# DISTRIBUZIONI DISCRETE 3

Note Title 11/11/2016

$$H(b,r,n)(\{k\}) = \frac{\binom{b}{k}\binom{r}{n-k}}{\binom{b+r}{n}} \qquad k=0,1,\ \_\_\_\ ,n-1,n$$

$$\mathbb{P}_X = H(b,r,n) \quad \Rightarrow \quad \mathbb{E}[X] = \frac{nb}{b+r} \qquad p := \frac{b}{b+r} \qquad np$$

$$\mathbb{E}[X^2] = \sum_{k=0}^{n} k^2 \frac{\binom{b}{k}\binom{r}{n-k}}{\binom{b+r}{n}} =$$

$$= \frac{1}{\binom{b+r}{n}} \sum_{k=0}^{n} k(k-1)\binom{b}{k}\binom{r}{n-k} + \overbrace{\frac{1}{\binom{b+r}{n}} \sum_{k=0}^{n} k \binom{b}{k}\binom{r}{n-k}}^{\mathbb{E}[X] = \frac{nb}{b+r}}$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \underbrace{k(k-1)\binom{b}{k}}\binom{r}{n-k}\right) z^n =$$

$$= \left(\sum_{n=0}^{\infty} n(n-1)\binom{b}{n} z^n\right)\left(\sum_{n=0}^{\infty}\binom{r}{n} z^n\right) =$$

$$= z^2\left(\sum_{n=0}^{\infty}\binom{b}{n}\frac{d^2}{dz^2} z^n\right)(1+z)^r = \qquad \sum_{n=0}^{\infty}\binom{b}{n} z^n = (1+z)^b$$

$$= z^2\left(\frac{d^2}{dz^2}(1+z)^b\right)(1+z)^r =$$

$$= b(b-1)z^2(1+z)^{b-2}(1+z)^r = b(b-1)z^2(1+z)^{b+r-2}$$

$$= b(b-1)z^2 \sum_{j=0}^{\infty}\binom{b+r-2}{j} z^j \qquad n := j+2$$

$$= \sum_{n=2}^{\infty} b(b-1)\binom{b+r-2}{n-2} z^n$$

$$\mathbb{E}[X^2] = \frac{1}{\binom{b+r}{n}} b(b-1)\binom{b+r-2}{n-2} + \frac{nb}{b+r} =$$

$$= \frac{b(b-1)(b+r-2)!}{(n-2)!\,\cancel{(b+r-n)!}} \cdot \frac{n!\,\cancel{(b+r-n)!}}{(b+r)!} + \frac{nb}{b+r}$$

$$= \frac{b(b-1)\,n(n-1)}{(b+r)(b+r-1)} + \frac{nb}{b+r} =$$

$$= \frac{nb}{(b+r)(b+r-1)} \left\{ \overset{bn-n-\cancel{b}+\cancel{1}+\cancel{b}+r-\cancel{1}}{(b-1)(n-1) + b+r-1} \right\}$$

$$= \frac{nb}{(b+r)(b+r-1)} (bn-n+r)$$

$$\mathrm{Var}[X] = E[X^2] - (E[X])^2 = \frac{nb(bn-n+r)}{(b+r)(b+r-1)} - \left(\frac{nb}{b+r}\right)^2$$

$$= \frac{nb}{(b+r)^2(b+r-1)} \left\{ (\cancel{bn}-n+r)(b+r) - nb(\cancel{b+r}-1) \right\}$$

