DISTRIBUZIONI A.C.

DISTRIBUZIONE UNIFORME SU UN INTERVALLO $U([a,b])$

Associata alla densità $f(x) = \frac{1}{b-a} \mathbb{1}_{(a,b)}(x)$

$$\int_{\mathbb{R}} f(x)\,dx = \frac{1}{b-a}\int_{\mathbb{R}} \mathbb{1}_{(a,b)}(x)\,dx = \frac{1}{b-a}\int_a^b 1\,dx = \frac{b-a}{b-a} = 1$$

Supponiamo che X sia v.a. con $P_X = U([a,b])$

$$E[|X|] = \int_{\mathbb{R}} |x| f(x)\,dx = \int_{\mathbb{R}} \frac{|x|}{b-a} \mathbb{1}_{(a,b)}(x)\,dx < +\infty$$

$$E[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_{\mathbb{R}} \frac{x}{b-a} \mathbb{1}_{(a,b)}(x)\,dx =$$

$$= \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\,\frac{x^2}{2}\Big|_{x=a}^{x=b} = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$

$$E[X^2] = \int_{\mathbb{R}} x^2 f(x)\,dx = \frac{1}{b-a}\int_a^b x^2\,dx = \frac{1}{b-a}\,\frac{x^3}{3}\Big|_{x=a}^{x=b}$$

$$= \frac{a^2+ab+b^2}{3}$$

$$Var[X] = \frac{a^2+ab+b^2}{3} - \frac{\overset{a^2+2ab+b^2}{(a+b)^2}}{4} = \frac{a^2-2ab+b^2}{12}$$

$$= \frac{(b-a)^2}{12}$$

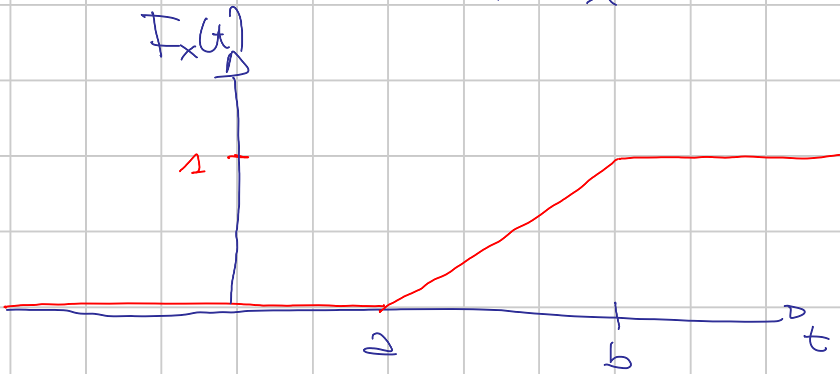
$$F_X(t) = \int_{-\infty}^{t} \frac{1}{b-a} \mathbb{1}_{(a,b)}(x)\,dx$$

$\Rightarrow \quad F_X(t) = 0 \qquad t \le a$

$t \in (a,b) \qquad F_X(t) = \int_a^t \frac{1}{b-a}\,dx = \frac{x}{b-a}\Big|_{x=a}^{x=t} = \frac{t-a}{b-a}$

$t \geq b \qquad F_X(t) = \int_{-\infty}^{t} f(x)\,dx = \int_{\mathbb{R}} f(x)\,dx = 1$

$$F_X(t) = \begin{cases} 0 & t \le a \\ \frac{t-a}{b-a} & t \in (a,b) \\ 1 & t \geq b \end{cases}$$



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DISTRIBUZIONE GAUSSIANA o NORMALE DI PARAMETRI $m \in \mathbb{R}$ e $\sigma > 0$ $\qquad N(m, \sigma^2)$

È la distribuzione associata alla densità

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-m)^2}{2\sigma^2}\right) \qquad x \in \mathbb{R}$$

$$\int_{\mathbb{R}} f(x)\,dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} \exp\left(-\left(\frac{x-m}{\sigma\sqrt{2}}\right)^2\right) dx$$

$t = \frac{x-m}{\sigma\sqrt{2}} \qquad x = m + t\sigma\sqrt{2} \qquad dx = \sigma\sqrt{2}\,dt$

$$= \frac{1}{\sqrt{\cancel{2}\pi\cancel{\sigma^2}}}\,\cancel{\sigma}\cancel{\sqrt{2}}\,\underbrace{\int_{\mathbb{R}} \exp(-t^2)\,dt}_{\sqrt{\pi}} = 1$$

Nel caso particolare $m=0$ $\sigma=1$ ho la densità

$$f_0(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) \qquad x\in\mathbb{R}$$

$N(0,1)$ si dice DISTRIBUZIONE GAUSSIANA (o NORMALE) STANDARD.

