FORMULA DI COMPOSIZIONE, COVARIANZA

Note Title 21/11/2016

$X = (X_1, \ldots, X_N) : \Omega \to \mathbb{R}^N$ v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

$\forall A \in \mathcal{B}(\mathbb{R}^N) \quad X^{-1}(A) = \{X \in A\} \in \mathcal{E}$

$\mathbb{P}_X : A \in \mathcal{B}(\mathbb{R}^N) \mapsto \mathbb{P}(X \in A) \in \mathbb{R}$

$(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N), \mathbb{P}_X)$ è una misura di probabilità

$\mathbb{P}_X$ DISTRIBUZIONE DELLA v.a. X o DISTRIBUZIONE CONGIUNTA di $X_1, \ldots, X_N$

$\mathbb{P}_{X_1}, \ldots, \mathbb{P}_{X_N}$ su $\mathcal{B}(\mathbb{R})$ DISTRIBUZIONI MARGINALI

$X : \Omega \to \mathbb{R}^N$ è v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

$\varphi : \mathbb{R}^N \to \mathbb{R}$ è funzione di Borel nonnegativa

$\Rightarrow \varphi \circ X$ è una v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

$$\int_\Omega \varphi \circ X(\omega)\, \mathbb{P}(d\omega) = \int_{\mathbb{R}^N} \varphi(t_1 \ldots t_N)\, \mathbb{P}_X(dt_1 \ldots dt_N)$$

FORMULA DI COMPOSIZIONE

$X : \Omega \to \mathbb{R}^N$ v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

$\varphi : \mathbb{R}^N \to \mathbb{R}^k$ funzione di Borel

$\psi : \mathbb{R}^k \to \mathbb{R}$ funzione di Borel, nonnegativa

Allora

$\psi \circ \varphi \circ X : \Omega \to \mathbb{R}$ è una v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

$$\int_{\mathbb{R}^k} \psi(t_1 \ldots t_k)\, \mathbb{P}_{\varphi \circ X}(dt_1 \ldots dt_k) = \int_\Omega \psi \circ \varphi \circ X(\omega)\, \mathbb{P}(d\omega)$$

$$= \int_{\mathbb{R}^N} (\psi \circ \varphi)(s_1 \ldots s_N)\, \mathbb{P}_X(ds_1 \ldots ds_N).$$

DIM

$$\int_{\mathbb{R}^N} \psi(t_1 \dots t_N)\, \mathbb{P}_{\varphi\circ X}(dt_1 \dots dt_N) = \int_\Omega \psi\circ(\varphi\circ X)(\omega)\,\mathbb{P}(d\omega)$$

$$= \int_\Omega (\psi\circ\varphi)\circ X(\omega)\,\mathbb{P}(d\omega) = \int_{\mathbb{R}^N} (\psi\circ\varphi)(s_1 \dots s_N)\,\mathbb{P}_X(ds_1 \dots ds_N)$$

— o —

X, Y v.a. su Ω a valori in $\mathbb{R}^N$

$X : \Omega \to \mathbb{R}^N$ $\qquad (X, Y) : \omega \in \Omega \mapsto (X(\omega), Y(\omega)) \in \mathbb{R}^{2N}$

$Y : \Omega \to \mathbb{R}^N$ $\qquad \mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$

$\varphi : (x, y) \in \mathbb{R}^N \times \mathbb{R}^N \longmapsto x + y \in \mathbb{R}^N$

$\varphi(x_1 \dots x_N, y_1 \dots y_N) = (x_1+y_1, x_2+y_2, \dots, x_N+y_N)$

$\psi : \mathbb{R}^N \to \mathbb{R}$ funzione di Borel nonnegativa.

$$\int_{\mathbb{R}^{2N}} \psi\circ\varphi(x_1 \dots x_N, y_1 \dots y_N)\,\mathbb{P}_{X,Y}(dx_1 \dots dx_N\, dy_1 \dots dy_N) =$$

$$= \int_{\mathbb{R}^N} \psi(t_1 \dots t_N)\,\mathbb{P}_{X+Y}(dt_1 \dots dt_N)$$

$$\int_{\mathbb{R}^{2N}} \psi(x_1+y_1, x_2+y_2, \dots x_N+y_N)\,\mathbb{P}_{X,Y}(dx_1 \dots dx_N\, dy_1 \dots dy_N) =$$

$$= \int_{\mathbb{R}^N} \psi(t_1, \dots t_N)\,\mathbb{P}_{X+Y}(dt_1 \dots dt_N)$$

