = \frac{1}{\cancel{9}}\frac{b(b-1)}{\cancel{(b+r)(b+r-1)}}\;\frac{\cancel{9}\cancel{(b+r)(b+r-1)}}{b(b-1)+4br+4r(r-1)}$$

Domanda 2) (9 punti) La v.a. (X, Y) ha distribuzione assolutamente continua con densità

$$f(x,y) = \begin{cases} c & x^2+y^2 \le 1, \quad 0 < y < |x|, \\ 2c & x^2+y^2 \le 1, \quad y \ge |x|, \\ 0 & \text{altrimenti.} \end{cases}$$

Calcolare la distribuzione congiunta e le distribuzioni marginali della v.a. $(R, T) := \left(\sqrt{X^2+Y^2}, \frac{Y}{X}\right)$.

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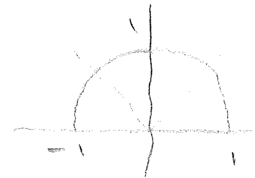
Svolgimento

$$1 = \int_{\mathbb{R}^2} f(x,y)\,dx\,dy = \frac{\pi}{8}c + \frac{\pi}{4}2c + \frac{\pi}{8}c = c\,\frac{\pi}{4}3 \Rightarrow$$



$$c = \frac{4}{3\pi}$$

$(R,T) = \varphi \circ (X,Y)$ dove $\varphi : (x,y) \in \mathbb{R}^2 \mapsto \left(\sqrt{x^2+y^2}, \frac{y}{x}\right) \in \mathbb{R}^2$

ψ funzione test

$$\int_{\mathbb{R}^2} \psi(r,t)\,\mathbb{P}_{R,T}(dr\,dt) = \int_{\mathbb{R}^2} (\psi \circ \varphi)(x,y)\,\mathbb{P}_{X,Y}(dx\,dy) = \int_{\mathbb{R}^2} \psi\left(\sqrt{x^2+y^2}, \frac{y}{x}\right) f(x,y)\,dx\,dy$$

= coordinate polari $x = \rho\cos\theta$, $y = \rho\sin\theta$

$$= \int_0^1 d\rho \int_0^{\pi/4} \rho\,\psi(\rho, \operatorname{tg}\theta)\frac{4}{3\pi}\,d\theta + \int_0^1 d\rho \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \rho\,\psi(\rho, \operatorname{tg}\theta)\frac{8}{3\pi}\,d\theta +$$

$$+ \int_0^1 d\rho \int_{\frac{3}{4}\pi}^{\pi} \rho\,\psi(\rho, \operatorname{tg}\theta)\frac{4}{3\pi}\,d\theta \qquad \boxed{t = \operatorname{tg}\theta}$$

$$\int_{(0,1)\times(0,1)} \rho\frac{4}{3\pi}\psi(\rho,t)\frac{1}{1+t^2}\,d\rho\,dt + \int_{(0,1)\times\left((-\infty,-1)\cup(1,+\infty)\right)} \rho\frac{8}{3\pi}\psi(\rho,t)\frac{1}{1+t^2}\,d\rho\,dt + \int_{(0,1)\times(-1,0)} \rho\frac{4}{3\pi}\psi(\rho,t)\frac{1}{1+t^2}\,d\rho\,dt$$

$$= \int_{\mathbb{R}^2} \psi(\rho,t)\,g(\rho,t)\,d\rho\,dt$$

$$g(\rho,t) = \begin{cases} \dfrac{4\rho}{\pi 3(1+t^2)} & (\rho,t) \in (0,1)\times(-1,1) \\ \dfrac{8\rho}{\pi 3(1+t^2)} & (\rho,t) \in (0,1)\times\left(\mathbb{R}\setminus(-1,1)\right) \\ 0 & \rho \ge 1,\ \rho \le 0 \end{cases}$$

$h(e)$ densità di R

$$h(e) = \int_{\mathbb{R}} g(e,t)\,dt \Rightarrow h(e) = 0 \quad \text{per } e \geq 1 \vee e \geq 0$$

Per $e \in (0,1)$
$$h(e) = \int_{-1}^{1} \frac{4e}{\pi 3(1+t^2)}\,dt + \int_{(-\infty,-1)\cup(1,+\infty)} \frac{8e}{\pi 3(1+t^2)}\,dt =$$

$$= \int_0^1 \frac{8e}{\pi 3(1+t^2)}\,dt + \int_1^{+\infty} \frac{16e}{\pi 3(1+t^2)}\,dt$$

$$= \frac{8e}{3\pi} \operatorname{atan}(t)\Big|_{t=0}^{t=1} + \frac{16e}{3\pi} \operatorname{atan}(t)\Big|_{t=1}^{t\to+\infty}$$

$$\frac{\cancel{8}^{2} e}{3\cancel{\pi}} \frac{\cancel{\pi}}{\cancel{4}} + \frac{16e}{3\pi}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{2}{3}e + \frac{\cancel{16}^{4}}{3} e \frac{1}{\cancel{4}} = 2e$$

$$h(e) = 2e\,\mathbb{1}_{(0,1)}(e)$$

$k(t)$ densità di T

$$k(t) = \int_{\mathbb{R}} g(e,t)\,de = \begin{cases} \int_0^1 \frac{4e}{3\pi(1+t^2)}\,de & t \in (-1,1) \\ \int_0^1 \frac{8e}{3\pi(1+t^2)}\,de & t \in \mathbb{R} \setminus (-1,1) \end{cases}$$

$$= \begin{cases} \frac{2}{3\pi(1+t^2)} & t \in (-1,1) \\ \frac{4}{3\pi(1+t^2)} & t \in \mathbb{R} \setminus (-1,1) \end{cases}$$