Sia X_0 una v.a t.c. $\mathbb{P}_{X_0} = N(0,1)$

$$\mathbb{E}[|X|] = \int_{\mathbb{R}} |x| f_0(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |x| \exp\left(\frac{-x^2}{2}\right) dx < +\infty$$

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f_0(x)\,dx = 0$$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f_0(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^2 \exp\left(\frac{-x^2}{2}\right) dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (-x) \Big(\underbrace{-x \exp\left(\frac{-x^2}{2}\right)}_{= \frac{d}{dx} \exp\left(-\frac{x^2}{2}\right)}\Big) dx$$

$$0 - 0 = 0$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \cancel{(-x) \exp\left(\frac{-x^2}{2}\right)} \Bigg|_{x\to-\infty}^{x\to+\infty} + \int_{\mathbb{R}} \exp\left(\frac{-x^2}{2}\right) dx \right\}$$

$$= \frac{1}{\sqrt{\cancel{2}\pi}} \int_{\mathbb{R}} \exp(-t^2) \cancel{\sqrt{2}}\, dt = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

$t = \frac{x}{\sqrt{2}}$

$x = t\sqrt{2}$

$dx = \sqrt{2}\,dt$

Se X_0 è t.c. $\mathbb{P}_{X_0} = N(0,1) \Rightarrow \begin{cases} \mathbb{E}[X] = 0 \\ \mathrm{Var}[X] = 1 \end{cases}$

Sia X_0 con $\mathbb{P}_X = N(0,1)$ e siano $m\in\mathbb{R}$ e $\sigma>0$

Considero la v.a $X := m + \sigma X_0$

$$f(x) = \frac{1}{\sigma} f_0\left(\frac{x-m}{\sigma}\right) = \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-m)^2}{2\sigma^2}\right)$$

$$\Rightarrow \mathbb{P}_X = N(m,\sigma^2)$$

$\forall Y$ v.a. A.C. $\mathbb{P}_Y = N(m,\sigma^2) \Rightarrow \mathbb{E}[Y] = \mathbb{E}[m+\sigma X_0]$

$Var[Y] = Var[m+\sigma X_0]$

$$\mathbb{E}[Y] = \mathbb{E}[m+\sigma X_0] = m+\sigma\,\mathbb{E}[X_0] = m+\sigma\cdot 0 = m$$
$$Var[Y] = Var[m+\sigma X_0] = \sigma^2\, Var[X_0] = \sigma^2\cdot 1 = \sigma^2$$

Sia Φ la legge associata a $N(0,1)$, allora
$$\Phi(t) + \Phi(-t) = 1 \quad \forall t \in \mathbb{R}$$

LEMMA Sia $\mathbb{P}_X$ una distribuzione A.C. la cui densità f è una funzione pari, allora $F_X(t) + F_X(-t) = 1$

$$F_X(-t) = \int_{-\infty}^{-t} f(x)\,dx \qquad y=-x$$

$$= \int_{+\infty}^{t} -f(-y)\,dy = \int_{t}^{+\infty} f(y)\,dy = \int_{\mathbb{R}} f(y)\,dy - \int_{-\infty}^{t} f(y)\,dy$$

$$= 1 - F_X(t)$$

DISTRIBUZIONE ESPONENZIALE DI PARAMETRO $\lambda > 0$

$\exp(\lambda)$

Associata alla densità $f(x) = \begin{cases} \lambda e^{-\lambda x} & x>0 \\ 0 & x\le 0 \end{cases}$

$$\int_{\mathbb{R}} f(x)\,dx = \int_0^{+\infty} \lambda e^{-\lambda x}\,dx = \cancel{\lambda}\cdot \frac{-1}{\cancel{\lambda}} e^{-\lambda x} \Big|_{x=0}^{x\to+\infty}$$

$$= 0 - (-1) = 1$$

N.B. Se X è v.c. $\mathbb{P}_X = \exp(\lambda) \Rightarrow X > 0$ $\mathbb{P}$ q.c.

