SOMMA m V.A INDIPENDENTI _ ESERCIZI

Note Title 30/11/2016

X, Y v.a indipendenti su $(\Omega, \mathcal{E}, \mathbb{P})$

$\mathbb{P}_X = B(n, p) \quad \mathbb{P}_Y = B(m, p)$

$k \in \mathbb{Z} \quad \mathbb{P}(X+Y=k) = \sum_{j=-\infty}^{+\infty} \mathbb{P}(X=j)\,\mathbb{P}(Y=k-j)$

$$= \sum_{j=0}^{k} \binom{n}{j} \cancel{p^{j}} (1-p)^{n-\cancel{j}} \binom{m}{k-j} p^{k-\cancel{j}} (1-p)^{m-(k-\cancel{j})}$$

$$= p^k (1-p)^{n+m-k} \sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j}$$

$k \geq 0$

$$\sum_{j=0}^{\infty} \binom{n}{j} z^j = (1+z)^n$$

$$\sum_{j=0}^{\infty} \binom{m}{j} z^j = (1+z)^m$$

$$(1+z)^n \cdot (1+z)^m = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} \right) z^k$$

$$\| \qquad\qquad \overline{=}$$

$$(1+z)^{n+m} = \sum_{k=0}^{\infty} \binom{n+m}{k} z^k$$

$$\Rightarrow \mathbb{P}(X+Y=k) = \binom{n+m}{k} p^k (1-p)^{n+m-k} \qquad k=0,\dots,n+m$$

$$\mathbb{P}_{X+Y} = B(n+m, p).$$

— o —

monete su cui esce Testa con probabilità p a ciascun lancio

Faccio n lanci.

$(\Omega, \mathcal{E}, \mathbb{P}) \qquad \Omega = \{0,1\}^n \qquad$ "1" = Testa

"0" = croce

$\mathcal{E} = \mathcal{P}(\Omega) \qquad \mathbb{P}(\{(\omega_1, \dots, \omega_n)\}) = p^{\#\text{"}1\text{"}} (1-p)^{n-\#\text{"}1\text{"}}$

$= p^{\sum \omega_i} (1-p)^{n-\sum \omega_i}$

$i=1 \,\_\_\_\, n \qquad X_i(\omega)=\omega_i \qquad \mathbb{P}_{X_i}=B(p) \quad \forall i=1\_n$

$X = \#$ Totale di Teste ottenute

$$X(\omega) = \sum_{i=1}^{n} \omega_i = \sum_{i=1}^{n} X_i(\omega)$$

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} p = np$$

Le X_i sono indipendenti $\Rightarrow$

$$Var[X] = Var\Big[\sum_{i=1}^{n} X_i\Big] = \sum_{i=1}^{n} Var[X_i] + 2\sum_{i<j} \overbrace{Cov(X_i,X_j)}^{=0 \ \forall i \neq j}$$

$$= \sum_{i=1}^{n} p(1-p) = np(1-p)$$

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X e Y indipendenti con distribuzione esponenziale

$Z := \min\{X,Y\}$

$\mathbb{P}_X = \exp(\lambda) \qquad f:(x,y)\in\mathbb{R}^2 \mapsto \min\{x,y\}\in\mathbb{R}$

$\mathbb{P}_Y = \exp(\mu) \qquad \Rightarrow \quad Z = f\circ(X,Y)$

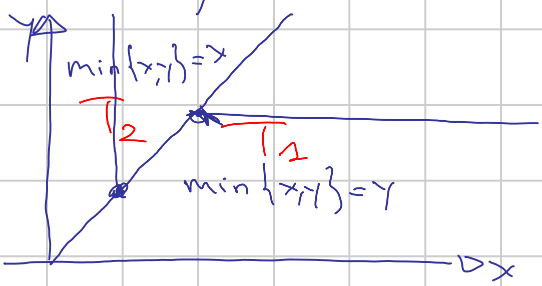
$\psi : \mathbb{R}\to\mathbb{R}$ di Borel non negativa

