THE FIXED POINT INDEX OF THE POINCARÉ TRANSLATION OPERATOR ON DIFFERENTIABLE MANIFOLDS

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1. Introduction

The fixed point index of the Poincaré translation operator associated to an ordinary differential equation is a very useful tool for establishing the existence of periodic solutions.

In this chapter we focus on ODEs on differentiable manifolds embedded in Euclidean spaces. Our purpose is twofold: on the one hand we aim to provide a short and accessible introduction to some topological tools (such as the Topological Degree, the Degree of a tangent vector field and the Fixed Point Index) that are useful in Nonlinear Analysis; on the other hand we offer a unifying approach to several results about the fixed point index of the Poincaré translation operator that were previously scattered among a number of publications.

Our main concern will be a formula for the computation of the fixed point index of the flow operator induced on a manifold by a first order autonomous ordinary differential equation. Other formulas for the fixed point index of the translation operator associated with non-autonomous equations will be deduced as consequences. We emphasize that other results, unrelated to our approach, but still involving the fixed point index of the Poincaré translation operator have been successfully exploited, for instance, by Srzednicki (see e.g. [Srz2, Srz3]).

The chapter is organized as follows. In Section 2 we recall first some elements of calculus on finite dimensional manifolds. Then, in this context, we introduce the notions of Fixed Point Index of a map and of Degree of a tangent vector field. In particular, we show a simple axiomatic approach to the fixed point index theory for maps on a manifold based on just three axioms (see Theorem 2.23 below). In the same section we discuss some basics regarding first order differential equations on differentiable manifolds.

In Section 3, we consider the flow on a manifold M induced by an autonomous differential equation of the form

$$\dot{x} = g(x),$$

where g is a vector field tangent to M. Then, in the spirit of an earlier result by Krasnosel'skii [Kra], we provide a relation, valid for sufficiently short times, between the degree of a tangent vector field and the fixed point index of its associated flow. From this result, we deduce a formula (see Theorem 3.8 below) for the computation, in terms of the degree of g, of the fixed point index of the flow operator associated with the above equation. This formula is valid whenever the fixed point index of the flow is well defined (and not merely for "sufficiently short" times). The idea behind the proof stems, besides the quoted result by Krasnosel'skii, from more recent results by Capietto-Mawhin-Zanolin ([Maw, CaMaZa]). Later, in the same section, we consider non-autonomous equations and derive some other formulas for the fixed point index of the (Poincaré) T-translation operator ($T > 0$ given) associated with the equations

$$\dot{x} = \lambda f(t,x), \quad \dot{x} = g(x) + \lambda f(t,x),$$

for $\lambda > 0$ sufficiently small, where g and f are tangent vector fields on M, and f is T-periodic. Similar results can be deduced for an equation of the form

$$\dot{x} = a(t)h(x) + \lambda f(t,x).$$

where h is a tangent vector field on M, and $a\colon \mathbb{R} \to \mathbb{R}$ is a T-periodic function with nonzero average. However, as far as we know, there is no general result in the literature encompassing all the different situations reflected by the above three equations.

Section 4, finally, collects a number of simple consequences of the formulas obtained in the previous section. Particular emphasis is given to results describing the structure of pairs (λ, p), where p is an initial point of a T-periodic solution to the equations considered in Section 3. As an illustrative application, we prove a continuation result for the T-periodic solutions of the equation $\dot{x} = f(t,x)$ when f is a T-periodic tangent vector field.

It should be remarked that the selection of applications collected in this chapter is very restrictive and made with the sole purpose of highlighting the role of the fixed point index of the translation operator. In fact, in order to remain within reasonable space limits, the most general situations are not sought for, and many interesting applications have been omitted. Among these, we mention multiplicity results for T-periodic solutions, guiding-functions-like and continuation-like existence results. Moreover, since a second order equation on a manifold can be seen as a particular first order equation on the tangent bundle (see e.g. [Fur]), a number of results could also be deduced for second order ODEs.

