SOMMA DI V.A. INDIPENDENTI

Note Title 28/11/2016

X, Y v.a. indipendenti su $(\Omega, \mathcal{E}, \mathbb{P})$, entrambe discrete

$X(\Omega) = \{x_i\}_{i\in I}$ $\quad Y(\Omega) = \{y_j\}_{j\in J}$

$p_i = \mathbb{P}(X = x_i)$ $\quad q_j = \mathbb{P}(Y = y_j)$

$(X, Y)(\Omega) \subseteq X(\Omega) \times Y(\Omega)$

$\mathbb{P}((X,Y) = (x_i, y_j)) = \mathbb{P}(X = x_i, Y = y_j) = \mathbb{P}(X = x_i)\,\mathbb{P}(Y = y_j)$

$$\mathbb{P}((X,Y) = (x_i, y_j)) = p_i q_j$$

— o —

X, Y indipendenti su $(\Omega, \mathcal{E}, \mathbb{P})$ t.c.

$\mathbb{P}_{X,Y} = h(x,y)\,dx\,dy$ cioè (X,Y) ha distribuzione A.C.

$\varphi : \mathbb{R}^2 \to \mathbb{R}$ di Borel nonnegativa

$$\int_{\mathbb{R}^2} \varphi(x,y)\,h(x,y)\,dx\,dy = \int_{\mathbb{R}^2} \varphi(x,y)\,\mathbb{P}_{X,Y}(dx\,dy) =$$

$$= \int_{\mathbb{R}^2} \varphi(x,y)\,(\mathbb{P}_X \times \mathbb{P}_Y)(dx\,dy) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi(x,y)\,\mathbb{P}_X(dx) \right) \mathbb{P}_Y(dy) = (*)$$

So che $\mathbb{P}_X = f(x)\,dx$ $\quad f(x) = \int_{\mathbb{R}} h(x,y)\,dy$

$\mathbb{P}_Y = g(y)\,dy$ $\quad g(y) = \int_{\mathbb{R}} h(x,y)\,dx$

$$(*) \quad \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi(x,y)\,f(x)\,dx \right) g(y)\,dy$$

$\forall \varphi$ di Borel non negativa

$$\int_{\mathbb{R}^2} \varphi(x,y)\,h(x,y)\,dx\,dy = \int_{\mathbb{R}^2} \varphi(x,y)\,f(x)\,g(y)\,dx\,dy$$

$\Rightarrow h(x,y) = f(x)\,g(y)$ $\quad$ Lebesgue-q.o. in $\mathbb{R}^2$

Supponiamo X e Y entrambe con distribuzioni A.C e che siano indipendenti.

$$\mathbb{P}_{X,Y} = \mathbb{P}_X \times \mathbb{P}_Y \qquad \mathbb{P}_X = f(x)\,dx \qquad \mathbb{P}_Y = g(y)\,dy$$

$\forall \varphi : \mathbb{R}^2 \to \mathbb{R}$ di Borel nonnegativa

$$\int_{\mathbb{R}^2} \varphi(x,y)\,\mathbb{P}_{X,Y}(dx\,dy) = \int_{\mathbb{R}^2} \varphi(x,y)\,(\mathbb{P}_X \times \mathbb{P}_Y)(dx\,dy) =$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi(x,y)\,\mathbb{P}_X(dx) \right) \mathbb{P}_Y(dy) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi(x,y) f(x)\,dx \right) g(y)\,dy =$$

$$= \int_{\mathbb{R}^2} \varphi(x,y) f(x) g(y)\,dx\,dy \implies \mathbb{P}_{X,Y} \text{ è AC con densità } h(x,y) = f(x)\,g(y)$$

— o —

X e Y v.a. indipendenti su $(\Omega, \mathcal{E}, \mathbb{P})$

$X(\Omega),\ Y(\Omega) \subseteq \mathbb{Z}$

$k \in \mathbb{Z} \qquad p_k := \mathbb{P}(X = k) \qquad q_k := \mathbb{P}(Y = k)$