$$\int_{\mathbb{R}} \psi(t)\,\mathbb{P}_Z(dt) = \int_{\mathbb{R}^2} \psi\circ f(x,y)\,\mathbb{P}_{X,Y}(dx\,dy) =$$

$$= \int_{\mathbb{R}^2} \psi(\min\{x,y\})\,(\mathbb{P}_X \times \mathbb{P}_Y)(dx\,dy) =$$

$$= \int_{\mathbb{R}^2} \psi(\min\{x,y\})\,\lambda e^{-\lambda x}\,\mathbb{1}_{(0,+\infty)}(x)\,\mu e^{-\mu y}\,\mathbb{1}_{(0,+\infty)}(y)\,dx\,dy$$

$$= \int_{(0,+\infty)^2} \psi(\min\{x,y\})\,\lambda\mu\, e^{-\lambda x} e^{-\mu y}\,dx\,dy$$



$$= \int_{T_1} \varphi(y)\lambda\mu e^{-\lambda x} e^{-\mu y}\,dx\,dy + \int_{T_2} \varphi(x)\lambda\mu e^{-\lambda x} e^{-\mu y}\,dx\,dy$$

$$= \int_0^{+\infty}\left(\int_y^{+\infty} \varphi(y)\lambda\mu e^{-\lambda x} e^{-\mu y}\,dx\right)dy + \int_0^{+\infty}\left(\int_x^{+\infty} \varphi(x)\lambda\mu e^{-\lambda x} e^{-\mu y}\,dy\right)dx$$

$$= \int_0^{+\infty} \varphi(y)\mu e^{-\mu y}\left(\int_y^{+\infty} \lambda e^{-\lambda x}\,dx\right)dy + \int_0^{+\infty} \varphi(x)\lambda e^{-\lambda x}\left(\int_x^{+\infty} \mu e^{-\mu y}\,dy\right)dx$$

$$= \int_0^{+\infty} \varphi(y)\mu e^{-\mu y}\left(-e^{-\lambda x}\right)\Big|_{x=y}^{y\to+\infty} dy + \int_0^{+\infty} \varphi(x)\lambda e^{-\lambda x}\left(-e^{-\mu y}\right)\Big|_{y=x}^{y\to+\infty} dx$$

$$= \int_0^{+\infty} \varphi(y)\mu e^{-\mu y} e^{-\lambda y}\,dy + \int_0^{+\infty} \varphi(x)\lambda e^{-\lambda x} e^{-\mu x}\,dx$$

$$= \int_0^{+\infty} \varphi(t)\, e^{-(\mu+\lambda)t}(\mu+\lambda)\,dt$$

$$= \int_{\mathbb{R}} \varphi(t)(\mu+\lambda)e^{-(\mu+\lambda)t}\,\mathbb{1}_{(0,+\infty)}(t)\,dt$$

$$\mathbb{P}_Z = \exp(\mu+\lambda)$$

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Ex 5/5

$\mathbb{P}(Y\le t)$

$t<0$ $\quad \{Y\le t\} = \{X\in(-1,1),\ X\le t\} =$

$\quad = \{-1<X<1,\ X\le t\}$

$t\le -1$ $\quad \{Y\le t\} = \phi \implies F_Y(t) = \mathbb{P}(Y\le t) = 0$

$t\in(-1,0)$ $\quad \{Y\le t\} = \{-1<X\le t\} = \{X\le t\}\setminus\{X\le -1\}$

$\quad F_Y(t) = \mathbb{P}(Y\le t) = \mathbb{P}(X\le t) - \mathbb{P}(X\le -1) =$

$\quad = \Phi(t) - \Phi(-1) =$

$\Phi(t) + \Phi(-t) = 1 \quad \forall t\in\mathbb{R} \implies$ $\quad 1-\Phi(-t) - (1-\Phi(1))$

