ESERCIZI

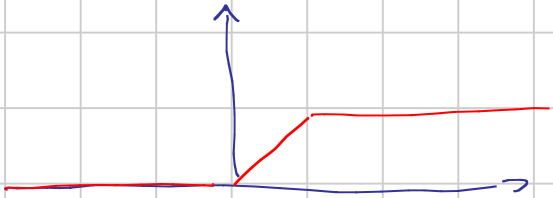
Titolo nota 09/12/2015

FOGLIO 7 - Ex 2

$\mathbb{P}_X = B(p)$

$$\mathbb{P}(Y \le t \mid X=1) = \begin{cases} 0 & t<0 \\ t & t\in[0,1) \\ 1 & t \geq 1 \end{cases}$$

$$\mathbb{P}(Y \le t \mid X=0) = \begin{cases} 0 & t<-1 \\ t+1 & t\in[-1,0) \\ 1 & t \geq 0 \end{cases}$$



$(\mathbb{P}_{\{X=1\}})_Y = \mathcal{U}((0,1))$

$(\mathbb{P}_{\{X=0\}})_Y = \mathcal{U}((-1,0))$

$$\mathbb{P}(Y\le t) = \mathbb{P}(Y\le t \mid X=1)\,\mathbb{P}(X=1) + \mathbb{P}(Y \le t \mid X=0)\,\mathbb{P}(X=0)$$

Si trova $\mathbb{P}_Y = (1-p)\,\mathcal{U}((-1,0)) + p\,\mathcal{U}((0,1))$

$\mathbb{E}[Y \mid X=t]$

$\mathbb{E}[Y \mid X=0]$ $\qquad$ $\mathbb{E}[Y \mid X=1]$

$$\mathbb{E}[Y \mid X=0] = \frac{1}{\mathbb{P}(X=0)} \int_{\{X=0\}} Y(\omega)\,\mathbb{P}(d\omega) = \int_\Omega Y(\omega)\,\mathbb{P}_{\{X=0\}}(d\omega)$$

$$= \int_{\mathbb{R}} y \,\underbrace{(\mathbb{P}_{\{X=0\}})_Y}_{\mathcal{U}((-1,0)) = \mathbb{1}_{(-1,0)}(y)\,dy}(dy)$$

$$= \int_{-1}^{0} y\,dy = \frac{y^2}{2}\Big|_{y=-1}^{y=0} = 0 - \frac{1}{2} = -\frac{1}{2}$$

$$\mathbb{E}[Y \mid X=1] = \frac{1}{\mathbb{P}(X=1)} \int_{\{X=1\}} Y(\omega)\,\mathbb{P}(d\omega) = \int_\Omega Y(\omega)\,\mathbb{P}_{\{X=1\}}(d\omega) =$$

$$= \int_{\mathbb{R}} y\, \underbrace{(\mathbb{P}_{\{X=1\}})_Y}_{\mathcal{U}((0,1)) = \mathbb{1}_{(0,1)}(y)\,dy}(dy) = \int_0^1 y\,dy = \frac{y^2}{2}\Big|_{y=0}^{y=1} = \frac{1}{2}$$

FOGLIO 7 - EX 5

$\mathbb{P}_Y = \text{Poisson}(\lambda) \qquad X(\Omega) = \{0,1\} \qquad \mathbb{P}(X=1 \mid Y=k) = \frac{1}{k+1} \quad \forall k \geq 0$

$\mathbb{P}(X=1) = \frac{1-e^{-\lambda}}{\lambda} \qquad \mathbb{P}_X = B\left(\frac{1-e^{-\lambda}}{\lambda}\right)$

$\mathbb{E}[X \mid Y=t] \qquad \mathbb{E}[X \mid Y=k] \quad k \in \mathbb{N}_0$

$$\mathbb{E}[X \mid Y=k] = \int_\Omega X(\omega)\, \mathbb{P}_{\{Y=k\}}(d\omega) = 0\, \mathbb{P}_{\{Y=k\}}(X=0) + 1\, \mathbb{P}_{\{Y=k\}}(X=1)$$
$$= \frac{1}{k+1}$$