2. Notation and Preliminaries

2.1. **Tangent cones and tangent spaces to subsets of $\mathbb{R}^k$.** In this subsection we introduce the notions of tangent cones and tangent spaces to arbitrary subsets of $\mathbb{R}^k$. We also recall the concept of C^r map, $r \in \mathbb{N}\cup\{\infty\}$, between arbitrary subsets of Euclidean spaces and discuss the notion of Fréchet derivative in this context. These concepts, which are well-known for maps defined on open sets, need an extended definition in the general case (see e.g. [Mil]). Roughly speaking, the extension of the notion of C^r map is obtained by forcing down the *hereditary property* of C^r maps on open sets, i.e., by requiring that the restriction of a C^r map to any subset of its domain is still a C^r map. The following definition also preserves the *local property* of C^r maps, i.e., any map which is locally C^r is C^r.

Definition 2.1. A map $f\colon X \to Y$, from a subset of $\mathbb{R}^k$ into a subset of $\mathbb{R}^s$, is said to be C^r, $r \in \mathbb{N} \cup \{\infty\}$, if for any $p \in X$ there exists a C^r map $g\colon U \to \mathbb{R}^s$, defined on an open neighborhood of p, such that $f(x) = g(x)$ for all $x \in U \cap X$.

In other words, $f\colon X \to Y$ is C^r if it can be locally extended as a map into $\mathbb{R}^s$ (and not merely into Y) to a C^r map defined on an open subset of $\mathbb{R}^k$. To understand why one must seek the extension of f as a map into $\mathbb{R}^s$, observe that the identity $i\colon [0,1] \to [0,1]$ is not the restriction of any C^1 function $g\colon U \to [0,1]$ defined on an open neighborhood U of $[0,1]$.

Remark 2.2. Using the well-known fact that any family of open subsets of $\mathbb{R}^k$ admits a subordinate smooth partition of unity, it is easy to show that any C^r map on $X \subseteq \mathbb{R}^k$ is actually the restriction of a C^r map defined on an open neighborhood of X.

As a straightforward consequence of the definition one gets that, given $X \subseteq \mathbb{R}^k$, the identity $i\colon X \to X$ is a smooth map. Moreover, we observe that the composition of C^r maps between arbitrary Euclidean sets is again a C^r map, since the same is true for maps defined on open sets. Thus, one can view Euclidean sets as objects of a category, whose morphisms are C^r maps.

Definition 2.3. A C^r map $f\colon X \to Y$, from a subset X of $\mathbb{R}^k$ into a subset Y of $\mathbb{R}^s$, is said to be a *C^r-diffeomorphism* if it is bijective and f^{-1} is C^r. In this case X and Y are said to be *C^r-diffeomorphic.*

A straightforward consequence of the definition of diffeomorphism (and of the hereditary property of C^r maps) is that the restriction of a C^r-diffeomorphism is again a C^r-diffeomorphism onto its image.

Remark 2.4. An example of a C^r-diffeomorphism is given by the graph map associated with a C^r map $f\colon X \to \mathbb{R}^s$ defined on an arbitrary subset of $\mathbb{R}^k$. In fact, let

$$G_f = \big\{(x,y) \in \mathbb{R}^k \times \mathbb{R}^s : x \in X,\ y = f(x)\big\}$$

denote the graph of f. The map $\hat{f}\colon X \to G_f$, defined by $\hat{f}(x) = \big(x, f(x)\big)$, is clearly C^r and bijective. Observe now that $\hat{f}^{-1}$ is just the restriction to G_f of the projection $(x,y) \mapsto x$ of $\mathbb{R}^k \times \mathbb{R}^s$ onto the first factor, which is a linear map (and, consequently, smooth). This proves that *the graph of a C^r map is C^r-diffeomorphic to its domain.*

We are ready to give the definitions of tangent vector, tangent cone, and tangent space to an arbitrary subset $X \subseteq \mathbb{R}^k$ at a point $p \in X$.

In the sequel $|\cdot|$ denotes the canonical Euclidean norm on $\mathbb{R}^k$.

Definition 2.5. Let X be a subset of $\mathbb{R}^k$ and take $p \in X$. A unit vector $v \in S^{k-1} = \{x \in \mathbb{R}^k : |x| = 1\}$ is said to be *tangent* to X at p if there exists a sequence $\{p_n\}$ in $X \setminus \{p\}$ such that $p_n \to p$ and $(p_n - p)/|p_n - p| \to v$.