$t \in (-1,0) \qquad F_Y(t) = \Phi(1) - \Phi(-t)$

$t=0 \qquad \{Y \le t\} = \{Y \le 0\} = \{X \in (-1,1),\ X \le 0\} \cup \{|X| \ge 1\}$

$$F_Y(0) = \mathbb{P}(Y \le 0) = \mathbb{P}(-1 < X \le 0) + \mathbb{P}(X \ge 1) + \mathbb{P}(X \le -1)$$
$$= \mathbb{P}(X \le 0) - \cancel{\mathbb{P}(X \le -1)} + \mathbb{P}(X \ge 1) + \cancel{\mathbb{P}(X \le -1)}$$
$$= \frac{1}{2} + 1 - \mathbb{P}(X < 1) = \frac{3}{2} - \Phi(1)$$

$t > 0 \qquad \{Y \le t\} = \{X \in (-1,1),\ X \le t\} \cup \{|X| \ge 1\}$ $\circledast$

$t \ge 1 \qquad \{Y \le t\} = \Omega \qquad F_Y(t) = 1$

$t \in (0,1) \quad \circledast \quad \{Y \le t\} = \{-1 < X \le t\} \cup \{X \ge 1\} \cup \{X \le -1\}$
$$= \{X \le t\} \cup \{X \ge 1\}$$

$$F_Y(t) = \mathbb{P}(X \le t) + \mathbb{P}(X \ge 1) =$$
$$= \mathbb{P}(X \le t) + 1 - \mathbb{P}(X < 1)$$
$$= \Phi(t) + 1 - \Phi(1)$$

$$F_Y(t) = \begin{cases} 0 & t \le -1 \\ \Phi(1) - \Phi(-t) & t \in (-1,0) \\ \Phi(t) + 1 - \Phi(1) & t \in [0,1) \\ 1 & t \ge 1 \end{cases}$$

$F_Y(0^+) = F_Y(0) = \frac{3}{2} - \Phi(1) \qquad F_Y(0^-) = \Phi(1) - \frac{1}{2}$

$$\frac{3}{2} - \Phi(1) = \Phi(1) - \frac{1}{2}$$

$\Phi(1) = 1 \qquad$ FALSA

EX 6/5

$$\mathbb{P}(Y=-1) = \mathbb{P}(X \leq \mu - \sigma) \qquad X_0 \text{ t.c. } \mathbb{P}_{X_0} = N(0,1)$$
$$= \mathbb{P}(\mu + \sigma X_0 \leq \mu - \sigma) \qquad X \Rightarrow \mu + \sigma X_0$$
$$= \mathbb{P}(X_0 \leq -1) = \Phi(-1) = 1 - \Phi(1)$$

$$\mathbb{P}(Y=0) = \mathbb{P}(\mu - \sigma < X < \mu + \sigma) =$$
$$= \mathbb{P}(\cancel{\mu} - \sigma < \cancel{\mu} + \sigma X_0 < \cancel{\mu} + \sigma)$$
$$= \mathbb{P}(-1 < X_0 < 1) = \mathbb{P}(X_0 < 1) - \mathbb{P}(X_0 \leq -1)$$
$$= \Phi(1) - \Phi(-1) = \Phi(1) - (1 - \Phi(1))$$
$$= 2\Phi(1) - 1$$

$$\mathbb{P}(Y=1) = 1 - \mathbb{P}(Y=0) - \mathbb{P}(Y=-1) =$$
$$1 - (2\Phi(1) - \cancel{1}) - (\cancel{1} - \Phi(1))$$
$$= 1 - \Phi(1)$$

$$\mathbb{P}(Y=-1) = \mathbb{P}(Y=1) = 1 - \Phi(1) \qquad \mathbb{P}(Y=0) = 2\Phi(1) - 1$$