$\mathbb{E}[Y \mid X=t] \qquad \mathbb{E}[Y \mid X=0] \qquad \mathbb{E}[Y \mid X=1]$

$$\mathbb{E}[Y \mid X=1] = \int_\Omega Y(\omega)\, \mathbb{P}_{\{X=1\}}(d\omega) = \sum_{k=0}^{\infty} k\, \mathbb{P}_{\{X=1\}}(Y=k) =$$
$$= \sum_{k=0}^{\infty} k\, \mathbb{P}(Y=k \mid X=1)$$

$$\mathbb{P}(Y=k \mid X=1) = \frac{\overbrace{\mathbb{P}(X=1 \mid Y=k)}^{=\frac{1}{k+1}}\ \overbrace{\mathbb{P}(Y=k)}^{=\frac{e^{-\lambda}\lambda^k}{k!}}}{\underbrace{\mathbb{P}(X=1)}_{=\frac{1-e^{-\lambda}}{\lambda}}} = \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}} \frac{\lambda^k}{(k+1)!}$$

$$\mathbb{E}[Y \mid X=1] = \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}} \sum_{k=0}^{\infty} \frac{k \lambda^k}{(k+1)!} = \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{k=0}^{\infty} \frac{\overset{k+1-1}{\overset{=}{k}}\, \lambda^{k+1}}{(k+1)!}$$
$$= \frac{e^{-\lambda}}{1-e^{-\lambda}} \left\{ \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{k!} - \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{(k+1)!} \right\}$$
$$= \frac{e^{-\lambda}}{1-e^{-\lambda}} \left\{ \lambda e^{\lambda} - (e^{\lambda} - 1) \right\}$$

$$\mathbb{E}[Y \mid X=0] = \int_\Omega Y(\omega)\, \mathbb{P}_{\{X=0\}}(d\omega) = \sum_{k=0}^{\infty} k\, \mathbb{P}_{\{X=0\}}(Y=k)$$
$$= \sum_{k=0}^{\infty} k\, \mathbb{P}(Y=k \mid X=0)$$

$$\mathbb{P}(X=1 \mid Y=k) = \frac{1}{k+1} \qquad \mathbb{P}(X=0 \mid Y=k) = 1 - \frac{1}{k+1} = \frac{k}{k+1}$$

$$\mathbb{P}(Y=k \mid X=0) = \frac{\overbrace{\mathbb{P}(X=0 \mid Y=k)}^{=k/(k+1)}\ \mathbb{P}(Y=k)}{\mathbb{P}(X=0)} \quad \mathbb{P}(Y=k) = e^{-\lambda}\frac{\lambda^k}{k!} \quad \mathbb{P}(X=0) = 1 - \mathbb{P}(X=1)$$

$$\mathbb{P}(X=1) = \frac{1-e^{-\lambda}}{\lambda} \qquad \mathbb{P}(X=0) = \frac{\lambda - 1 + e^{-\lambda}}{\lambda}$$

$$\mathbb{P}(Y=k \mid X=0) = \frac{\lambda}{\lambda-1+e^{-\lambda}} \frac{k}{k+1} e^{-\lambda}\frac{\lambda^k}{k!} = \frac{\lambda e^{-\lambda}}{\lambda-1+e^{-\lambda}} \frac{k\lambda^k}{(k+1)!}$$

$$\mathbb{E}[Y \mid X=0] = \frac{e^{-\lambda}}{\lambda-1+e^{-\lambda}} \sum_{k=0}^{\infty} \frac{k^2 \lambda^{k+1}}{(k+1)!} \qquad k+1=j \qquad k=j-1$$

$$= \frac{e^{-\lambda}}{\lambda-1+e^{-\lambda}} \sum_{j=2}^{+\infty} \frac{(j-1)^2\lambda^j}{j!} \qquad (j-1)^2 = j(j-1) - 1(j-1) = j(j-1) - j + 1$$

$$= \frac{e^{-\lambda}}{\lambda-1+e^{-\lambda}} \sum_{j=2}^{\infty} \left(\frac{\lambda^{j-2}\lambda^2}{(j-2)!} - \frac{\lambda\lambda^{j-1}}{(j-1)!} + \frac{\lambda^j}{j!} \right)$$

$$= \frac{e^{-\lambda}}{\lambda-1+e^{-\lambda}} \left\{ \lambda^2 e^{\lambda} - \lambda(e^{\lambda} - \cancel{1}) + e^{\lambda} - 1 - \cancel{\lambda} \right\}$$

$$= \frac{e^{-\lambda}}{\lambda-1+e^{-\lambda}} \left(e^{\lambda}(\lambda^2 - \lambda + 1) - 1 \right)$$