$\mathbb{P}(X + Y = k) \quad k \in \mathbb{Z}$ perché $(X+Y)(\Omega) \subseteq \mathbb{Z}$

$$\{X + Y = k\} = \bigcup_{j=-\infty}^{+\infty} \{X = j,\ Y = k-j\}$$

$$\mathbb{P}(X+Y=k) = \sum_{j=-\infty}^{+\infty} \mathbb{P}(X=j,\ Y=k-j) =$$

$$= \sum_{j=-\infty}^{+\infty} \mathbb{P}(X=j)\,\mathbb{P}(Y=k-j) = \sum_{j=-\infty}^{+\infty} p_j\, q_{k-j}$$

— o —

X, Y indipendenti su $(\Omega, \mathcal{E}, \mathbb{P})$, entrambe con distribuzione A.C.

$$\mathbb{P}_X = f(x)\,dx$$

$$\mathbb{P}_Y = g(y)\,dy$$

$\psi:\mathbb{R}\to\mathbb{R}$ di Borel non negativa $\qquad \varphi\circ(X,Y)=X+Y$

$\varphi(x,y)=x+y$

$$\int_{\mathbb{R}}\psi(t)\,\mathbb{P}_{X+Y}(dt)=$$

$$=\int_{\mathbb{R}^2}\psi(\varphi(x,y))\,\mathbb{P}_{X,Y}(dx\,dy)=\int_{\mathbb{R}^2}\psi(x+y)f(x)g(y)\,dx\,dy$$

$$\begin{cases} s=x\\ t=x+y\end{cases},\qquad \begin{cases} x=s\\ y=t-s\end{cases}\qquad J=\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}$$

$$\det J=1$$

$$\int_{\mathbb{R}^2}\psi(t)f(s)g(t-s)\,1\,ds\,dt$$

$$\int_{\mathbb{R}}\psi(t)\,\mathbb{P}_{X+Y}(dt)=\int_{\mathbb{R}^2}\psi(t)f(s)g(t-s)\,ds\,dt=$$

$$=\int_{\mathbb{R}}\psi(t)\left(\int_{\mathbb{R}}f(s)g(t-s)\,ds\right)dt$$

$\Rightarrow \mathbb{P}_{X+Y}=h(t)\,dt$ dove $h(t):=\int_{\mathbb{R}}f(s)g(t-s)\,ds$

— o —

ESEMPI

X,Y i.i.d $\qquad \mathbb{P}_X=\mathbb{P}_Y=U((a,b))=\underbrace{\frac{1}{b-a}\mathbb{1}_{(a,b)}(x)\,dx}_{f(x)}$

$$h(t)=\int_{\mathbb{R}}f(x)g(t-x)\,dx=$$

$$=\int_{\mathbb{R}}\frac{1}{(b-a)^2}\mathbb{1}_{(a,b)}(x)\,\mathbb{1}_{(a,b)}(t-x)\,dx$$

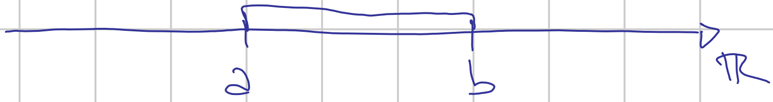
$$=\frac{1}{(b-a)^2}\int_a^b\mathbb{1}_{(a,b)}(t-x)\,dx$$

$y=t-x$

$x=t-y$

$dx=(-1)\,dy$

$$= \frac{1}{(b-a)^2}\int_{t-a}^{t-b} -\mathbb{1}_{(a,b)}(y)\,dy = \frac{1}{(b-a)^2}\int_{t-b}^{t-a} \mathbb{1}_{(a,b)}(y)\,dy$$



