INTEGRAZIONE E DISTRIBUZIONE

Note Title 04/11/2016

Se X è una v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ allora $\mathbb{E}[X]$ esiste ssе $\exists \int_{\mathbb{R}} t\, \mathbb{P}_X(dt)$ e in tal caso $\mathbb{E}[X] = \int_{\mathbb{R}} t\, \mathbb{P}_X(dt)$.

— o —

Se X è una v.a. discreta con $X(\Omega) = \{t_j\}_{j\in\mathbb{N}}$ e $p_j := \mathbb{P}(X = t_j)$, e se $\mathbb{E}[X]$ esiste finito, allora

$$\mathbb{E}[|X|] = \sum_{j\in\mathbb{N}} |t_j| p_j \quad \text{e} \quad \mathbb{E}[X] = \sum_{j\in\mathbb{N}} t_j p_j$$

Più in generale si può dimostrare che se X è v.a. discreta con $X(\Omega) = \{t_j\}_{j\in\mathbb{N}}$ e definito $p_j := \mathbb{P}(X = t_j)$ e, se $\varphi : \mathbb{R} \to \mathbb{R}$ è una funzione di Borel non negativa, allora

$$\mathbb{E}[\varphi \circ X] = \sum_{j\in\mathbb{N}} \varphi(t_j) p_j$$

(DIM PER ESERCIZIO)

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$X : \Omega \to \mathbb{R}$ con distribuzione A.C. e densità $f : \mathbb{R} \to [0, +\infty)$. Allora $\forall \varphi : \mathbb{R} \to \mathbb{R}$ di Borel non negativa, si ha

$$\mathbb{E}[\varphi \circ X] = \int_{\mathbb{R}} \varphi(t) f(t)\, dt$$

DIM 1° caso φ semplice non negativa

$$\varphi(t) = \sum_{i=1}^{n} c_i \mathbb{1}_{E_i}(t) \qquad \{E_i\}_{i=1}^{n} \text{ partizione di } \mathbb{R} \text{ in boreliani}$$

$$\mathbb{E}[\varphi \circ X] = \int_{\Omega} \varphi(X(\omega))\, \mathbb{P}(d\omega) = \int_{\mathbb{R}} \varphi(t)\, \mathbb{P}_X(dt) =$$

$$= \int_{\mathbb{R}} \sum_{i=1}^{n} c_i \mathbb{1}_{E_i}(t) \mathbb{P}_X(dt) = \sum_{i=1}^{n} c_i \mathbb{P}_X(E_i)$$

$$\mathbb{P}_X(E_i) = \mathbb{P}(X \in E_i) = \int_{E_i} f(t)\,dt \Rightarrow$$

$$\mathbb{E}[\varphi \circ X] = \sum_{i=1}^{n} c_i \int_{E_i} f(t)\,dt = \sum_{i=1}^{n} c_i \int_{\mathbb{R}} f(t) \mathbb{1}_{E_i}(t)\,dt =$$

$$= \int_{\mathbb{R}} f(t) \underbrace{\left(\sum_{i=1}^{n} c_i \mathbb{1}_{E_i}(t) \right)}_{= \varphi(t)} dt = \int_{\mathbb{R}} f(t) \varphi(t)\,dt$$

2° caso φ di Borel nonnegativa

So che $\exists \{\varphi_k\}_{k=1}^{\infty}$ con φ_k funzione semplice non negative

t.c. $0 \le \varphi_k(t) \le \varphi(t) \quad \forall t \in \mathbb{R}$

$\varphi_k(t) \le \varphi_{k+1}(t) \quad \forall t \in \mathbb{R} \quad \forall k \in \mathbb{N}$

$\lim_{k \to \infty} \varphi_k(t) = \varphi(t) \quad \forall t \in \mathbb{R}$

$$\mathbb{E}[\varphi_k \circ X] = \int_{\mathbb{R}} \varphi_k(t) f(t)\,dt \qquad \forall k \in \mathbb{N}$$

Pongo $\psi_k := \varphi_k \circ X \Rightarrow \psi_k$ è funzione semplice su $(\Omega, \mathcal{E}, \mathbb{P})$

$0 \le \psi_k(\omega) \le \psi_{k+1}(\omega)$

$\lim_{k \to \infty} \psi_k(\omega) = (\varphi \circ X)(\omega)$

Applico Beppo Levi a ψ_k e a $\mathbb{P}$

$\mathbb{E}[\varphi_k \circ X] \to \mathbb{E}[\varphi \circ X]$

D'altra parte f è nonnegativa $\Rightarrow$ applico Beppo Levi a

$g_k(t) = \varphi_k(t) f(t)$ con $\mathcal{L}^1$

$$\int_{\mathbb{R}} \varphi_k(t) f(t)\,dt \to \int_{\mathbb{R}} \varphi(t) f(t)\,dt$$

Per l'unicità del limite ottengo la tesi.