FOGLIO 7 - EX 6

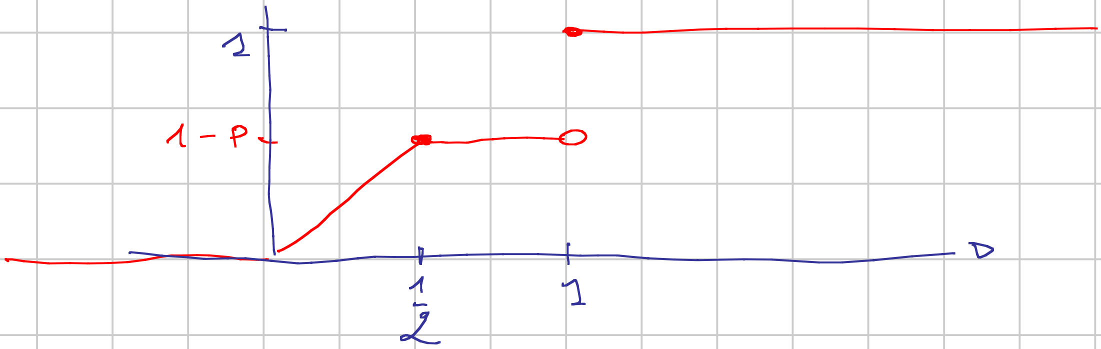
$\mathbb{P}_X = U((0,\tfrac{1}{2}))$ $\quad \mathbb{P}_Y = B(p)$ $\quad$ X e Y sono indipendenti

$Z := \max\{X,Y\}$

$$F_Z(t) = \mathbb{P}(\max\{X,Y\} \le t) = \mathbb{P}(X \le t, Y \le t) = \mathbb{P}(X \le t)\mathbb{P}(Y \le t)$$

$$F_X(t) = \begin{cases} 0 & t < 0 \\ 2t & 0 \le t < \frac{1}{2} \\ 1 & t \geq \frac{1}{2} \end{cases} \qquad F_Y(t) = \begin{cases} 0 & t < 0 \\ 1-p & 0 \le t < 1 \\ 1 & t \geq 1 \end{cases}$$

$$F_Z(t) = \begin{cases} 0 & t < 0 \\ 2(1-p)t & 0 \le t < \frac{1}{2} \\ 1-p & \frac{1}{2} \le t < 1 \\ 1 & t \geq 1 \end{cases}$$



$$\mathbb{P}_Z = c_1 U((0,\tfrac{1}{2})) + c_2 \delta_1 = (1-p) U((0,\tfrac{1}{2})) + p\delta_1$$

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ψ funzione di Borel nonnegativa

$\varphi : (x,y) \in \mathbb{R}^2 \longmapsto \max(x,y) \in \mathbb{R}$ $\qquad Z = \varphi \circ (X,Y)$

$$\int_{\mathbb{R}} \psi(t)\, \mathbb{P}_Z(dt) = \int_{\mathbb{R}} \psi(t)\, \mathbb{P}_{\varphi\circ(X,Y)}(dt) = \int_{\mathbb{R}^2} \psi(\varphi(x,y))\, \mathbb{P}_{(X,Y)}(dx\,dy)$$

$$= \int_{\mathbb{R}^2} \psi(\varphi(x,y))\, \mathbb{P}_X(dx)\, \mathbb{P}_Y(dy) = \qquad \mathbb{P}_X(dx) = 2\,\mathbb{1}_{(0,\frac{1}{2})}(x)\,dx$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \psi(\varphi(x,y))\, 2\,\mathbb{1}_{(0,\frac{1}{2})}(dx) \right) \mathbb{P}_Y(dy)$$