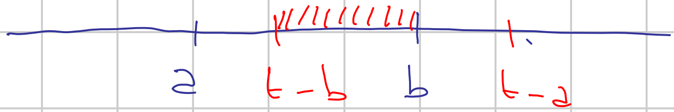
$(t-b,\ t-a)$

1° CASO $t-a<a$ $\quad h(t)=0$ $\quad t<2a$

2° CASO

$$h(t)=\frac{1}{(b-a)^2}(t-2a)$$

3° CASO



$$h(t)=\frac{1}{(b-a)^2}(2b-t)$$

4° CASO $t-b>b$ $\quad h(t)=0$ $\quad t>2b$

2° CASO $t-b<a<t-a<b$

$t<a+b$

$t>2a$

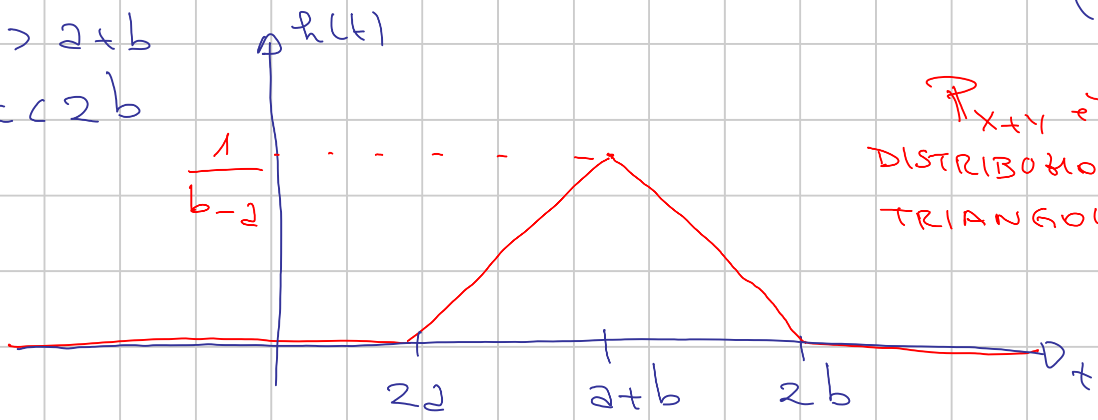
$$t\in(2a,a+b)\quad h(t)=\frac{t-2a}{(b-a)^2}$$

3° CASO $a<t-b<b<t-a$

$$t\in(a+b,2b)\quad h(t)=\frac{2b-t}{(b-a)^2}$$

$t>a+b$

$t<2b$



P_{X+Y} è una DISTRIBUZIONE TRIANGOLARE

X e Y gaussiane indipendenti

$\mathbb{P}_X = N(\mu_1, \sigma_1^2)$ $\qquad$ $\mathbb{P}_Y = N(\mu_2, \sigma_2^2)$

$\tilde X = X - \mu_1$ $\qquad$ $\tilde Y = Y - \mu_2$

$$\mathbb{P}_X = f(x)\,dx = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(\frac{-(x-\mu_1)^2}{2\sigma_1^2}\right) dx$$

$\mathbb{P}_{\tilde X} = \tilde f(x)\,dx$ $\qquad$ $\tilde f(x) = \frac{1}{|1|} f\left(\frac{x+\mu_1}{1}\right) = f(x+\mu_1)$

$$= \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(x+\cancel{\mu_1}-\cancel{\mu_1})^2}{2\sigma_1^2}\right)$$

$\mathbb{P}_{\tilde X} = N(0, \sigma_1^2) = \tilde f(x)\,dx$

Analogamente $\mathbb{P}_{\tilde Y} = N(0, \sigma_2^2) = \tilde g(y)\,dy$

$$X+Y = (\tilde X + \mu_1) + (\tilde Y + \mu_2) = (\tilde X + \tilde Y) + (\mu_1 + \mu_2)$$

$\mathbb{P}_{\tilde X + \tilde Y} = h(x)\,dx$ $\qquad$ $h(x) = \int_{\mathbb{R}} \tilde f(y)\, \tilde g(x-y)\,dy =$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(\frac{-y^2}{2\sigma_1^2}\right) \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(\frac{-(x-y)^2}{2\sigma_2^2}\right) dy =$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} \int_{\mathbb{R}} \exp\left(\frac{-y^2}{2\sigma_1^2} - \frac{(x-y)^2}{2\sigma_2^2}\right) dy$$