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_0^{+\infty} x \underbrace{\lambda e^{-\lambda x}}_{-e^{-\lambda x}}\,dx =$$

$$= \cancel{x\left(-e^{-\lambda x}\right)}^{\,0\,-\,0}\Big|_{x=0}^{x\to+\infty} + \int_0^{+\infty} e^{-\lambda x}\,dx = -\frac{1}{\lambda} e^{-\lambda x}\Big|_{x=0}^{x\to+\infty}$$

$$= -\frac{1}{\lambda}(0-1) = \frac{1}{\lambda}$$

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)\,dx = \int_0^{+\infty} x^2 \underbrace{\lambda e^{-\lambda x}}_{\frac{d}{dx}\left(-e^{-\lambda x}\right)}\,dx =$$

$$= \underset{0\,-\,0}{\cancel{x^2\left(-e^{-\lambda x}\right)}}\Big|_{x=0}^{x\to+\infty} + \int_0^{+\infty} 2x e^{-\lambda x}\,dx =$$

$$= \frac{2}{\lambda}\int_0^{+\infty} x\lambda e^{-\lambda x}\,dx = \frac{2}{\lambda}\cdot\frac{1}{\lambda} = \frac{2}{\lambda^2}$$

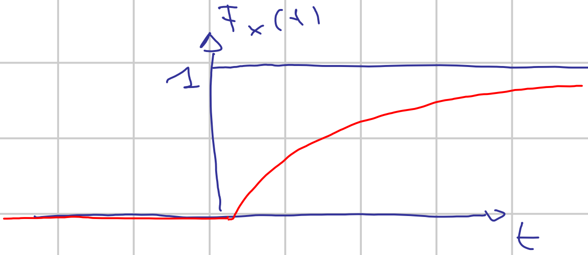
$$\Rightarrow \mathrm{Var}[X] = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

$$F_X(t) = \int_{-\infty}^{t} f(x)\,dx \quad \Rightarrow \quad F_X(t) = 0 \qquad t \le 0$$

$$t > 0 \qquad F_X(t) = \int_{-\infty}^{0} 0\,dx + \int_0^t \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_{x=0}^{x=t} =$$

$$= -e^{-\lambda t} + 1$$

$$F_X(t) = \begin{cases} 0 & t \le 0 \\ 1 - e^{-\lambda t} & t > 0 \end{cases}$$



Per $t, s \geq 0$ $\mathbb{P}(X \le t+s \mid X \geq s) =$

$$= \frac{\mathbb{P}(s \le X \le t+s)}{\mathbb{P}(X \geq s)} = \frac{\mathbb{P}(X \le t+s) - \mathbb{P}(X \le s)}{1 - \mathbb{P}(X \le s)}$$

$$= \frac{F_X(t+s) - F_X(s)}{1 - F_X(s)} = \frac{\left(\cancel{1} - e^{-\lambda(t+s)}\right) - \left(\cancel{1} - e^{-\lambda s}\right)}{\cancel{1} - \left(\cancel{1} - e^{-\lambda s}\right)} =$$

$$= \frac{e^{-\lambda s} - e^{-\lambda t - \lambda s}}{e^{-\lambda s}} = 1 - e^{-\lambda t} = F_X(t) = \mathbb{P}(X \le t)$$

Se X è una v.a. t.c. $\mathbb{P}_X = \exp(\lambda)$ $\forall \lambda > 0$, allora

$$\mathbb{P}(X \le t+s \mid X \ge s) = \mathbb{P}(X \le t) \quad \forall t, s \ge 0$$

(PROPRIETÀ DI MANCANZA DI MEMORIA)

TEOREMA Sia X v.a. non negativa t.c. $F_X(0) < 1$. Allora sono fatti equivalenti:

1) X è priva di memoria
2) $\mathbb{P}_X = \exp(\lambda)$ dove $\lambda = -\log(1 - F_X(1))$

N.B. Se $\mathbb{P}_X = \exp(\lambda)$ $F_X(1) = 1 - e^{-\lambda}$

$e^{-\lambda} = 1 - F_X(1)$ $\lambda = -\log(1 - F_X(1))$

LEMMA Sia $\alpha : [0, +\infty) \to \mathbb{R}$ con

1) α continua o α monotona non decrescente
2) $\alpha(t+s) = \alpha(t) + \alpha(s)$ $\forall t, s \in [0, +\infty)$