$$\mathbb{P}_Y(dy) = (1-p)\delta_0 + p\delta_1$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \psi(\underbrace{\varphi(x,y)}_{=\max(x,y)})\, \mathbb{P}_Y(dy) \right) 2\,\mathbb{1}_{(0,\frac{1}{2})}(x)\,(dx)$$

$$= \int_{\mathbb{R}} \left((1-p)\,\psi(\max(x,0)) + p\,\psi(\max(x,1)) \right) 2\,\mathbb{1}_{(0,\frac{1}{2})}(x)\,dx$$

$$= 2\int_0^{1/2} \left((1-p)\,\psi(x) + p\,\psi(1) \right) dx$$

$$= \int_0^{1/2} \left(2(1-p)\,\psi(x) + 2p\,\psi(1) \right) dx$$

$$= (1-p) \int_{\mathbb{R}} \psi(x)\, \underbrace{2\,\mathbb{1}_{(0,\frac{1}{2})}(x)\,dx}_{\text{densità di } U((0,\frac{1}{2}))} + \cancel{\tfrac{1}{2}}\,\cancel{2}\,p\,\psi(1)$$

$$= (1-p)\int_{\mathbb{R}} \varphi(x)\, U((0,\tfrac{1}{2}))(dx) + p\int_{\mathbb{R}} \varphi(x)\,\delta_1(dx)$$

$$= \int_{\mathbb{R}} \varphi(x) \underbrace{\left((1-p)U((0,\tfrac{1}{2})) + p\delta_1 \right)}_{\mathbb{P}_Z} (dx)$$

FOGLIO 7 - EX 9

$\mathbb{P}_Y = B(p)$

$$\mathbb{P}(X \le t \mid Y=0) = \begin{cases} 0 & t \le 0 \\ 1-e^{-\lambda t} & t>0 \end{cases} \qquad \left(\mathbb{P}_{\{Y=0\}}\right)_X = \exp(\lambda)$$

$$\mathbb{P}(X \le t \mid Y=1) = \begin{cases} 0 & t \le 0 \\ 1-e^{-\mu t} & t>0 \end{cases} \qquad \left(\mathbb{P}_{\{Y=1\}}\right)_X = \exp(\mu)$$

$\mathbb{P}_X = \,?$ $\qquad \mathbb{E}[X \mid Y=t]$ e $\mathbb{E}[Y \mid X=t]$

$$F_X(t) = \mathbb{P}(X \le t) = \mathbb{P}(X \le t \mid Y=0)\,\mathbb{P}(Y=0) + \mathbb{P}(X\le t \mid Y=1)\,\mathbb{P}(Y=1)$$

$$= \begin{cases} 0 & t \le 0 \\ (1-p)(1-e^{-\lambda t}) + p(1-e^{-\mu t}) & t>0 \end{cases}$$

$$\mathbb{P}_X = (1-p)\exp(\lambda) + p\exp(\mu) \quad \leftarrow$$

$$f_X(t) = \left((1-p)\lambda e^{-\lambda t} + p\mu e^{-\mu t} \right) \mathbb{1}_{(0,+\infty)}(t)$$

$$\mathbb{E}[X \mid Y=0] = \int_{\Omega} X(\omega)\, \mathbb{P}_{\{Y=0\}}(d\omega) = \int_{\mathbb{R}} x \left(\mathbb{P}_{\{Y=0\}}\right)_X (dx) =$$

$$= \int_{\mathbb{R}} x\lambda e^{-\lambda x}\, \mathbb{1}_{(0,+\infty)}(x)\, dx = \frac{1}{\lambda}$$

$$\mathbb{E}[X \mid Y=1] = \int_{\Omega} X(\omega)\, \mathbb{P}_{\{Y=1\}}(d\omega) = \int_{\mathbb{R}} x \left(\mathbb{P}_{\{Y=1\}}\right)_X (dx)$$

$$= \int_{\mathbb{R}} x\mu e^{-\mu x}\, \mathbb{1}_{(0,+\infty)}(x)\, dx = \frac{1}{\mu}$$

$$\mathbb{E}[Y|X=t] =: \varphi(t)$$

$$\underbrace{\int_{X^{-1}(A)} Y(\omega)\,\mathbb{P}(d\omega)}\ = \int_A \varphi(t)\,\mathbb{P}_X(dt) \qquad \forall A \in \mathcal{B}(\mathbb{R})$$

$$= \int_A \varphi(t)\left((1-p)\lambda e^{-\lambda t} + p\mu e^{-\mu t}\right)\mathbb{1}_{(0,+\infty)}(t)\,dt$$