Considero l'argomento dell'esponenziale

$$\frac{-1}{2\sigma_1^2\sigma_2^2}\left\{\sigma_2^2 y^2 + \sigma_1^2 x^2 - 2\sigma_1^2 xy + \sigma_1^2 y^2\right\} =$$

$$= \frac{-1}{2\sigma_1^2\sigma_2^2}\left\{(\sigma_1^2 + \sigma_2^2) y^2 - 2\sigma_1^2 xy + \sigma_1^2 x^2\right\}$$

$$= \frac{-1}{2\sigma_1^2\sigma_2^2}\left\{ \left(y\sqrt{\sigma_1^2+\sigma_2^2}\right)^2 - 2\left(y\sqrt{\sigma_1^2+\sigma_2^2}\right)\frac{\sigma_1^2 x}{\sqrt{\sigma_1^2+\sigma_2^2}} + \right.$$

$$\left. + \frac{\sigma_1^4 x^2}{\sigma_1^2+\sigma_2^2} - \frac{\sigma_1^4 x^2}{\sigma_1^2+\sigma_2^2} + \sigma_1^2 x^2 \right\} =$$

$$= \frac{-1}{2\sigma_1^2\sigma_2^2}\left\{ \left(y\sqrt{\sigma_1^2+\sigma_2^2} - \frac{\sigma_1^2 x}{\sqrt{\sigma_1^2+\sigma_2^2}} \right)^2 + \frac{\sigma_1^2 x^2\left(\cancel{-\sigma_1^2} + \cancel{\sigma_1^2} + \sigma_2^2\right)}{\sigma_1^2+\sigma_2^2} \right\}$$

$$h(x) = \frac{1}{2\pi\sigma_1\sigma_2}\int_{\mathbb{R}} \exp\left(\frac{-\left(y\sqrt{\sigma_1^2+\sigma_2^2} - \frac{\sigma_1^2 x}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2}{2\sigma_1^2\sigma_2^2} - \frac{x^2}{2(\sigma_1^2+\sigma_2^2)} \right) dy$$

$$= \frac{1}{2\pi\sigma_1\sigma_2}\exp\left(\frac{-x^2}{2(\sigma_1^2+\sigma_2^2)}\right)\int_{\mathbb{R}} \exp\left(- \left(\frac{y\sqrt{\sigma_1^2+\sigma_2^2} - \frac{\sigma_1^2 x}{\sqrt{\sigma_1^2+\sigma_2^2}}}{\sigma_1\sigma_2\sqrt{2}} \right)^2 \right) dy$$

$$t = \frac{1}{\sigma_1\sigma_2\sqrt{2}}\left(y\sqrt{\sigma_1^2+\sigma_2^2} - \frac{\sigma_1^2 x}{\sqrt{\sigma_1^2+\sigma_2^2}} \right),$$

$$dt = \frac{1}{\sigma_1\sigma_2\sqrt{2}}\sqrt{\sigma_1^2+\sigma_2^2}\, dy$$

$$= \frac{1}{2\pi\cancel{\sigma_1}\cancel{\sigma_2}}\exp\left(\frac{-x^2}{2(\sigma_1^2+\sigma_2^2)}\right)\frac{\cancel{\sigma_1}\cancel{\sigma_2}\sqrt{2}}{\sqrt{\sigma_1^2+\sigma_2^2}}\underbrace{\int_{\mathbb{R}} \exp(-t^2)\,dt}_{\sqrt{\pi}} =$$

$$= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}\exp\left(\frac{-x^2}{2(\sigma_1^2+\sigma_2^2)}\right)$$

$$\mathbb{P}_{\tilde{X}+\tilde{Y}} = N(0, \sigma_1^2+\sigma_2^2)$$

$$X + Y = (\tilde{X}+\tilde{Y}) + (\mu_1+\mu_2)$$

$\mathbb{P}_{X+Y} = k(t)dt \qquad k(t) = \frac{1}{|\lambda|} h\left(\frac{t-(\mu_1+\mu_2)}{1}\right) = h(t-(\mu_1+\mu_2))$

$$= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} \exp\left(\frac{-(t-(\mu_1+\mu_2))^2}{2(\sigma_1^2+\sigma_2^2)}\right)$$

$$\mathbb{P}_{X+Y} = N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$$

— o —

X e Y v.a. indipendenti su $(\Omega, \mathcal{E}, \mathbb{P})$

$\mathbb{P}_X = \Gamma(\alpha, \lambda) \qquad \alpha, \beta, \lambda > 0 \qquad \mathbb{P}_X = f(x)\,dx$