Allora $\alpha(t) = t\,\alpha(1)$ $\forall t \in [0, +\infty)$

DIM Scelgo $t = s = 0$ $\alpha(0) = 2\alpha(0) \Rightarrow \alpha(0) = 0$

$t = s = 1$ $\alpha(2) = \alpha(1) + \alpha(1) = 2\alpha(1)$

$t = 2$ $s = 1$ $\alpha(3) = \alpha(2) + \alpha(1) = 3\alpha(1)$

$t_1, \dots, t_n \ge 0 \Rightarrow \alpha\left(\sum_{i=1}^{n} t_i\right) = \sum_{i=1}^{n} \alpha(t_i)$ ←

Per $n = 2$ è vero per ipotesi

Per induzione: supponiamo sia vero fino a n addendi.

$t_1 \,\_\_\, t_n, t_{n+1} \geq 0 \quad \alpha\left(\sum_{i=1}^{n+1} t_i\right) = \alpha\left(\sum_{i=1}^{n} t_i + t_{n+1}\right) =$

$$= \alpha\left(\sum_{i=1}^{n} t_i\right) + \alpha(t_{n+1}) = \sum_{i=1}^{n} \alpha(t_i) + \alpha(t_{n+1}) = \sum_{i=1}^{n+1} \alpha(t_i)$$

In particolare, $t_1 = \dots = t_n = 1$ mi dà

$$\alpha(n) = \sum_{i=1}^{n} \alpha(1) = n\alpha(1)$$

$t_1 = \dots = t_k = \frac{1}{n} \quad \alpha\left(\frac{k}{n}\right) = \sum_{i=1}^{k} \alpha\left(\frac{1}{n}\right) = k\alpha\left(\frac{1}{n}\right)$

Se scelgo $k = n \quad \alpha(1) = n\alpha\left(\frac{1}{n}\right) \quad \alpha\left(\frac{1}{n}\right) = \frac{1}{n}\alpha(1)$

Con k e n generici in $\mathbb{N} \Rightarrow \alpha\left(\frac{k}{n}\right) = k\alpha\left(\frac{1}{n}\right) = \frac{k}{n}\alpha(1)$

$\Rightarrow \alpha(q) = q\alpha(1) \quad \forall q \in [0, +\infty) \cap \mathbb{Q}$

Sia $x \in [0, +\infty)$ Allora so che $\exists \{q_n\}_{n=1}^{\infty} \subseteq [0, +\infty) \cap \mathbb{Q}$
t.c. $\lim_{n \to \infty} q_n = x$

Allora, se α è continua io ho

$$\alpha(x) = \lim_{n \to \infty} \alpha(q_n) = \lim_{n \to \infty} q_n \alpha(1) = x\alpha(1)$$

Se α è monotona non decrescente, comunque
$\exists \{q_n\}_{n=1}^{\infty}, \{r_n\}_{n=1}^{\infty} \subset [0, +\infty) \cap \mathbb{Q}$ t.c.

$$q_n \nearrow x \qquad r_n \downarrow x$$

$$\Rightarrow q_n \alpha(1) = \alpha(q_n) \leq \alpha(x) \leq \alpha(r_n) = r_n \alpha(1)$$

$\forall n \in \mathbb{N} \quad q_n \alpha(1) \leq \alpha(x) \leq r_n \alpha(1)$
Passando a limite su $n \quad x\alpha(1) \leq \alpha(x) \leq x\alpha(1)$

$$= 0 \qquad \alpha(x) = x\,\alpha(1) \qquad \forall x \in [0, +\infty)$$

DIM DEL TEO Supponiamo che X manchi di memoria

$$\mathbb{P}(X \leq t+s \mid X \geq t) = \mathbb{P}(X \leq s) \qquad \forall t, s \geq 0$$

$$\frac{\mathbb{P}(X \leq t+s,\ X \geq t)}{\mathbb{P}(X \geq t)} = \mathbb{P}(X \leq s)$$

$$= \frac{\mathbb{P}(X \leq t+s) - \mathbb{P}(X < t)}{1 - \mathbb{P}(X < t)} = \mathbb{P}(X \leq s)$$