$$\cancel{-nb} - nr + br + r^2 \cancel{+nb}$$

$$\frac{nbr}{(b+r)^2(b+r-1)} (b+r-n)$$

$$= n \frac{b}{b+r} \frac{r}{b+r} \frac{1-\frac{n}{b+r}}{1-\frac{1}{b+r}}$$

$$p := \frac{b}{b+r}$$

$$\Rightarrow \frac{r}{b+r} = 1-p$$

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$$Y = X - 1$$

$$P_X = G'(p) \qquad G'(p)(\{k\}) = p(1-p)^k \quad k = 0,1,2 \ldots$$

$$F_X(k) = P(X \le k) = \sum_{j=0}^{k} P(X=j) = \sum_{j=0}^{k} p(1-p)^j =$$

$$= p\,\frac{1-(1-p)^{k+1}}{1-(1-p)} = 1-(1-p)^{k+1} \qquad k = 0,1,2,\ldots$$

$$i,j = 0,1,2,\ldots \qquad P(X \le i+j \mid X \ge j) =$$

$$\frac{\mathbb{P}(X \le i+j,\ X \ge j)}{\mathbb{P}(X \ge j)} = \frac{\mathbb{P}(X \le i+j) - \mathbb{P}(X \le j-1)}{1 - \mathbb{P}(X \le j-1)} =$$

$$= \frac{\left(\cancel{1} - (1-p)^{i+j+1}\right) - \left(\cancel{1} - (1-p)^{j}\right)}{\cancel{1} - \left(\cancel{1} - (1-p)^{j}\right)} =$$

$$= \frac{(1-p)^{j} - (1-p)^{i+j+1}}{(1-p)^{j}} = 1 - (1-p)^{i+1} = F_X(i) = \mathbb{P}(X \le i)$$

$$\Rightarrow (\star)\quad \mathbb{P}(X \le i+j \mid X \ge j) = \mathbb{P}(X \le i) \quad \forall i,j = 0,1,2,\ldots$$

PROP Sia $X$ v.a. concentrata su $\mathbb{N}_0$ t.c. $\mathbb{P}(X=0) = p \in (0,1)$

Allora $\mathbb{P}_X = G'(p)$ SSE $X$ soddisfa $(\star)$

DIM Scelgo $i=0$ $\quad \mathbb{P}(X \le j \mid X \ge j) = \mathbb{P}(X \le 0)$

$$\frac{\mathbb{P}(X=j)}{\mathbb{P}(X \ge j)} = p \qquad \mathbb{P}(X \le 0) = \mathbb{P}(X=0) = p \in (0,1)$$

$$\mathbb{P}(X=j) = p\,\mathbb{P}(X \ge j) \quad \forall j = 0,1,2,\ldots$$

$$\mathbb{P}(X=j+1) = p\,\mathbb{P}(X \ge j+1)$$

$$\mathbb{P}(X=j) - \mathbb{P}(X=j+1) = p\,\mathbb{P}(X=j) \qquad \mathbb{P}(X \ge j) - \mathbb{P}(X \ge j+1) = \mathbb{P}(X=j)$$

$$\mathbb{P}(X=j+1) = (1-p)\,\mathbb{P}(X=j) \quad \forall j = 0,1,2,\ldots$$

$j=0 \quad \mathbb{P}(X=1) = (1-p)p$

$j=1 \quad \mathbb{P}(X=2) = (1-p)(1-p)p = p(1-p)^2$

Per induzione si dimostra $\mathbb{P}(X=k) = p(1-p)^k$ $\quad \forall k = 0,1,2,\ldots$

Contare il # di insuccessi prima di ottenere l'n-esimo successo

$$\{X=k\} \qquad X(\omega)=k \qquad \omega=\{\omega_i\}_{i=0}^{\infty} \subset \{0,1\}^{\infty}$$

$E_{n+k,A}$ $\quad A=\{\omega=(\omega_1 \ldots \omega_{n+k}) : \omega_{n+k}=1$, in $(\omega_1, \ldots \omega_{n+k-1})$ ci sono $k$ "0" e $n-1$ "1" $\}$

$$\mathbb{P}(X=k) = \mathbb{P}_{n+k}(A) = p\,(p)^{n-1}(1-p)^k \binom{n+k-1}{k}$$
$$= \binom{n+k-1}{k} p^n (1-p)^k$$

Chiamo DISTRIBUZIONE BINOMIALE NEGATIVA DI PARAMETRI $-n$ E $p$, la distribuzione $B(-n,p)$ concentrata su $\mathbb{N}_0$ t.c.