$A \in \mathcal{B}(\mathbb{R}^N) \qquad \psi = \mathbb{1}_A$

$$\int_{\mathbb{R}^N} \mathbb{1}_A(t_1 \dots t_N)\,\mathbb{P}_{X+Y}(dt_1 \dots dt_N) = \mathbb{P}(X+Y \in A)$$

$$\mathbb{P}(X+Y \in A) = \int_{\mathbb{R}^{2N}} \mathbb{1}_A(x_1+y_1, \dots, x_N+y_N)\,\mathbb{P}_{X,Y}(dx_1 \dots dx_N\, dy_1 \dots dy_N)$$

$$= \mathbb{P}_{X,Y}(\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x + y \in A\})$$

$$A = \prod_{i=1}^{N} (-\infty, t_i]$$

$$\{X+Y \in A\} = \{X_i + Y_i \leq t_i \quad \forall i = 1 \dots N\}$$

$$\mathbb{P}(X+Y \in A) = F_{X+Y}(t_1 \dots t_N)$$

$$F_{X+Y}(t_1 \dots t_N) = \mathbb{P}_{X,Y}\left(\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : x_i + y_i \leq t_i \quad \forall i = 1 \dots N\}\right)$$

$N = 1$

$$F_{X+Y}(t) = \mathbb{P}_{X,Y}\left(\{(x,y) \in \mathbb{R}^2 : x+y \leq t\}\right)$$

$$\overline{(X,Y)(\Omega) \subseteq \underset{\sim}{X(\Omega)} \times \underset{\sim}{Y(\Omega)}}$$

Una misura μ su $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$ si dice ASSOLUTAMENTE CONTINUA (RISPETTO A $\mathcal{L}^n$) se $\exists\, f : \mathbb{R}^N \to [0, +\infty]$, $\mathcal{L}^n$-misurabile t.c. $\int_{\mathbb{R}^N} f(x)\,dx = 1$ e t.c.

$$\mu(A) = \int_A f(x)\,dx \qquad \forall A \in \mathcal{B}(\mathbb{R}^N)$$

In modo del tutto analogo al caso scalare si dimostra che:

Sia X v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ $\quad X : \Omega \to \mathbb{R}^N$,

Allora $\mathbb{P}_X$ è A.C. con densità $f(x)$ sse

$$\int_\Omega (\varphi \circ X)(\omega)\,\mathbb{P}(d\omega) = \int_{\mathbb{R}^N} \varphi(x) f(x)\,dx$$

$$= \int_{\mathbb{R}^N} \varphi(x)\,\mathbb{P}_X(dx)$$

PROP Siano $X, Y : \Omega \to \mathbb{R}$ v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

Allora

$$\mathbb{E}[|XY|] \leq \sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]}$$

DIM 1° CASO $\mathbb{E}[X^2] \cdot \mathbb{E}[Y^2] = 0 \Rightarrow$ almeno uno dei due fattori è 0. Slg suppongo $\mathbb{E}[X^2] = 0 \Rightarrow X^2 = 0$ $\mathbb{P}$-qc. $\Rightarrow |XY| = 0$ $\mathbb{P}$.qc. $\Rightarrow \mathbb{E}[|XY|] = 0$ ok

2° CASO $\mathbb{E}[X^2] \cdot \mathbb{E}[Y^2] \neq 0$ ma uno dei due valori attesi è $+\infty$. Slg. suppongo $\mathbb{E}[X^2] = +\infty$ $\Rightarrow$ 2° membro $= +\infty$ ok

3° CASO $\mathbb{E}[X^2]\,\mathbb{E}[Y^2] \neq 0$ e entrambi finiti.

$$\alpha(\omega) = \frac{|X(\omega)|}{\sqrt{\mathbb{E}[X^2]}} \qquad \beta(\omega) = \frac{|Y(\omega)|}{\sqrt{\mathbb{E}[Y^2]}}$$

$$(\alpha(\omega) - \beta(\omega))^2 \geq 0 \qquad 2\alpha(\omega)\beta(\omega) \leq \alpha^2(\omega) + \beta^2(\omega)$$