$$\mathbb{E}[Y] = -1\,\mathbb{P}(Y=-1) + \cancel{0\,\mathbb{P}(Y=0)} + 1\,\mathbb{P}(Y=1)$$
$$= (-1+1)\,\mathbb{P}(Y=1) = 0$$
$$\mathrm{Var}[Y] = \mathbb{E}[Y^2] = (-1)^2\,\mathbb{P}(Y=-1) + 0^2\,\mathbb{P}(Y=0) +$$
$$+ (1)^2\,\mathbb{P}(Y=1)$$
$$= 2\,\mathbb{P}(Y=1) = 2(1 - \Phi(1))$$

EX 3/6

$$Z = X + Y \quad = \quad \mathbb{E}[Z] = \mathbb{E}[X] + \mathbb{E}[Y] = \frac{1}{\lambda} + p$$

$$t \in \mathbb{R} \quad \mathbb{P}(Z \leq t) = \qquad \{Y=0\} \quad \{Y=1\}$$
$$= \mathbb{P}(Z \leq t, Y=0) + \mathbb{P}(Z \leq t, Y=1)$$
$$= \mathbb{P}(X + Y \leq t, Y=0) + \mathbb{P}(X+Y \leq t, Y=1)$$

$$= \mathbb{P}(X \le t, Y=0) + \mathbb{P}(X \le t-1, Y=1)$$
$$= \mathbb{P}(X \le t)\mathbb{P}(Y=0) + \mathbb{P}(X \le t-1)\mathbb{P}(Y=1)$$
$$= \mathbb{P}(X \le t)(1-p) + \mathbb{P}(X \le t-1)p$$

$$F_X(t) = \begin{cases} 0 & t<0 \\ 1-e^{-\lambda t} & t \ge 0 \end{cases}$$

$t<0$ $\quad \mathbb{P}(Z \le t) = 0$

$t \in [0,1)$ $\quad \mathbb{P}(Z \le t) = (1-p)(1-e^{-\lambda t})$

$t \ge 1$ $\quad \mathbb{P}(Z \le t) = (1-p)(1-e^{-\lambda t}) + p(1-e^{-\lambda(t-1)})$

$\mathbb{P}_X = f(x)\,dx$ $\qquad f(x) = \begin{cases} \lambda e^{-\lambda x} & x>0 \\ 0 & x \le 0 \end{cases}$

$\mathbb{P}_{X^2} = g(x)\,dx$

$$g(x) = \begin{cases} \frac{1}{2\sqrt{x}}\left(f(\sqrt{x}) + \cancel{f(-\sqrt{x})}\right) & x>0 \\ 0 & x \le 0 \end{cases}$$

$x>0$

$$g(x) = \frac{1}{2\sqrt{x}} \lambda e^{-\lambda\sqrt{x}} = \frac{\lambda}{2} x^{\frac{1}{2}-1} e^{-\lambda x^{1/2}}$$

EX 5|6

$\{X+Y=k\}_{k=0}^{\infty}$ è una partizione di Ω in eventi

$i \in \mathbb{N}_0$

$$\mathbb{P}(Y=i) = \sum_{k=0}^{\infty} \mathbb{P}(Y=i \mid X+Y=k)\,\mathbb{P}(X+Y=k)$$
$$= \sum_{k=i}^{+\infty} \binom{k}{i} p^i (1-p)^{k-i} e^{-\lambda} \frac{\lambda^k}{k!}$$
$$= \sum_{k=i}^{\infty} \frac{\cancel{k!}}{i!(k-i)!} p^i (1-p)^{k-i} e^{-\lambda} \frac{\lambda^k}{\cancel{k!}} =$$
$$= \left(\frac{p}{1-p}\right)^i \frac{1}{i!} e^{-\lambda} \sum_{k=i}^{\infty} \frac{(1-p)^k \lambda^k}{(k-i)!} \qquad j = k-i \quad k \ge i$$
$$= \left(\frac{p}{1-p}\right)^i \frac{1}{i!} e^{-\lambda} \sum_{j=0}^{\infty} \frac{((1-p)\lambda)^j \cdot ((1-p)\lambda)^i}{j!} \qquad j \ge 0$$