If p is isolated in X, then the *tangent cone* of X at p, C_pX, is just the trivial subspace $\{0\}$ of $\mathbb{R}^k$. If p is an accumulation point of X, then C_pX is the cone generated by the set of tangent unit vectors, i.e.,

$$C_pX = \big\{\lambda v : \lambda \geq 0,\ v \in S^{k-1} \text{ is tangent to } X \text{ at } p\big\}.$$

The *tangent space* of X at p, T_pX, is the vector subspace of $\mathbb{R}^k$ spanned by C_pX.

It is fairly easy to check that the above definition of tangent cone is equivalent to the classical one introduced by Bouligand in [Bou] (see also [Sev, p. 149], for a precursor of this notion).

Observe that, because of the compactness of the unit sphere S^{k-1}, if p is an accumulation point of X, there exists at least one unit vector tangent to X at p. Moreover, the notion of tangent cone is local; that is, if two sets X and Y coincide in a neighborhood of a common point p, they have the same tangent cone. Another important property is the translation invariance: $C_pX = C_{x+p}(x + X)$, $\forall x \in \mathbb{R}^k$.

The following result is useful for the computation of the tangent cone to a set defined by inequalities. The easy proof, based on the Inverse Function Theorem, is left to the reader.

Theorem 2.6. *Let $f\colon U \to \mathbb{R}^s$ be a C^1 map defined on an open subset of $\mathbb{R}^k$. Let $Y \subseteq \mathbb{R}^s$ and $p \in f^{-1}(Y)$. Assume that p is a regular point of f; i.e., the derivative $f'(p)\colon \mathbb{R}^k \to \mathbb{R}^s$ of f at p is surjective. Then*

$$C_p\big(f^{-1}(Y)\big) = \big\{v \in \mathbb{R}^k : f'(p)v \in C_{f(p)}Y\big\} = f'(p)^{-1}(C_{f(p)}Y).$$

Given a C^1 map $f\colon X \to Y$ and a point $p \in X$, we shall define a linear operator $f'(p)$ from T_pX into $T_{f(p)}Y$, called the derivative of f at p, which maps the tangent cone of X at p into the tangent cone of Y at $f(p)$. This derivative will turn out to satisfy the usual functorial properties of the Fréchet derivative. To achieve this, we need the following three lemmas. The first one extends the well-known fact that the Fréchet derivative can be computed as a directional derivative. Its elementary proof is left to the reader.

Lemma 2.7. *Let $f\colon U \to \mathbb{R}^s$ be defined on an open subset of $\mathbb{R}^k$ and differentiable at $p \in U$. If $v \in S^{k-1}$ is a unit vector, then*

$$f'(p)v = \lim_{n\to\infty} \frac{f(p_n) - f(p)}{|p_n - p|},$$

where $\{p_n\}$ is any sequence in $U \setminus \{p\}$ such that $p_n \to p$ and $(p_n - p)/|p_n - p| \to v$.

Lemma 2.8. *Let $f\colon U \to \mathbb{R}^s$ be defined on an open subset of $\mathbb{R}^k$ and differentiable at $p \in U$. If f maps a subset X of U containing p into a subset Y of $\mathbb{R}^s$, then $f'(p)$ maps C_pX into $C_{f(p)}Y$. Consequently, because of the linearity of $f'(p)$, it also maps T_pX into $T_{f(p)}Y$.*

Proof. It is sufficient to show that if $v \in S^{k-1}$ is tangent to X at p, then $f'(p)v$ is tangent to Y at $f(p)$. For this, let $\{p_n\}$ be a sequence in $X \setminus \{p\}$ such that $p_n \to p$ and $(p_n - p)/|p_n - p| \to v$. By Lemma 2.7, we have $(f(p_n) - f(p))/|p_n - p| \to f'(p)v$. If $f'(p)v = 0$ there is nothing to prove since $0 \in C_{f(p)}Y$ by the definition of tangent cone. On the other hand, if $f'(p)v \neq 0$, we have $f(p_n) \neq f(p)$, for n large enough. Thus, for such n's, we can write

$$\frac{f(p_n) - f(p)}{|f(p_n) - f(p)|} = \frac{|p_n - p|}{|f(p_n) - f(p)|}\frac{f(p_n) - f(p)}{|p_n - p|}.$$