So che $\mathbb{P}(X < t) = \lim_{x \to t^-} \mathbb{P}(X \leq x) = \lim_{x \to t^-} F_X(x) =: F_X(t^-)$

$$F_X(t+s) - F_X(t^-) = F_X(s)\left(1 - F_X(t^-)\right)$$

$$F_X(t) - F_X(t^-) = \mathbb{P}(X = t) =: \delta_X(t)$$

$$F_X(t+s) - \left(F_X(t) - \delta_X(t)\right) = F_X(s)\left(1 - \left(F_X(t) - \delta_X(t)\right)\right)$$

$$F_X(t+s) - F_X(t) + \delta_X(t) = F_X(s)\left(1 - F_X(t) + \delta_X(t)\right)$$

Passo a limite per $s \to 0^+$

$$\cancel{F_X(t)} - \cancel{F_X(t)} + \delta_X(t) = F_X(0)\left(1 - F_X(t) + \delta_X(t)\right)$$

$$\delta_X(t)\left(1 - F_X(0)\right) = F_X(0)\left(1 - F_X(t)\right)$$

$$\mathbb{P}(X = t) = \frac{1}{1 - F_X(0)} F_X(0)\left(1 - F_X(t)\right)$$

Poichè $F_X(0) \neq 1$ e F_X è continua da destra

$\Rightarrow \exists \bar{t} > 0$ t.c. $F_X(t) < 1 \quad \forall t \in [0, \bar{t})$

Se fosse $F_X(0) \neq 0$ avrei $\mathbb{P}(X = t) \neq 0 \quad \forall t \in [0, \bar{t})$

ASSURDO $\Rightarrow F_X(0) = 0$

$\Rightarrow \mathbb{P}(X = t) = 0 \quad \forall t \in [0, +\infty) \Rightarrow F_X$ è continua su $[0, +\infty)$

$$F_X(t+s) - F_X(t) = F_X(s)\left(1 - F_X(t)\right)$$

$$g(t) = 1 - F_X(t)$$

$$\left(1 - g(t+s)\right) - \left(1 - g(t)\right) = \left(1 - g(s)\right)\left(1 - \left(1 - g(t)\right)\right)$$

$$\Rightarrow \cancel{g(t)} - g(t+s) = g(t)\left(\cancel{1} - g(s)\right)$$

$$g(t+s) = g(t)g(s) \qquad g(t) = 1 - F_X(t)$$

Supponiamo $\exists \bar{t}$ t.c. $F_X(\bar{t}) = 1$ cioè $g(\bar{t}) = 0$

$t = s = \frac{1}{2}\bar{t} \qquad 0 = g(\bar{t}) = \left(g\left(\frac{\bar{t}}{2}\right)\right)^2 \Rightarrow g\left(\frac{\bar{t}}{2}\right) = 0$

$t = s = \frac{1}{4}\bar{t} \qquad 0 = g\left(\frac{\bar{t}}{2}\right) = \left(g\left(\frac{\bar{t}}{4}\right)\right)^2 \Rightarrow g\left(\frac{\bar{t}}{4}\right) = 0$

Per induzione si dimostra $g\left(\frac{\bar{t}}{2^n}\right) = 0 \quad \forall n \in \mathbb{N}$

$\Rightarrow \lim_{n \to \infty} g\left(\frac{\bar{t}}{2^n}\right) = g(0) \Rightarrow g(0) = 0 \qquad F_X(0) = 1$

ASSURDO

$\Rightarrow g(t) > 0 \quad \forall t \in [0, +\infty)$

$$g(t+s) = g(t)g(s)$$

$$\log\left(1 - F_X(t+s)\right) = \log\left(1 - F_X(t)\right) + \log\left(1 - F_X(s)\right)$$

$\ell(t) := \log\left(1 - F_X(t)\right)$ è continua

$$\alpha(t+s) = \alpha(t) + \alpha(s)$$

$\Rightarrow$ per il lemme $\alpha(t) = t\,\alpha(1)$ $\forall t \geq 0$

$$\log(1-F_X(t)) = t \log(1-F_X(1))$$

$$1-F_X(t) = (1-F_X(1))^t = e^{t\log(1-F_X(1))} = e^{-\lambda t}$$

$$\lambda := -\log(1-F_X(1)) > 0$$

$$F_X(t) = 1 - e^{-\lambda t} \qquad \forall t \geq 0$$

$$\Downarrow$$

$$\mathbb{P}_X = \exp(\lambda) \qquad \lambda = -\log(1-F_X(1))$$