$$= \left(\frac{p}{\cancel{1-p}}\right)^i \frac{1}{i!} e^{-\lambda} \cancel{(1-p)^i} \lambda^i \underbrace{\sum_{j=0}^{\infty} \frac{((1-p)\lambda)^j}{j!}}_{= e^{(1-p)\lambda} = e^{\lambda - \lambda p}}$$

$$= e^{-\lambda p} \frac{(\lambda p)^i}{i!} \quad \Rightarrow \quad \mathbb{P}_Y = \text{Poisson}(\lambda p),$$

$$\mathbb{P}(X=j) = \sum_{k=0}^{\infty} \mathbb{P}(X=j, X+Y=k) =$$

$$= \sum_{k=0}^{\infty} \mathbb{P}(Y=k-j, X+Y=k) =$$

$$= \sum_{k=0}^{\infty} \mathbb{P}(Y=k-j \mid X+Y=k)\, \mathbb{P}(X+Y=k)$$

$$= \sum_{k=j}^{+\infty} \binom{k}{k-j} p^{k-j} (1-p)^j e^{-\lambda} \frac{\lambda^k}{k!} =$$

$$= \sum_{k=j}^{+\infty} \frac{\cancel{k!}}{(k-j)!\, j!} p^{k-j} (1-p)^j e^{-\lambda} \frac{\lambda^k}{\cancel{k!}}$$

$$= \left(\frac{1-p}{p}\right)^j \frac{1}{j!} e^{-\lambda} \sum_{k=j}^{\infty} \frac{1}{(k-j)!} (p\lambda)^k =$$

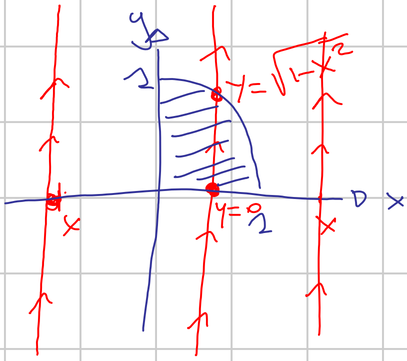
$s = k-j$

$k = s+j$

$$= \left(\frac{1-p}{p}\right)^j \frac{1}{j!} e^{-\lambda} \sum_{s=0}^{\infty} \frac{1}{s!} (p\lambda)^s \cdot (p\lambda)^j$$

$$= \left(\frac{1-p}{\cancel{p}}\right)^j \frac{1}{j!} e^{-\lambda} \cancel{p^j} \lambda^j e^{\lambda p}$$

$$= e^{-\lambda(1-p)} \frac{(\lambda(1-p))^j}{j!} \qquad \mathbb{P}_X = \text{Poisson}(\lambda(1-p)),$$



$$\mathbb{P}_{X,Y} = f(x,y)\,dx\,dy$$

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$$

$x < 0$, $x > 1$ $\quad f_X(x) = 0$

$x \in [0,1]$

$$f_X(x) = \int_0^{\sqrt{1-x^2}} \frac{2xy}{(x^2+y^2)^{3/2}}\,dy =$$

$$= x \int_0^{\sqrt{1-x^2}} 2y\,(x^2+y^2)^{-3/2}\,dy = x\,\frac{1}{-\frac{3}{2}+1}\,(x^2+y^2)^{-\frac{3}{2}+1}\Big|_{y=0}^{y=\sqrt{1-x^2}}$$

$$= -2x\,(x^2+y^2)^{-1/2}\Big|_{y=0}^{y=\sqrt{1-x^2}} = -2x\left(1-\frac{1}{x}\right) = 2-2x$$

$$= 2(1-x)$$