FORMULA DI COMPOSIZIONE

Sia $X : \Omega \to \mathbb{R}$ v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

Sia $\varphi : \mathbb{R} \to \mathbb{R}$ di Borel

Sia $\psi : \mathbb{R} \to \mathbb{R}$ di Borel non negativa.

Considero $\psi \circ \varphi \circ X$

$$\mathbb{E}[\psi \circ \varphi \circ X] = \mathbb{E}[\psi \circ (\varphi \circ X)] = \int_{\mathbb{R}} \psi(s)\, \mathbb{P}_{\varphi \circ X}(ds)$$

$$\mathbb{E}[(\psi \circ \varphi) \circ X] = \int_{\mathbb{R}} \psi(\varphi(t))\, \mathbb{P}_X(dt)$$

$$\Rightarrow \int_{\mathbb{R}} (\psi \circ \varphi)(t)\, \mathbb{P}_X(dt) = \int_{\mathbb{R}} \psi(s)\, \mathbb{P}_{\varphi \circ X}(ds)$$

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FORMULA DI CAVALIERI

Sia $(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato e sia $X : \Omega \to \mathbb{R}$ una v.a. non negativa.

Allora $$\mathbb{E}[X] = \int_0^{+\infty} \mathbb{P}(X > t)\, dt = \int_0^{+\infty} (1 - F_X(t))\, dt$$

1° CASO X semplice non negativa.

$$X(\omega) = \sum_{i=1}^n c_i \mathbb{1}_{E_i}(\omega) \qquad 0 \le c_1 < c_2 < \dots < c_n$$

$\{E_i\}_{i=1}^n$ partizione di Ω in eventi.

$$\{X > t\} = \bigcup_{i=1}^n \left(\{X > t\} \cap E_i \right)$$

$$\mathbb{P}(X > t) = \sum_{i=1}^n \mathbb{P}\left(\{X > t\} \cap E_i \right)$$

$$\int_0^{\infty} \mathbb{P}(X > t)\, dt = \int_0^{+\infty} \sum_{i=1}^n \mathbb{P}\left(\{X > t\} \cap E_i \right) dt =$$

$$= \sum_{i=1}^n \int_0^{+\infty} \mathbb{P}\left(\{X > t\} \cap E_i \right) dt \qquad \{X > t\} \cap E_i = \{X > t\} \cap \{X = c_i\}$$

$$= \begin{cases} \{X = c_i\} = E_i & c_i > t \\ \emptyset & c_i < t \end{cases}$$

$$= \sum_{i=1}^n \int_0^{c_i} \mathbb{P}(E_i)\, dt =$$

$$= \sum_{i=1}^{n} c_i \mathbb{P}(E_i) = \mathbb{E}[X].$$

2° CASO X v.a. non negativa. ($\mathcal{E}$-misurabile nonnegativa)

So che $\exists \{f_k\}_{k=1}^{\infty}$ $f_k : \Omega \to \mathbb{R}$ semplici non negative t.c.

$$0 \le f_k(\omega) \le f_{k+1}(\omega) \quad \forall k \in \mathbb{N} \quad \forall \omega \in \Omega$$

$$\lim_{k \to +\infty} f_k(\omega) = X(\omega) \quad \forall \omega \in \Omega.$$

$$\int_0^{+\infty} \mathbb{P}(f_k > t)\,dt = \int_\Omega f_k(\omega)\mathbb{P}(d\omega) \xrightarrow[\text{Beppo Levi}]{} \int_\Omega X(\omega)\mathbb{P}(d\omega) = \mathbb{E}[X]$$

$$\{f_k > t\} \subseteq \{f_{k+1} > t\}$$

$$\bigcup_{k=1}^{\infty} \{f_k > t\} = \{X > t\}$$

$$g_k(t) = \mathbb{P}(f_k > t) \qquad 0 \le g_k(t) \le g_{k+1}(t)$$

$$\lim_{k \to \infty} g_k(t) = \lim_{k \to \infty} \mathbb{P}(f_k > t) = \mathbb{P}\left(\bigcup_{k=1}^{\infty} \{f_k > t\}\right) = \mathbb{P}(X > t)$$

$$\int_0^{+\infty} \mathbb{P}(f_k > t)\,dt = \int_0^{+\infty} g_k(t)\,dt \longrightarrow \int_0^{+\infty} \mathbb{P}(X > t)\,dt$$