$\forall \omega \in \Omega$

$$2\int_\Omega \alpha(\omega)\beta(\omega)\,\mathbb{P}(d\omega) \leq \int_\Omega \alpha^2(\omega)\,\mathbb{P}(d\omega) + \int_\Omega \beta^2(\omega)\,\mathbb{P}(d\omega)$$

$$\frac{2}{\sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]}} \int_\Omega |XY|(\omega)\,\mathbb{P}(d\omega) \leq$$

$$\leq \frac{1}{\mathbb{E}[X^2]} \int_\Omega X^2(\omega)\,\mathbb{P}(d\omega) + \frac{1}{\mathbb{E}[Y^2]} \int_\Omega Y^2(\omega)\,\mathbb{P}(d\omega)$$

$$\frac{2}{\sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]}}\, \mathbb{E}[|XY|] \leq 1 + 1$$

$$\mathbb{E}[|XY|] \le \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$$

Supponiamo $\mathbb{E}[X^2]$, $\mathbb{E}[Y^2]$ finiti e non nulli
$\Rightarrow$ $\mathbb{E}[XY]$ esiste finito.
Ripeto la dimostrazione con

$$\alpha(\omega) = \frac{X(\omega)}{\sqrt{\mathbb{E}[X^2]}} \qquad \beta(\omega) = \frac{Y(\omega)}{\sqrt{\mathbb{E}[Y^2]}}$$

$$\mathbb{E}[XY] \le \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$$

Se vale l' $=$ $\Rightarrow$ $2\alpha(\omega)\beta(\omega) = \alpha^2(\omega) + \beta^2(\omega)$ $\mathbb{P}$-q.c.
cioè $\alpha(\omega) = \beta(\omega)$ $\mathbb{P}$-q.c.

$$\frac{X(\omega)}{\sqrt{\mathbb{E}[X^2]}} = \frac{Y(\omega)}{\sqrt{\mathbb{E}[Y^2]}} \qquad X(\omega) = C\, Y(\omega) \quad \mathbb{P}\text{-q.c.}$$

dove $C = \sqrt{\dfrac{\mathbb{E}[X^2]}{\mathbb{E}[Y^2]}} > 0$

— o —

$\mathcal{L}^p(\Omega, \mathbb{P}) = \{ X : \Omega \to \mathbb{R}$ v.a. t.c. $\mathbb{E}[|X|^p] < +\infty \}$
Ci interessano $p = 2$, $p = 1$

$$\mathbb{E}[|XY|] \le \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$$

Scelgo $Y \equiv 1$ $\quad \mathbb{E}[|X|] \le \sqrt{\mathbb{E}[X^2]\cdot 1} = \sqrt{\mathbb{E}[X^2]}$

$\Rightarrow$ Se $X \in \mathcal{L}^2(\Omega, \mathbb{P})$ $\Rightarrow$ $X \in \mathcal{L}^1(\Omega, \mathbb{P})$
cioè

$$\mathcal{L}^2(\Omega, \mathbb{P}) \subseteq \mathcal{L}^1(\Omega, \mathbb{P})$$

Gli insiemi $\mathcal{L}^p(\Omega, \mathbb{P})$ sono spazi vettoriali su $\mathbb{R}$.
Lo dimostro per $p = 2$.

$X, Y \in \mathcal{L}^2(\Omega, \mathbb{P}) \qquad \mathbb{E}[X^2] < +\infty \quad \mathbb{E}[Y^2] \leq +\infty$

$(X+Y)^2(\omega) = X^2(\omega) + Y^2(\omega) + 2XY(\omega)$

$$\mathbb{E}[(X+Y)^2] = \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\,\mathbb{E}[XY] \leq$$
$$\leq \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$$
$$= \left(\sqrt{\mathbb{E}[X^2]} + \sqrt{\mathbb{E}[Y^2]}\right)^2$$

$X \in \mathcal{L}^2(\Omega, \mathbb{P}) \quad a \in \mathbb{R}$

$\mathbb{E}[(aX)^2] = \mathbb{E}[a^2X^2] = a^2\,\mathbb{E}[X^2] < +\infty$.

$\Phi : (X, Y) \in \mathcal{L}^2(\Omega, \mathbb{P}) \times \mathcal{L}^2(\Omega, \mathbb{P}) \longmapsto \mathbb{E}[XY] \in \mathbb{R}$

è una forma bilineare simmetrica

semidefinita positiva.

Simmetrica: banale $\mathbb{E}[XY] = \mathbb{E}[YX]$

Bilineare: per la simmetria basta far vedere che è lineare sulla prima componente:

$X_1, X_2, Y \in \mathcal{L}^2(\Omega, \mathbb{P}) \qquad a_1, a_2 \in \mathbb{R}$

$$\Phi(a_1X_1 + a_2X_2, Y) = \mathbb{E}[(a_1X_1 + a_2X_2)Y] = \mathbb{E}(a_1X_1Y + a_2X_2Y)$$
$$= a_1\,\mathbb{E}[X_1Y] + a_2\,\mathbb{E}[X_2Y]$$
$$= a_1\,\Phi(X_1, Y) + a_2\,\Phi(X_2, Y)$$

$\Phi(X, X) = \mathbb{E}[X^2]$

$\Phi(X, X) = 0 \qquad \mathbb{E}[X^2] = 0 \quad \cancel{\Rightarrow} \quad X \equiv 0$

$\Rightarrow \quad X = 0 \quad \mathbb{P}$-q.c.