Domanda 3) (9 punti) Le v.a. X_1 e X_2 sono i.i.d. e seguono la distribuzione di Poisson di parametro λ. Sia $Y := X_1 + X_2$. Calcolare $\mathbb{E}[X_1|Y]$.

$$\sum_{k=0}^{\infty} \frac{k}{2} \mathbb{1}_{\{Y=k\}}(\omega)$$

Svolgimento Sappiamo che $\mathbb{P}_Y = \mathcal{P}(2\lambda)$ $\qquad \mathbb{P}(Y=k) = e^{-2\lambda} \dfrac{(2\lambda)^k}{k!}$

$$\mathbb{E}[X_1|Y](\omega) = \sum_{k=0}^{\infty} \mathbb{E}[X_1|Y=k] \mathbb{1}_{\{Y=k\}}(\omega)$$

$$\mathbb{E}[X_1|Y=k] = \frac{1}{\mathbb{P}(Y=k)} \int_{\Omega} X_1(\omega) \mathbb{1}_{\{Y=k\}}(\omega) \mathbb{P}(d\omega) =$$

$$X_1(\omega) = \sum_{j=0}^{\infty} j \mathbb{1}_{\{X_1=j\}}(\omega) \implies$$

$$\mathbb{E}[X_1|Y=k] = \frac{e^{2\lambda} k!}{(2\lambda)^k} \int_{\Omega} \sum_{j=0}^{\infty} j \mathbb{1}_{\{X_1=j, Y=k\}}(\omega) \mathbb{P}(d\omega) =$$

$$= \frac{e^{2\lambda} k!}{(2\lambda)^k} \sum_{j=0}^{\infty} j \mathbb{P}(X_1=j, Y=k) = \frac{e^{2\lambda} k!}{(2\lambda)^k} \sum_{j=0}^{\infty} j \mathbb{P}(X_1=j, X_2=k-j)$$

$$= \frac{e^{2\lambda} k!}{(2\lambda)^k} \sum_{j=0}^{k} j \mathbb{P}(X_1=j) \mathbb{P}(X_2=k-j) = \frac{\cancel{e^{2\lambda}} k!}{(2\cancel{\lambda})^k} \sum_{j=0}^{k} j \frac{\cancel{e^{-\lambda}} \cancel{\lambda^j}}{j!} \frac{\cancel{e^{-\lambda}} \cancel{\lambda^{k-j}}}{(k-j)!}$$

$$= \frac{1}{2^k} \sum_{j=0}^{k} j \binom{k}{j} = \frac{1}{2^k} \sum_{j=1}^{k} j \binom{k}{j} x^{j-1} \Big|_{x=1}$$

$$= \frac{1}{2^k} \left(\frac{d}{dx} \sum_{j=0}^{\infty} \binom{k}{j} x^j \right) \Big|_{x=1} = \frac{1}{2^k} \frac{d}{dx} (1+x)^k \Big|_{x=1} = \frac{1}{2^k} k (1+x)^{k-1} \Big|_{x=1} = \frac{k}{2}$$

$$\implies \mathbb{E}[X_1|Y](\omega) = \sum_{k=0}^{\infty} \frac{k}{2} \mathbb{1}_{\{Y=k\}}(\omega)$$

Domanda 4) (5 punti) La v.a. X è uniformemente distribuita sull'intervallo $[-L, L]$. Al variare di n tra gli interi positivi, calcolare distribuzione, valore atteso e varianza della v.a. $Y := X^n$.

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Svolgimento

$Y = \varphi \circ X \qquad \varphi(t) = t^n$

$$\int_{\mathbb{R}} \psi(y)\, \mathbb{P}_Y(dy) = \int_{\mathbb{R}} \psi(\varphi(t))\, \mathbb{P}_X(dt) = \int_{-L}^{L} \psi(t^n) \frac{1}{2L}\, dt$$

1) n pari $\Rightarrow$ " $= \int_0^L \psi(t^n) \frac{1}{L}\, dt \qquad y = t^n \qquad t = y^{1/n}$

$$= \int_0^{L^n} \psi(y) \frac{1}{L} \frac{1}{n} y^{\frac{1}{n}-1}\, dy \Rightarrow \mathbb{P}_Y = g(y)dy \qquad g(y) = \frac{y^{\frac{1-n}{n}}}{nL} \mathbb{1}_{[0,L^n]}(y)$$

$$\mathbb{E}[Y] = \mathbb{E}[X^n] = \int_{-L}^{L} \frac{1}{2L} x^n dx = \cancel{2} \int_0^L \frac{x^n}{\cancel{2}L} dx = \frac{L^{n+1}}{L(n+1)} = \frac{L^n}{n+1}$$

$$\mathbb{E}[Y^2] = \mathbb{E}[X^{2n}] = \frac{L^{2n}}{2n+1} \Rightarrow \text{Var}[Y] = L^{2n}\left(\frac{1}{2n+1} - \frac{1}{(n+1)^2}\right)$$

$$= \frac{n^2 L^{2n}}{(2n+1)(n+1)^2}$$

2) n dispari $= \int_{-L^n}^{L^n} \psi(y) \frac{1}{2L} \frac{1}{n} y^{\frac{1}{n}-1}\, dy \Rightarrow \mathbb{P}_Y = g(y)dy$

$$g(y) = \frac{y^{\frac{1-n}{n}}}{2nL} \mathbb{1}_{[-L^n,L^n]}(y)$$

$\mathbb{E}[Y] = 0$

$\Rightarrow \text{Var}[Y] = \mathbb{E}[Y^2] = \dfrac{L^{2n}}{2n+1}$