$$B(-n,p)(\{k\}) = \binom{n+k-1}{k} p^n (1-p)^k \qquad k=0,1,2,\ldots$$

$$\binom{n+k-1}{k}(-1)^k = \frac{(n+k-1)!\,(-1)^k}{k!\,(n-1)!} = \frac{1}{k!}\underbrace{(n+k-1)(n+k-2)\ldots n}_{k \text{ fattori}}(-1)^k$$

$$= \frac{1}{k!}(-n)(-n-1)\ldots(-n-k+2)(-n-k+1) = \binom{-n}{k}$$

$$(1+z)^{\alpha} = \sum_{k=0}^{\infty}\binom{\alpha}{k} z^k \qquad |z|<1$$

$$\sum_{k=0}^{\infty} B(-n,p)(\{k\}) = \sum_{k=0}^{\infty}\binom{n+k-1}{k} p^n \underbrace{(1-p)^k}_{(-1)^k z^k} = \quad z:=p-1$$

$$= p^n \sum_{k=0}^{\infty}\binom{-n}{k} z^k \Big|_{z=p-1} = p^n (1+z)^{-n}\Big|_{z=p-1} = 1$$

Se $\mathbb{P}_X = B(-n, p)$

$$E[X] = \frac{n(1-p)}{p} \qquad Var[X] = \frac{n(1-p)}{p^2}$$

FOGLIO 4 - EX 5

$$\int_{\mathbb{R}} f(x)\,dx = \frac{1}{\pi b}\int_{\mathbb{R}} \frac{1}{1+\left(\frac{x-a}{b}\right)^2}\,dx \qquad t = \frac{x-a}{b}$$

$$x = a + bt \qquad dx = b\,dt$$

$$= \frac{1}{\pi \cancel{b}}\int_{-\infty}^{+\infty} \frac{1}{1+t^2}\,\cancel{b}\,dt = \frac{1}{\pi}\left(\operatorname{arctg}(t)\right)\Big|_{t\to-\infty}^{t\to+\infty}$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1$$

$$\int_{\mathbb{R}} x f(x)\,dx$$

$$\int_{\mathbb{R}} |x| f(x)\,dx \qquad \int_{\mathbb{R}} (x f(x))^+\,dx$$

$$\int_{\mathbb{R}} (x f(x))^-\,dx$$

$$E[X^+] = \int_{\mathbb{R}} (x f(x))^+\,dx = \int_0^{+\infty} \frac{1}{\pi b}\,\frac{x}{1+\left(\frac{x-a}{b}\right)^2}\,dx \qquad t = \frac{x-a}{b}$$

$$x = a + bt \qquad dx = b\,dt$$

$$= \int_{-a/b}^{+\infty} \frac{1}{\pi \cancel{b}}\,\cancel{b}\,\frac{a+bt}{1+t^2}\,dt =$$

$$= \frac{1}{\pi}\left\{ a \operatorname{artg}(t) + \frac{b}{2}\log(1+t^2)\right\}\Big|_{t=-a/b}^{t\to+\infty} = +\infty$$

$$E[X^-] = \int_{\mathbb{R}} (x f(x))^-\,dx = \int_{-\infty}^{0} -x\,\frac{1}{\pi b}\,\frac{1}{1+\left(\frac{x-a}{b}\right)^2}\,dx$$

$$= \frac{1}{\pi \cancel{b}} \int_{-\infty}^{-a/b} -(a+bt) \frac{1}{1+t^2} \cancel{b}\, dt = \qquad x = a+bt \qquad t = \frac{x-a}{b}$$

$$= \frac{1}{\pi} \left\{ -a \operatorname{artg}(t) - \frac{b}{2} \log(1+t^2) \right\} \Big|_{t \to -\infty}^{t = -\frac{a}{b}} = +\infty$$