$N : X \in \mathcal{L}^2(\Omega, \mathbb{P}) \longmapsto \sqrt{\mathbb{E}[X^2]}$ è una seminorma

(DIM PER ESERCIZIO)

— o —

$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$\parallel$

$\mathbb{E}[(X - \mathbb{E}[X])^2]$

Se X è v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

Allora $\mathrm{Var}[X]$ esiste finita

SSE $X \in \mathcal{L}^2(\Omega, \mathbb{P})$

COVARIANZA DI DUE V.A.

Per $X, Y \in \mathcal{L}^2(\Omega, \mathbb{P})$ chiamo COVARIANZA DI X e Y il numero

$$Cov(X,Y) := \mathbb{E}\left[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])\right]$$

X e Y si dicono SCORRELATE se $Cov(X,Y) = 0$.

PROPRIETÀ

1) $Cov : (X,Y) \in \mathcal{L}^2(\Omega,\mathbb{P}) \times \mathcal{L}^2(\Omega,\mathbb{P}) \mapsto Cov(X,Y) \in \mathbb{R}$

è una forma bilineare simmetrica

2) $Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \quad \forall X, Y \in \mathcal{L}^2(\Omega,\mathbb{P})$

$a, b, c, d \in \mathbb{R} \quad X, Y \in \mathcal{L}^2(\Omega,\mathbb{P})$

$Cov(aX+b, cY+d) = ac\, Cov(X,Y)$

$\forall X \in \mathcal{L}^2(\Omega,\mathbb{P}) \quad Cov(X,X) = Var[X]$

OSSERVAZIONE $\quad X$ e Y sono scorrelate sse

$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

1) $Cov(X,Y) = Cov(Y,X)$ ovvio

$Cov(a_1X_1 + a_2X_2, Y) = ? \qquad a_1, a_2 \in \mathbb{R} \quad X_1, X_2, Y \in \mathcal{L}^2(\Omega,\mathbb{P})$

$$\mathbb{E}\left[\left((a_1X_1 + a_2X_2) - \mathbb{E}[a_1X_1 + a_2X_2]\right)(Y-\mathbb{E}[Y])\right] =$$

$$= \mathbb{E}\left[(a_1X_1 + a_2X_2 - a_1\mathbb{E}[X_1] - a_2\mathbb{E}[X_2])(Y-\mathbb{E}[Y])\right] =$$

$$= \mathbb{E}\left[\left(a_1(X_1-\mathbb{E}[X_1]) + a_2(X_2-\mathbb{E}[X_2])\right)(Y-\mathbb{E}[Y])\right] =$$

$$= \mathbb{E}\left[a_1(X_1-\mathbb{E}[X_1])(Y-\mathbb{E}[Y]) + a_2(X_2-\mathbb{E}[X_2])(Y-\mathbb{E}[Y])\right]$$

$$= a_1\mathbb{E}\left[(X_1-\mathbb{E}[X_1])(Y-\mathbb{E}[Y])\right] + a_2\mathbb{E}\left[(X_2-\mathbb{E}[X_2])(Y-\mathbb{E}[Y])\right]$$

$$= a_1 Cov(X_1,Y) + a_2 Cov(X_2,Y)$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] =$$

$$= \mathbb{E}[XY - \mathbb{E}[X]Y - \mathbb{E}[Y]X + \mathbb{E}[X]\mathbb{E}[Y]] =$$

$$= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \cancel{- \mathbb{E}[Y]\mathbb{E}[X]} + \cancel{\mathbb{E}[X]\mathbb{E}[Y]}$$

$$\mathrm{Cov}(aX+b,\ cY+d) = \mathbb{E}[(aX+b-\mathbb{E}[aX+b])(cY+d-\mathbb{E}[cY+d])]$$

$$= \mathbb{E}[(aX+\cancel{b}-a\mathbb{E}[X]-\cancel{b})(cY+\cancel{d}-c\mathbb{E}[Y]-\cancel{d})]$$

$$= \mathbb{E}[ac(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] =$$

$$= ac\,\mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] = ac\,\mathrm{Cov}(X,Y)$$

— o —

PROP $\forall X, Y \in \mathcal{L}^2(\Omega, \mathbb{P})$,

$$\mathrm{Cov}(X,Y) = \int_{\mathbb{R}^2} (x-\mathbb{E}[X])(y-\mathbb{E}[Y])\, \mathbb{P}_{X,Y}(dx\,dy)$$

$$= \int_{\mathbb{R}^2} xy\, \mathbb{P}_{X,Y}(dx\,dy) - \mathbb{E}[X]\mathbb{E}[Y].$$