$A = (-\infty, s]$ $\qquad X^{-1}(A) = \{X \le s\}$

$$\int_\Omega Y(\omega)\,\mathbb{1}_{\{X\le s\}}(\omega)\,\mathbb{P}(d\omega) = \int_{-\infty}^{s} \varphi(t)\left((1-p)\lambda e^{-\lambda t} + p\mu e^{-\mu t}\right)\mathbb{1}_{(0,+\infty)}(t)\,dt$$

$$Z(\omega) = Y(\omega)\,\mathbb{1}_{\{X\le s\}}(\omega) = \begin{cases} 1 & \text{se } Y(\omega)=1 \text{ e } X(\omega)\le s \\ 0 & \text{altrimenti} \end{cases}$$

$$= \mathbb{P}(Z=1) = \mathbb{P}(Y=1, X\le s) = \mathbb{P}(X\le s \mid Y=1)\,\mathbb{P}(Y=1)$$

$$= p\,\mathbb{P}(X\le s \mid Y=1)$$

$$p\,\underbrace{\mathbb{P}(X\le s\mid Y=1)}_{\begin{cases} 0 & s\le 0 \\ 1-e^{-\mu s} & s>0\end{cases}} = \int_{-\infty}^{s} \varphi(t)\left((1-p)\lambda e^{-\lambda t} + p\mu e^{-\mu t}\right)\underline{\mathbb{1}_{(0,+\infty)}(t)\,dt}$$

$\varphi(t)$ qualsiasi $\quad t \le 0$

$\mathcal{L}^1$ - q.o. $s>0$ $\qquad p\mu e^{-\mu s} = \varphi(s)\left((1-p)\lambda e^{-\lambda s} + p\mu e^{-\mu s}\right)$

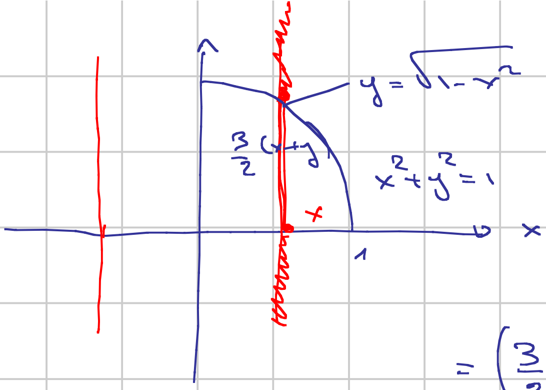
$$\varphi(s) = \frac{p\mu e^{-\mu s}}{(1-p)\lambda e^{-\lambda s} + p\mu e^{-\mu s}}$$

FOGLIO 7 - EX 18

$$f(x,y) = \begin{cases} \frac{3}{2}(x+y) & x^2+y^2\le 1,\ x>0,\ y>0 \\ 0 & \text{altrimenti} \end{cases}$$

$$\mathbb{E}[Y|X=x] = \int_{\mathbb{R}} y\, h(y|x)\,dy \qquad h(y|x) = \frac{f(x,y)}{f_X(x)}$$

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$$



$x<0,\ x>1 \quad f_X(x)=0$

$x\in[0,1]$

$$f_X(x) = \int_0^{\sqrt{1-x^2}} \frac{3}{2}(x+y)\,dy =$$

$$= \left(\frac{3}{2}xy + \frac{3}{4}y^2\right)\Big|_{y=0}^{y=\sqrt{1-x^2}} = \frac{3}{2}x\sqrt{1-x^2} + \frac{3}{4}(1-x^2)$$

$$h(y|x) = \frac{f(x,y)}{f_X(x)} \quad \text{per } x\in(0,1) \text{ vale } \frac{f(x,y)}{\frac{3}{2}x\sqrt{1-x^2}+\frac{3}{4}(1-x^2)}$$

$$= \frac{\frac{3}{2}(x+y)\,\mathbb{1}_{(0,\sqrt{1-x^2})}(y)}{\frac{3}{2}x\sqrt{1-x^2}+\frac{3}{4}(1-x^2)}$$