Therefore,

$$\lim_{n\to\infty} \frac{f(p_n) - f(p)}{|f(p_n) - f(p)|} = \frac{f'(p)v}{|f'(p)v|}.$$

And this shows that $f'(p)v = \lambda w$, where $\lambda > 0$ and $w \in S^{k-1}$ is tangent to Y at $f(p)$. $\square$

Lemma 2.9. *Let $f, g\colon U \to \mathbb{R}^s$ be defined on an open subset of $\mathbb{R}^k$ and differentiable at $p \in U$. Assume that f and g coincide on some subset of X containing p. Then $f'(p)$ and $g'(p)$ coincide on C_pX and, consequently, on T_pX.*

Proof. Let $\varphi\colon U \to \mathbb{R}^s$ be defined by $\varphi(x) = f(x) - g(x)$; so that φ maps X into the trivial subspace $Y = \{0\}$ of $\mathbb{R}^s$. Thus, by Lemma 2.8, we obtain

$$\varphi'(p)v = f'(p)v - g'(p)v = 0, \quad \forall v \in C_pX,$$

and the assertion is proven. □

Lemma 2.9 ensures that if $f\colon X \to \mathbb{R}^s$ is a C^1 map on a subset X of $\mathbb{R}^s$ and p is a point in X, then the restriction to T_pX of the derivative at p of any C^1 local extension of f to a neighborhood of p does not depend on the chosen extension. In other words, all the C^1 extensions of f to an open neighborhood of p have the same directional derivative along the vectors of the subspace T_pX. Moreover, Lemma 2.8 implies that if g is such an extension and f maps X into Y, then $g'(p)$ maps T_pX into $T_{f(p)}Y$. These two facts justify the following definition.

Definition 2.10. Let $f\colon X \to Y$ be a C^1 map from a subset X of $\mathbb{R}^k$ into a subset Y of $\mathbb{R}^s$. The *derivative* of f at p, $f'(p)\colon T_pX \to T_{f(p)}Y$, is the restriction to T_pX of the derivative at p of any C^1 extension of f to a neighborhood of p in $\mathbb{R}^k$.

We point out that this extended derivative inherits the two functorial properties of the classical derivative (the easy proof of this fact is left to the reader). As a consequence of this and Lemma 2.8 one gets the following result.

Theorem 2.11. *Let $f\colon X \subseteq \mathbb{R}^k \to Y \subseteq \mathbb{R}^s$ be a C^1-diffeomorphism. Then for any $p \in X$, $f'(p)\colon T_pX \to T_{f(p)}Y$ is an isomorphism mapping C_pX onto $C_{f(p)}Y$.*

Proof. To simplify the notation, put $q = f(p)$. By the definition of diffeomorphism we have $f^{-1} \circ f = i_X$ and $f \circ f^{-1} = i_Y$, where i_X and i_Y denote the identity on X and Y, respectively. Therefore, by the functorial properties of the extended derivative, the two compositions $(f^{-1})'(q)f'(p)$ and $f'(p)(f^{-1})'(q)$ coincide, respectively, with the identity on T_pX and T_qY. This means that $f'(p)$ is invertible and $f'(p)^{-1} = (f^{-1})'(q)$. The fact that C_pX and C_qY correspond to each other under $f'(p)$ is a direct consequence of Lemma 2.8. □

Let X be a subset of $\mathbb{R}^k$. We say that a point $p \in X$ is *singular* for X if $T_pX \neq C_pX$. In other words, since T_pX is the space spanned by C_pX, saying that p is a non-singular point for X means that C_pX is a vector space. The set of singular points of X will be denoted by δX.

For example, if X is an n-simplex in $\mathbb{R}^k$, δX is just the union of all the $(n-1)$-faces of X, $\delta\delta X$, denoted by $\delta^2 X$, is the union of all the $(n-2)$-faces of X, and so on.

Observe also that if X is an open subset of $\mathbb{R}^k$, then $\delta X = \emptyset$.

The following straightforward consequence of Theorem 2.11 shows that the concept of singular point is invariant under diffeomorphisms.