$\mathbb{P}_Y = \Gamma(\beta, \lambda) \qquad \mathbb{P}_Y = g(y)\,dy$

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

$$g(x) = \begin{cases} \frac{\lambda^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\lambda x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

$\mathbb{P}_{X+Y} = h(x)\,dx \qquad h(x) = \int_{\mathbb{R}} f(y)\, g(x-y)\, dy$

$$\begin{cases} y > 0 \\ x-y > 0 \end{cases} \qquad \begin{cases} y > 0 \\ y < x \end{cases} \qquad \begin{matrix} \emptyset & x \le 0 & h(x) = 0 \\ y \in (0, x) & x > 0 & \end{matrix}$$

$$x > 0 \quad h(x) = \int_0^x \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} \cancel{e^{-\lambda y}} \cdot \frac{\lambda^\beta}{\Gamma(\beta)} (x-y)^{\beta-1} e^{-\lambda(x-\cancel{y})} dy$$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} e^{-\lambda x} \int_0^x y^{\alpha-1} (x-y)^{\beta-1} dy$$

$y = xt \qquad dy = x\,dt \qquad \begin{matrix} y=0 \\ t=0 \end{matrix} \qquad \begin{matrix} y=x \\ t=1 \end{matrix}$

$$\frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} e^{-\lambda x} \int_0^1 x^{\alpha-\cancel{1}} t^{\alpha-1} \left(x(1-t)\right)^{\beta-1} \cancel{x}\, dt$$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} e^{-\lambda x} x^{\alpha+\beta-1} \underbrace{\int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt}_{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}} ,$$

$$h(x) = \begin{cases} 0 & x \le 0 \\ \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha+\beta)} x^{\alpha+\beta-1} e^{-\lambda x} & x > 0 \end{cases}$$

$$\Rightarrow P_{X+Y} = \Gamma(\alpha+\beta, \lambda)$$

$T :=$ # di lanci a cui ottengo la 1ª Testa

$P_T = G(p) \quad p = \frac{1}{2} \quad P(T=k) = p(1-p)^{k-1} \quad k \ge 1$

$P(T=k) = \left(\frac{1}{2}\right)^k$

$S :=$ # di lanci a cui ottengo il 1° "6"

$P_S = G(q) \quad q = \frac{1}{6} \quad P(S=k) = q(1-q)^{k-1} \quad k \ge 1$

$= \frac{1}{6}\left(\frac{5}{6}\right)^{k-1}$

$P(T=S) \qquad \{T=S\} = \bigcup_{k=1}^{\infty} \{T=k, S=k\}$

$P(T=S) = \sum_{k=1}^{\infty} P(T=k, S=k) =$ per l'indipendenza $=$

$$= \sum_{k=1}^{\infty} P(T=k) P(S=k) = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \frac{1}{6}\left(\frac{5}{6}\right)^{k-1} =$$

$$= \frac{1}{12} \sum_{k=1}^{\infty} \left(\frac{5}{12}\right)^{k-1} = \frac{1}{12} \sum_{j=0}^{\infty} \left(\frac{5}{12}\right)^{j} = \qquad j = k-1$$

$$= \frac{1}{12} \frac{1}{1-\frac{5}{12}} = \frac{1}{12}\frac{12}{7} = \frac{1}{7}$$