Per l'unicità del limite

$$\mathbb{E}[X] = \int_0^{+\infty} \mathbb{P}(X > t)\,dt$$

Considero $X^+ = \max\{0, X\}$ e $X^- = \max\{0, -X\}$

$$\mathbb{E}[X^+] = \int_0^{+\infty} \mathbb{P}(X^+ > t)\,dt$$

$$\{X^+ > t\} = \begin{cases} \Omega & t < 0 \\ \{X > t\} & t > 0 \end{cases}$$

$$\mathbb{E}[X^+] = \int_0^{+\infty} \mathbb{P}(X > t)\,dt = \int_0^{+\infty} (1 - \mathbb{P}(X \le t))\,dt$$
$$= \int_0^{+\infty} (1 - F_X(t))\,dt$$

$$\{X^- > t\} = \begin{cases} \Omega & t < 0 \\ ? & t > 0 \end{cases}$$

$$\{\max\{0, -X\} > t\} = \{-X > t\} = \{X < -t\}$$

$$\mathbb{E}[X^-] = \int_0^{+\infty} \mathbb{P}(X^- > t)\,dt = \int_0^{+\infty} \mathbb{P}(X < -t)\,dt$$

$$s = -t \qquad -\int_0^{-\infty} \mathbb{P}(X < s)\,ds$$

$$\mathbb{P}(X < s) = \mathbb{P}(X \le s) - \mathbb{P}(X = s)$$

$$\mathcal{L}^1\{s \in \mathbb{R} : \mathbb{P}(X = s) \neq 0\} = 0 \Rightarrow \mathbb{P}(X = s) = 0 \quad \mathcal{L}^1\text{-q.o.}$$

$$\mathbb{E}[X^-] = +\int_{-\infty}^{0} \mathbb{P}(X \le s)\,ds = \int_{-\infty}^{0} F_X(s)\,ds$$

$$\mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-] = \int_0^{+\infty} (1 - F_X(t))\,dt - \int_{-\infty}^{0} F_X(t)\,dt$$

Esercizio: calcolare il valore atteso del tempo di attesa al semaforo

A = estraggo 1B e 1R

A_1 = bianca da U_1 e rossa da U_2

A_2 = bianca da U_2 e rossa da U_1

$$\mathbb{P}(A \mid TT) = \frac{\binom{3}{1}\binom{7}{1}}{\binom{10}{2}} = \frac{21}{\binom{10}{2}}$$

$$\mathbb{P}(A \mid TC) = \mathbb{P}(A_1 \mid TC) + \mathbb{P}(A_2 \mid TC) =$$

$$= \frac{3}{10}\frac{5}{10} + \frac{5}{10}\frac{7}{10} = \frac{5}{10}\left(\frac{3}{10} + \frac{7}{10}\right) = \frac{5}{10} = \frac{1}{2}$$

$$\mathbb{P}(A \mid CC) = \frac{\binom{5}{1}\binom{5}{1}}{\binom{10}{2}} = \frac{25}{\binom{10}{2}}$$

$$P(A) = P(A|TT)P(TT) + P(A|TC)P(TC) + P(A|CC)P(CC)$$

$$\frac{21}{\binom{10}{2}} \cdot \frac{1}{4} + \frac{1}{2}\,\frac{1}{2} + \frac{25}{\binom{10}{2}} \cdot \frac{1}{4}$$

$$= \frac{21}{45} \cdot \frac{1}{4} + \frac{1}{4} + \frac{25}{45} \cdot \frac{1}{4} = \frac{1}{4 \cdot 45}\left(21 + 45 + 25\right)$$

$$= \frac{91}{180}$$

$$P(TC|A) = \frac{P(A|TC)P(TC)}{P(A)} = \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{91}{180}} = \frac{180}{2 \cdot 2}\,\frac{1}{91} = \frac{45}{91}$$

S = estraggo 2 palline dello stesso colore da U_1

D = estraggo 2 palline di colore diverso da U_1

G_1 = estraggo 1! gialla da U_2

$$P(G_1|S) = \frac{\binom{5}{1}\binom{5}{1}}{\binom{10}{2}} \qquad P(G_1|D) = \frac{\binom{5}{1}\binom{5}{3}}{\binom{10}{4}} = \frac{5\,\frac{5 \cdot \cancel{4}^{2}}{\cancel{2}}}{\binom{10}{4}}$$

$$P(S) = \frac{\binom{4}{2}\binom{6}{0}}{\binom{10}{2}} + \frac{\binom{6}{2}\binom{4}{0}}{\binom{10}{2}} = \frac{7}{15} \qquad P(D) = \frac{\binom{4}{1}\binom{6}{1}}{\binom{10}{2}} = \frac{24}{45} = \frac{8}{15}$$

$$P(G_1) = P(G_1|S)P(S) + P(G_1|D)P(D)$$

$$= \frac{\cancel{5}\cdot\cancel{5}}{\cancel{45}_{9}}\,\frac{7}{\cancel{15}_{3}} + \frac{50}{\binom{10}{4}} \cdot \frac{8}{15} \qquad \binom{10}{4} = \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8} \cdot 7}{4 \cdot \cancel{3} \cdot \cancel{2}} = 210$$