$$\mathbb{E}[Y|X=x] = \int_{\mathbb{R}} \frac{\frac{3}{2}y(x+y)\,\mathbb{1}_{(0,\sqrt{1-x^2})}(y)}{\frac{3}{2}x\sqrt{1-x^2}+\frac{3}{4}(1-x^2)}\,dy =$$

$$= \frac{1}{\frac{3}{2}x\sqrt{1-x^2}+\frac{3}{4}(1-x^2)} \int_0^{\sqrt{1-x^2}} \left(\frac{3}{2}xy + \frac{3}{2}y^2\right) dy =$$

$$= \frac{1}{\frac{3}{2}x\sqrt{1-x^2}+\frac{3}{4}(1-x^2)} \left(\frac{3}{4}xy^2 + \frac{1}{2}y^3\right)\Big|_{y=0}^{y=\sqrt{1-x^2}}$$

$$= \frac{1}{\frac{3}{2}x\sqrt{1-x^2}+\frac{3}{4}(1-x^2)} \left(\frac{3}{4}x(1-x^2) + \frac{1}{2}(1-x^2)\sqrt{1-x^2}\right)$$

FOGLIO 7 - EX 8

$X(\Omega)=Y(\Omega)=\mathbb{N}_0 \qquad \mathbb{P}_{X+Y} = \text{Poiss}(\lambda)$

$$\mathbb{P}(Y=i\,|\,X+Y=k) = \begin{cases} \binom{k}{i}p^i(1-p)^{k-i} & i=0,\dots k \quad i\le k \\ 0 & i>k \end{cases}$$

$$\left(\mathbb{P}_{\{X+Y=k\}}\right)_Y = B(k,p) \qquad \forall k\ge 0$$

$X(\Omega)=Y(\Omega)=\mathbb{N}_0 \qquad (X+Y)(\Omega)=\mathbb{N}_0$

$\{X+Y=k\} \quad k \in \mathbb{N}_0$

$$\mathbb{P}(Y=i) = \sum_{k=0}^{\infty} \mathbb{P}(Y=i \mid X+Y=k)\, \underbrace{\mathbb{P}(X+Y=k)}_{e^{-\lambda}\frac{\lambda^k}{k!}}$$

$$= \sum_{k=i}^{+\infty} \binom{k}{i} p^i (1-p)^{k-i} e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \sum_{k=i}^{+\infty} \frac{\cancel{k!}}{i!(k-i)!} p^i (1-p)^{k-i} e^{-\lambda} \frac{\lambda^k}{\cancel{k!}}$$

$$= \frac{1}{i!} p^i (1-p)^{-i} e^{-\lambda} \sum_{k=i}^{\infty} \frac{1}{(k-i)!} ((1-p)\lambda)^k$$

$$j = k-i \qquad k = i+j$$

$$= \frac{1}{i!} p^i \cancel{(1-p)^{-i}} e^{-\lambda} \sum_{j=0}^{\infty} \frac{1}{j!} ((1-p)\lambda)^j \, (\cancel{(1-p)}\lambda)^i$$

$$= \frac{(\lambda p)^i \cancel{e^{-\lambda}}}{i!} e^{\lambda(\cancel{1}-p)} = e^{-\lambda p} \frac{(\lambda p)^i}{i!}$$

$\Rightarrow \mathbb{P}_Y = \text{Poisson}(\lambda p)$

$$\{X=i\} = \bigcup_{k=0}^{\infty} \{X=i,\ X+Y=k\} = \bigcup_{k=0} \{Y=k-i,\ X+Y=k\}$$

$$\mathbb{P}(X=i) = \sum_{k=0}^{\infty} \mathbb{P}(Y=k-i \mid X+Y=k)\, \mathbb{P}(X+Y=k)$$

$0 \leq k-i \leq k \qquad k \geq i$

$$= \sum_{k=i}^{\infty} \binom{k}{k-i} p^{k-i} (1-p)^{\cancel{k}-(\cancel{k}-i)} e^{-\lambda} \frac{\lambda^k}{k!}$$

$\mathbb{P}_X = \text{Poiss}(\lambda(1-p))$