Theorem 2.12. *If $f\colon X \to Y$ is a C^r-diffeomorphism, then it maps δX onto δY. Consequently, for any $n \in \mathbb{N}$, $\delta^n X$ and $\delta^n Y$ are C^r-diffeomorphic.*

2.2. **Differentiable manifolds in Euclidean spaces.** A subset M of $\mathbb{R}^k$ is called a (boundaryless) m-dimensional (differentiable) manifold of class C^r, $r \in \mathbb{N} \cup \{\infty\}$, if it is locally C^r-diffeomorphic to $\mathbb{R}^m$; meaning that any point p of M admits a neighborhood (in M) which is C^r-diffeomorphic to an open subset of $\mathbb{R}^m$. A C^r-diffeomorphism $\varphi\colon W \to V \subseteq M$ from an open subset W of $\mathbb{R}^m$ onto an open subset V of M is called a *parametrization* (of class C^r of V). The inverse $\varphi^{-1}\colon V \to W$

of φ is called a *chart* or a *coordinate system* on V, and its component functions, $x_1, x_2, \dots, x_m$, are the *coordinate functions* of φ^{-1} on V.

As a straightforward consequence of the definition of differentiable manifold and Theorem 2.12, any point p of an m-dimensional C^1-manifold M is non-singular (i.e., $C_pM = T_pM$). Moreover, $\dim T_pM = m$. In fact, since this property is true for open subsets of $\mathbb{R}^m$, according to Theorem 2.11, it holds true for m-dimensional C^1-manifolds. Incidentally, observe that Theorem 2.11 provides a practical method for computing T_pM. That is, if $\varphi\colon W \to V$ is a any C^1-parametrization of a neighborhood V of p in M, then $T_pM = \operatorname{Im}\varphi'(w)$, where $\varphi(w) = p$.

The following direct consequence of the Implicit Function Theorem can be used to produce a large variety of examples of differentiable manifolds. It gives also a useful tool to compute the tangent space at any given point of a manifold. We recall first that if $f\colon U \to \mathbb{R}^s$ is a C^1 map on an open subset U of $\mathbb{R}^k$, an element $p \in U$ is said to be a *regular point* of f if the derivative $f'(p)$ of f at p is surjective. Non-regular points are called *critical (points)*. The *critical values* of f are those points of the target space $\mathbb{R}^s$ which lie in the image $f(C)$ of the set C of critical points. Any $y \in \mathbb{R}^s$ which is not in $f(C)$ is a regular value. Therefore, in particular, any element of $\mathbb{R}^s$ which is not in the image of f is a regular value. Notice that, in this terminology, the words "point" and "value" refer to the source and target spaces, respectively.

Theorem 2.13 (Regularity of the level set). *Let $f\colon U \to \mathbb{R}^s$ be a C^r map of an open subset of $\mathbb{R}^k$ into $\mathbb{R}^s$. If $0 \in \mathbb{R}^s$ is a regular value for f, then $f^{-1}(0)$ is a C^r-manifold of dimension $k - s$. Moreover, given $p \in f^{-1}(0)$, we have*

$$T_p\big(f^{-1}(0)\big) = \operatorname{Ker} f'(p).$$

Proof. Choose a point $p \in f^{-1}(0)$ and split $\mathbb{R}^k$ into the direct sum $\operatorname{Ker} f'(p) \oplus (\operatorname{Ker} f'(p))^{\perp}$. Since, by assumption, $f'(p)\colon \mathbb{R}^k \to \mathbb{R}^s$ is onto, the restriction of $f'(p)$ to $(\operatorname{Ker} f'(p))^{\perp}$ is an isomorphism. Observe that this restriction is just the (second) partial derivative, $\partial_2 f(p)$, of f at p with respect to the given decomposition. It follows, by the Implicit Function Theorem, that in a neighborhood of p, $f^{-1}(0)$ is the graph of a C^r map $\varphi\colon W \to \operatorname{Ker} f'(p)^{\perp}$ defined on an open subset W of $\operatorname{Ker} f'(p)$. Remark 2.4 implies that in this neighborhood $f^{-1}(0)$ is C^r-diffeomorphic to W. Thus $f^{-1}(0)$ is a C^r-manifold whose dimension is $\dim \operatorname{Ker} f'(p) = k - s$.