$$= \frac{7}{27} + \frac{\cancel{5}}{21} \cdot \frac{8}{\cancel{15}_{3}} = \frac{7}{27} + \frac{8}{63} = \frac{73}{189}$$

$$3^3 \qquad 3^2 \cdot 7$$

$$P(BB|G_1) = \frac{P(G_1|BB)P(BB)}{P(G_1)} = \frac{\cancel{25}^{5}}{\cancel{45}_{9}}\,\frac{\binom{4}{2}\binom{6}{0}}{\binom{10}{2}} \cdot \frac{\cancel{189}^{21}}{73}$$

$$= \frac{\cancel{5}\cdot\cancel{6}^{2}\cdot\cancel{21}\,7}{\cancel{45}_{\cancel{9}\,\cancel{3}}\cdot 73} = \frac{14}{73}$$

$$\mathbb{P}(B_1\cap B_2\cap R_3\cap R_4) = \mathbb{P}(R_4 \mid R_3\cap B_2\cap B_1)\,\mathbb{P}(R_3\cap B_2\cap B_1)$$

$$= \mathbb{P}(R_4\mid R_3\cap B_2\cap B_1)\,\mathbb{P}(R_3\mid B_2\cap B_1)\,\mathbb{P}(B_2\cap B_1)$$

$$= \mathbb{P}(R_4\mid R_3\cap B_2\cap B_1)\,\mathbb{P}(R_3\mid B_2\cap B_1)\,\mathbb{P}(B_2\mid B_1)\,\mathbb{P}(B_1)$$

$$= \frac{r+2}{(r+2)+(b+4)}\;\frac{r}{r+(b+4)}\;\frac{b+2}{r+(b+2)}\;\frac{b}{r+b}$$

$$= \frac{br\,(b+2)(r+2)}{(b+r)(b+r+2)(b+r+4)(b+r+6)}.$$

$$\mathbb{P}\big((B_1\cap R_2)\cup(R_1\cap B_2)\big) = \mathbb{P}(B_1\cap R_2) + \mathbb{P}(R_1\cap B_2)$$

$$= \mathbb{P}(R_2\mid B_1)\,\mathbb{P}(B_1) + \mathbb{P}(B_2\mid R_1)\,\mathbb{P}(R_1)$$

$$= \frac{r}{r+(b+2)}\;\frac{b}{r+b} + \frac{b}{b+(r+2)}\;\frac{r}{r+b} = \frac{2br}{(b+r)(b+r+2)}$$

$$\mathbb{P}(A).\mathbb{P}\big((B\cap C)\cup(B\cap E)\cup(D\cap E)\big) =$$

$$= \mathbb{P}(A)\Big\{\mathbb{P}(B\cap C) + \mathbb{P}\big((B\cap E)\cup(D\cap E)\big) - \mathbb{P}\big((B\cap C)\cap[(B\cap E)\cup(D\cap E)]\big)\Big\}$$

$$= \mathbb{P}(A)\Big\{\mathbb{P}(B)\,\mathbb{P}(C) + \mathbb{P}(B\cap E) + \mathbb{P}(D\cap E) - \mathbb{P}(B\cap D\cap E) -$$

$$- \Big[\mathbb{P}(B\cap C)\,\big(\mathbb{P}((B\cap E)\cup(D\cap E))\big)\Big]\Big\} =$$

$$= \mathbb{P}(A)\Big\{\mathbb{P}(B)\,\mathbb{P}(C) + \mathbb{P}(B)\mathbb{P}(E) + \mathbb{P}(D)\mathbb{P}(E) - \mathbb{P}(B)\mathbb{P}(D)\mathbb{P}(E) -$$

$$- \mathbb{P}(B)\mathbb{P}(C)\Big[\mathbb{P}(B\cap E) + \mathbb{P}(D\cap E) - \mathbb{P}(B\cap D\cap E)\Big]\Big\}$$

$$
\begin{aligned}
= \mathbb{P}(A) \Big\{ & \mathbb{P}(B)\mathbb{P}(C) + \mathbb{P}(B)\mathbb{P}(E) + \mathbb{P}(D)\mathbb{P}(E) - \mathbb{P}(B)\mathbb{P}(D)\mathbb{P}(E) \\
& - \mathbb{P}(B)\mathbb{P}(C)\left[\mathbb{P}(B)\mathbb{P}(E) + \mathbb{P}(D)\mathbb{P}(E) - \mathbb{P}(B)\mathbb{P}(D)\mathbb{P}(E)\right] \Big\}
\end{aligned}
$$