$$\mathbb{P}(T<S) \qquad \{T<S\} = \bigcup_{k=2}^{\infty} \{S=k, T<k\}$$

$$\mathbb{P}(T<S) = \sum_{k=2}^{\infty} \mathbb{P}(S=k, T<k) = \text{per l'indipendenza} =$$

$$= \sum_{k=2}^{\infty} \mathbb{P}(S=k)\mathbb{P}(T<k)$$

$$\{T<k\} = \bigcup_{j=1}^{k-1} \{T=j\}$$

$$\mathbb{P}(T<k) = \sum_{j=1}^{k-1} \mathbb{P}(T=j) = \sum_{j=1}^{k-1} \left(\frac{1}{2}\right)^{j} \qquad s=j-1$$

$$= \sum_{s=0}^{k-2} \left(\frac{1}{2}\right)^{s} \cdot \frac{1}{2} = \cancel{\frac{1}{2}} \frac{1-\left(\frac{1}{2}\right)^{k-1}}{\cancel{1-\frac{1}{2}}} = 1-\left(\frac{1}{2}\right)^{k-1}$$

$$\mathbb{P}(T<S) = \sum_{k=2}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{k-1}\left(1-\left(\frac{1}{2}\right)^{k-1}\right) =$$

$$= \frac{1}{6}\sum_{k=2}^{\infty}\left(\frac{5}{6}\right)^{k-1} - \frac{1}{6}\sum_{k=2}^{\infty}\left(\frac{5}{12}\right)^{k-1} \qquad \begin{array}{l} s=k-2 \\ k-1 = s+1 \end{array}$$

$$= \frac{1}{6}\sum_{s=0}^{\infty}\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)^{s} - \frac{1}{6}\sum_{s=0}^{\infty}\left(\frac{5}{12}\right)\left(\frac{5}{12}\right)^{s} =$$

$$= \frac{5}{36}\frac{1}{1-\frac{5}{6}} - \frac{5}{72}\frac{1}{1-\frac{5}{12}} =$$

$$= \frac{5}{\cancel{36}_{6}}\cdot\cancel{6} - \frac{5}{6\cdot\cancel{12}}\frac{\cancel{12}}{7} = \frac{5(7-1)}{6\cdot 7} = \frac{5}{7}$$

$$\mathbb{P}(T>S) = 1-\mathbb{P}(T\le S) = \qquad \{T\le S\} = \{T<S\}\cup\{T=S\}$$

$$= 1 - \{ \mathbb{P}(T < S) + \mathbb{P}(T = S) \}$$

$$= 1 - \left(\frac{5}{7} + \frac{1}{7} \right) = \frac{1}{7}$$

X e Y indipendenti, entrambe di Poisson

$\mathbb{P}_X = \mathcal{P}(\lambda)$ $\qquad \lambda, \mu > 0$

$\mathbb{P}_Y = \mathcal{P}(\mu)$

$$\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!} \qquad \forall k \geq 0$$

$$\mathbb{P}(Y = k) = e^{-\mu} \frac{\mu^k}{k!} \qquad \forall k \geq 0$$

$$\mathbb{P}(X + Y = k) = \sum_{j=-\infty}^{+\infty} \mathbb{P}(X = j)\, \mathbb{P}(Y = k - j)$$

$$\begin{cases} j \geq 0 \\ k - j \geq 0 \quad j \leq k \end{cases} \qquad j = 0, \_\_ k \qquad k \geq 0$$

$$\mathbb{P}(X + Y = k) = \sum_{j=0}^{k} e^{-\lambda} \frac{\lambda^j}{j!} e^{-\mu} \frac{\mu^{k-j}}{(k-j)!} \frac{k!}{k!} =$$

$$= \frac{e^{-(\lambda+\mu)}}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda^j \mu^{k-j} = \frac{e^{-(\lambda+\mu)}}{k!} (\lambda+\mu)^k$$

$$\mathbb{P}(X + Y = k) = \frac{e^{-(\lambda+\mu)}}{k!} (\lambda+\mu)^k \qquad \forall k \geq 0$$

$$\Rightarrow \mathbb{P}_{X+Y} = \mathcal{P}(\lambda + \mu)$$