To prove that $T_p(f^{-1}(0)) = \operatorname{Ker} f'(p)$ observe first that $T_p(f^{-1}(0)) \subseteq \operatorname{Ker} f'(p)$. In fact, f maps $f^{-1}(0)$ into $\{0\}$ and, consequently, $f'(p)$ maps $T_p(f^{-1}(0))$ into $T_0(\{0\}) = \{0\}$. The equality follows by computing the dimensions of the two spaces. $\square$

Theorem 2.13 can be partially inverted, in the sense that any C^r differentiable manifold in $\mathbb{R}^k$ can be locally regarded as a regular level set (i.e., as the inverse image of a regular value of a C^r map on an open subset of $\mathbb{R}^k$). In fact, the following theorem holds.

Theorem 2.14. *Let M be an m-dimensional manifold of class C^r in $\mathbb{R}^k$. Then, given $p \in M$, there exists a map $f\colon U \to \mathbb{R}^{k-m}$, C^r on a neighborhood U of p in $\mathbb{R}^k$, which defines $M \cap U$ as a regular level set.*

Proof. Let $\varphi\colon W \to \mathbb{R}^k$ be a C^r-parametrization of M around p and let $w = \varphi^{-1}(p)$. Consider any linear map $L\colon \mathbb{R}^{k-m} \to \mathbb{R}^k$ such that $\operatorname{Im} L \oplus T_pM = \mathbb{R}^k$ (this is clearly possible since $\dim T_pM = m$), and define $g\colon W \times \mathbb{R}^{k-m} \to \mathbb{R}^k$ by setting $g(x,y) = \varphi(x) + Ly$. The derivative of g at $(w,0) \in W \times \mathbb{R}^{k-m}$ is given by

$$g'(w,0)(h,k) = \varphi'(w)h + Lk,$$

which is surjective (therefore an isomorphism), since $\operatorname{Im}\varphi'(0) = T_pM$. By the Inverse Function Theorem, g is a C^r-diffeomorphism of a neighborhood of $(w,0)$ in $W \times \mathbb{R}^{k-m}$ onto a neighborhood U of p in $\mathbb{R}^k$. Let ψ be the inverse of such a diffeomorphism and define $f\colon U \to \mathbb{R}^{k-m}$ as the composition $\pi_2 \circ \psi$ of ψ with the projection $\pi_2\colon W \times \mathbb{R}^{k-m} \to \mathbb{R}^{k-m}$ of $W \times \mathbb{R}^{k-m}$ onto the second factor. We see that f satisfies the assertion. □

We point out that there are differentiable manifolds in $\mathbb{R}^k$ which cannot be globally defined as regular level sets. One can prove, in fact, that when this happens, the manifold must be orientable (the definition of orientability and the proof of this assertion would carry us too far away). As an intuitive example consider a Möbius strip M embedded in $\mathbb{R}^3$ and assume $M = f^{-1}(0)$, where $f\colon U \to \mathbb{R}$ is a C^1 map on an open subset of $\mathbb{R}^3$. If $0 \in \mathbb{R}$ were a regular value for f, the gradient of f at any point $p \in f^{-1}(0)$, $\nabla f(p)$, would be nonzero. Therefore, the map $\nu\colon M \to \mathbb{R}^3$, given by $\nu(p) = \nabla f(p)/|\nabla f(p)|$, would be a continuous normal unit vector field on M, and this is well-known to be impossible on the Möbius strip (a one-sided surface).

Now we want to define an "embedded" notion of tangent bundle TM associated with a C^r manifold M in $\mathbb{R}^k$. We will prove that if $r \geq 2$, TM is a C^{r-1} differentiable manifold in $\mathbb{R}^k \times \mathbb{R}^k$. In order to do this, we shall define the concept of tangent bundle for any subset of $\mathbb{R}^k$, and prove that when two sets X and Y are C^r-diffeomorphic, the corresponding tangent bundles are C^{r-1}-diffeomorphic.

Definition 2.15. Given $X \subseteq \mathbb{R}^k$, the subset

$$TX = \big\{(x,y) \in \mathbb{R}^k \times \mathbb{R}^k : x \in X,\ y \in T_xX\big\}$$

of $\mathbb{R}^k \times \mathbb{R}^k$ is called the *tangent bundle* of X. The canonical projection $\pi\colon TX \to X$ is the restriction to TX of the projection of $\mathbb{R}^k \times \mathbb{R}^k$ onto the first factor (thus, π is always a smooth map).