DIM $\mathrm{Cov}(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] =$

$$\varphi : (x,y) \in \mathbb{R}^2 \mapsto (x-\mathbb{E}[X])(y-\mathbb{E}[Y]) \in \mathbb{R}$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}[\varphi\circ(X,Y)] = \mathbb{E}[\varphi^+\circ(X,Y)] - \mathbb{E}[\varphi^-\circ(X,Y)]$$

$$\mathbb{E}[\varphi^+\circ(X,Y)] = \int_{\mathbb{R}^2} \varphi^+(x,y)\, \mathbb{P}_{X,Y}(dx\,dy)$$

$$\mathbb{E}[\varphi^-\circ(X,Y)] = \int_{\mathbb{R}^2} \varphi^-(x,y)\, \mathbb{P}_{X,Y}(dx\,dy)$$

$$\mathrm{Cov}(X,Y) = \int_{\mathbb{R}^2} (\varphi^+ - \varphi^-)(x,y)\, \mathbb{P}_{X,Y}(dx\,dy)$$

$$= \int_{\mathbb{R}^2} (x - \mathbb{E}[X])(y - \mathbb{E}[Y])\, \mathbb{P}_{X,Y}(dx\,dy)$$

VARIANZA DELLA SOMMA

$X, Y \in \mathcal{L}^2(\Omega, \mathbb{P})$ $\quad \mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}(X,Y)$

$X_1, X_2, \dots, X_N \in \mathcal{L}^2(\Omega, \mathbb{P})$

$$\mathrm{Var}\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} \mathrm{Var}[X_i] + 2 \sum_{1 \le i < j \le N} \mathrm{Cov}(X_i, X_j)$$

DIM $\quad \mathrm{Var}[X+Y] = \mathbb{E}\left[(X+Y - \mathbb{E}[X+Y])^2\right] =$

$$= \mathbb{E}\left[\left(X+Y - \mathbb{E}[X] - \mathbb{E}[Y]\right)^2\right] =$$

$$= \mathbb{E}\left[\left((X - \mathbb{E}[X]) + (Y - \mathbb{E}[Y])\right)^2\right]$$

$$= \mathbb{E}\left[(X - \mathbb{E}[X])^2 + (Y - \mathbb{E}[Y])^2 + 2(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right]$$

$$= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] + \mathbb{E}\left[(Y - \mathbb{E}[Y])^2\right] + 2\,\mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right]$$

$$= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}(X,Y)$$

COEFFICIENTE DI CORRELAZIONE

$X, Y \in \mathcal{L}^2(\Omega, \mathbb{P})$ $\quad \mathrm{Var}[X], \mathrm{Var}[Y] > 0$

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}} = \frac{\mathrm{Cov}(X,Y)}{\sigma(X)\sigma(Y)}$$

$X, Y \in \mathcal{L}^2(\Omega, \mathbb{P})$ $\qquad \mathbb{E}[XY] \le \sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]}$

$\tilde{X} = X - \mathbb{E}[X]$ $\qquad Cov(X,Y) \le \sqrt{Var[X]\;Var[Y]}$

$\tilde{Y} = Y - \mathbb{E}[Y]$

$\hat{X} = -\tilde{X}$, $\hat{Y} = +\tilde{Y}$ $\qquad \mathbb{E}[\hat{X}\hat{Y}] = -\mathbb{E}[\tilde{X}\tilde{Y}] = -Cov(X,Y)$

$$-Cov(X,Y) \le \sqrt{Var[X]\,Var[Y]}$$

$$|Cov(X,Y)| \le \sqrt{Var[X]\,Var[Y]}$$

$\Rightarrow \quad Corr(X,Y) \in [-1,1]$

$Corr(X,Y) = 1$ $\qquad \mathbb{E}[\tilde{X}\tilde{Y}] = \sqrt{\mathbb{E}[\tilde{X}^2]\,\mathbb{E}[\tilde{Y}]^2}$

$$\Downarrow$$

$\tilde{X} = c\,\tilde{Y}$ $\qquad c > 0$

$$X - \mathbb{E}[X] = c\,(Y - \mathbb{E}[Y])$$

$$X = cY + (\mathbb{E}[X] - c\,\mathbb{E}[Y])$$

$Corr(X,Y) = -1 \Rightarrow X = cY + D$ $\qquad D \in \mathbb{R}$, $c < 0$