$a = 0 \qquad b = 1 \qquad f(x) = \frac{1}{\pi} \frac{1}{1+x^2} \quad x \in \mathbb{R}$

$$\nexists \int_{\mathbb{R}} \frac{x}{\pi} \cdot \frac{1}{1+x^2} \, dx$$

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FOGLIO 3 EX 6

$X_1$ = v.a. che guarda l'estratta da $U_1$

$X_2$ = " da $U_2$

$X = \max\{X_1, X_2\}$

$X(\Omega) = \{1, \_\_ n\}$

$k \in \{1, \_\_ n\} \qquad \{X = k\} = \{X_1 = k, X_2 \le k\} \cup \{X_1 < k, X_2 = k\}$

$$\mathbb{P}(X=k) = \mathbb{P}(X_1 = k, X_2 \le k) + \mathbb{P}(X_1 \le k-1, X_2 = k)$$
$$= \mathbb{P}(X_1 = k)\mathbb{P}(X_2 \le k) + \mathbb{P}(X_1 \le k-1)\mathbb{P}(X_2 = k)$$
$$= \frac{1}{n}\frac{k}{n} + \frac{k-1}{n}\frac{1}{n} = \frac{2k-1}{n^2} \qquad \forall k = 1, \_\_ n$$

$$\mathbb{E}[X] = \sum_{k=0}^{n} k \frac{2k-1}{n^2} = \frac{2}{n^2}\sum_{k=0}^{n} k^2 - \frac{1}{n^2}\sum_{k=0}^{n} k =$$

$$= \frac{2}{n\cancel{2}} \frac{\cancel{n}(n+1)(2n+1)}{6} - \frac{1}{n\cancel{2}} \frac{\cancel{n}(n+1)}{2} =$$

$$= \frac{1}{6n}\left\{ 2(n+1)(2n+1) - 3(n+1) \right\} = \frac{n+1}{6n}\left(4n+2-3\right) = \frac{(n+1)(4n-1)}{6n}$$

Ex 8 Foglio 3

$x>0 \qquad g(x) = \frac{1}{2\sqrt{x}} \left( f(\sqrt{x}) + f(-\sqrt{x}) \right)$

$f(\sqrt{x}) \neq 0$ sse $x \in (0,1)$ $\qquad g(x) = 0 \quad x \geq 1$

$f(-\sqrt{x}) \neq 0$ sse $x \in (0,1)$

$x \in (0,1) \qquad f(\sqrt{x}) = f(-\sqrt{x}) = |\sqrt{x}| = \sqrt{x}$

$$g(x) = \frac{1}{2\sqrt{x}} \left( \sqrt{x} + \sqrt{x} \right) = 1$$

$g(x) = \mathbb{1}_{(0,1)}(x) \qquad \mathbb{P}_X(A) = \mathcal{L}^1(A) \quad \forall A \in \mathcal{B}([0,1])$

$\mathbb{P}(X = n) = p(1-p)^{n-1}$ $\qquad$ 6B 6R

3+3

$$p = \frac{\binom{6}{3}\binom{6}{3}}{\binom{12}{6}}$$

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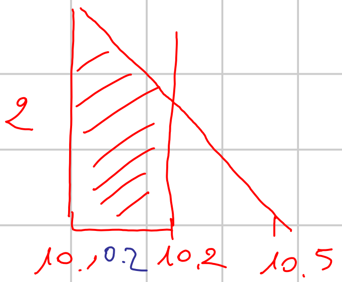
$$\mathbb{P}(X \in [9.8, 10.2]) = \int_{9.8}^{10.2} f(x)\,dx = 2\int_{10}^{10.2} f(x)\,dx =$$




$$= \cancel{2}\,\cancel{\tfrac{1}{2}} \cdot 0.2 \left( 2 + \frac{3}{5} \cdot 2 \right) = \frac{4}{10} \cdot \frac{16}{10} = \frac{64}{100}$$

$$= 0.64$$

$p = 0.64 \qquad n = 20$

$$B(n,p)(\{19,20\}) = B(n,p)(\{19\}) + B(n,p)(\{20\}) =$$

$$\binom{20}{19} p^{19}(1-p) + \binom{20}{20} p^{20}(1-p)^0 =$$

$$= 20\,p^{19}(1-p) + p^{20} = p^{19}\left(20(1-p) + p\right)$$

$$= (0.64)^{19}\left(20 \cdot 0.36 + 0.64\right) = (0.64)^{19}\;7.84$$

$$7.2 + 0.64$$