Definition 2.16. Let $f\colon X \to Y$ be a C^r map from a subset X of $\mathbb{R}^k$ into a subset Y of $\mathbb{R}^s$ and assume $1 \leq r \leq \infty$. The *tangent map* of f, $Tf\colon TX \to TY$, is given by

$$Tf(x,y) = \big(f(x), f'(x)y\big).$$

As pointed out in Remark 2.2, one may regard a C^r map $f\colon X \to Y$ as the restriction of a C^r map $g\colon U \to \mathbb{R}^s$ defined on an open neighborhood U of X. Consequently, if $r \geq 1$, $Tg\colon TU \to T\mathbb{R}^s$, given by $(x,y) \mapsto \big(g(x), g'(x)y\big)$, is a C^{r-1} map from the open neighborhood $TU = U \times \mathbb{R}^k$ of TX into $T\mathbb{R}^s = \mathbb{R}^s \times \mathbb{R}^s$. This proves that Tf, which is just the restriction to TX of Tg, is a C^{r-1} map.

Clearly, if $f\colon X \to Y$ and $g\colon Y \to Z$ are C^r maps, one has $T(g \circ f) = Tg \circ Tf$. Moreover, if $i\colon X \to X$ is the identity on X, then $Ti\colon TX \to TX$ is the identity on TX. Therefore, one may regard T as a covariant functor from the category of Euclidean sets with C^r maps into the category of Euclidean sets with C^{r-1} maps. This implies that if $f\colon X \to Y$ is a C^r-diffeomorphism with $r \geq 2$, then $Tf\colon TX \to TY$ is a C^{r-1}-diffeomorphism. Therefore, if M is a C^r manifold of dimension m, since it is locally C^r-diffeomorphic to the open subsets of $\mathbb{R}^m$, its tangent bundle TM is a C^{r-1} manifold of dimension $2m$. Moreover, if $\varphi\colon W \to$

$V \subseteq M$ is a parametrization of an open set V in M, $T\varphi\colon W \times \mathbb{R}^m \to TV \subseteq TM$ is a parametrization of the open set $TV = \pi^{-1}(V)$ of TM.

Definition 2.17. Let X be a subset of $\mathbb{R}^k$. A *tangent vector field* on X is a continuous map $g\colon X \to \mathbb{R}^k$ with the property that $g(x) \in T_xX$ for all $x \in X$. The tangent vector field g on X is said to be *inward* if $g(x) \in C_xX$ for all $x \in X$.

Usually, in differential geometry, a tangent vector field on a differentiable manifold M is defined as a section of the tangent bundle TM. That is, a map $w\colon M \to TM$ with the property that the composition $\pi \circ w\colon M \to M$ of w with the bundle projection π is the identity on M. However, in our "embedded" situation (i.e., M in $\mathbb{R}^k$) this "abstract" definition turns out to be redundant. In fact, observe that, for M embedded in $\mathbb{R}^k$, a map $w\colon M \to \mathbb{R}^k \times \mathbb{R}^k$ is a section of TM if and only if for all $x \in M$ one has $w(x) = \big(x, g(x)\big)$, with $g(x) \in T_xM$. Therefore, forgetting x in the pair $(x, g(x))$, one may accept the simpler definition given above.

An important example of a tangent vector field to a differentiable manifold $M \subseteq \mathbb{R}^k$ is the *gradient* of a C^1 function $f\colon M \to \mathbb{R}$. This is defined by assigning to any point $x \in M$ the unique vector $\nabla f(x) \in T_xM$ such that

$$\langle \nabla f(x), v\rangle = f'(x)v, \quad \forall v \in T_xM,$$

where $\langle \cdot, \cdot \rangle$ denotes the (canonical) inner product on $\mathbb{R}^k$.

We now recall a classical theorem that will be essential for our definition of fixed point index. For its proof see e.g. [Hir, Chapter 4, §5].

Theorem 2.18 (tubular neighborhoods). *Let $M \subseteq \mathbb{R}^k$ be a smooth manifold. Then there exists a neighborhood W of M such that any point $x \in W$ possesses a unique closest point $r(x)$ in M. Moreover, the map $r\colon W \to M$ is a smooth submersion (i.e., it has surjective derivative at any point).*

Note that the map r in the above theorem is a *retraction*, i.e. it has the property that $r(x) = x$ for all $x \in M$. Thus any manifold in $\mathbb{R}^k$ is a retract of one of its neighborhoods.

2.3. **The Brouwer degree and the fixed point index.** For the sake of simplicity, from now on "differentiable manifold" or, briefly, "manifold" will mean "smooth manifold" embedded in some Euclidean space. Moreover, any map between manifolds is assumed to be (at least) continuous.

Before introducing the fixed point index on manifolds, we need to discuss briefly a slightly extended notion of Brouwer degree in Euclidean spaces. For more details about degree, the reader is referred to [Llo, Mil, Nir].

Let V be an open subset of $\mathbb{R}^k$. A pair (g, V) is *admissible (for the Brouwer degree)* if g is an $\mathbb{R}^k$-valued (continuous) map whose domain $\mathcal{D}(g)$ contains V and such that $g^{-1}(0) \cap V$ is compact. In particular this holds if (g, V) is *strongly admissible*; that is, if V is bounded, g is defined at least on the closure $\overline{V}$ of V, and $g(x) \neq 0$ for all x in the boundary ∂V of V. On the set of admissible pairs there is defined an integer-valued function $\deg_B$, called *Brouwer degree*, satisfying the following three fundamental properties.

Normalization. *For the identity I on $\mathbb{R}^k$ one has* $\deg_B(I, \mathbb{R}^k) = 1$.

Additivity. *Given an admissible pair (g, V), if V_1 and V_2 are two disjoint open subsets of V such that $g^{-1}(0) \cap V \subseteq V_1 \cup V_2$, then*

$$\deg_B(g, V) = \deg_B(g|_{V_1}, V_1) + \deg_B(g|_{V_2}, V_2).$$

Homotopy invariance. *If* $V \subseteq \mathbb{R}^k$ *is an open subset of* $\mathbb{R}^k$*, and* $h\colon V\times[0,1] \to \mathbb{R}^k$ *is an* admissible homotopy*, i.e.* $h^{-1}(0)$ *is compact, then*

$$\deg_B\big(h(\cdot,0),V\big) = \deg_B\big(h(\cdot,1),V\big).$$

As a consequence of a well known result of Amann and Weiss (see [AmWa]), the function $\deg_B$ is uniquely determined by the above properties. This is because these properties imply the Amann-Weiss axioms, which are stated for strongly admissible pairs. Thus, henceforth, the three fundamental properties will be still referred to as *Amann-Weiss axioms* (for admissible pairs).

Below, we list some other important properties of the Brouwer degree, which can be easily derived from the Amann-Weiss axioms.

Excision. *Given an admissible pair* (g,V) *and an open subset* V_1 *of* V *containing* $g^{-1}(0)\cap V$*, one has* $\deg_B(g,V) = \deg_B(g|_{V_1},V_1)$.

By excision, taking $V_1 = V$, we get the following property which shows that the degree is independent of the behavior of g outside V.

Localization. *If* (g,V) *is admissible, then* $\deg_B(g,V) = \deg_B(g|_V,V)$.

Solution. *If* (g,V) *is admissible and* $\deg_B(g,V)\neq 0$*, then* $g^{-1}(0)\cap V$ *in nonempty.*

Boundary dependence. *If* (g_1,V) *and* (g_2,V) *are strongly admissible and* $g_1|_{\partial V} = g_2|_{\partial V}$*, then* $\deg_B(g_1,V) = \deg_B(g_2,V)$.

Remark 2.19. If (g,V) is admissible, g is C^1 on V and 0 is a regular value for g in V, then $g^{-1}(0)\cap V$ is a finite set. It can be shown (see e.g. [Llo]) that in this case

$$\deg_B(g,V) = \sum_{x\in g^{-1}(0)\cap V} \operatorname{sign}\det g'(x).$$

We now present the fixed point index in the context of differentiable manifolds. Let $M\subseteq\mathbb{R}^k$ be a manifold, U an open subset of M, and f an M-valued (continuous) map whose domain $\mathcal{D}(f)\subseteq M$ contains U. The fixed point index of f in U, $\operatorname{ind}(f,U)$, is a well defined integer whenever the set $\operatorname{fix}(f,U)$ of fixed points of f in U is compact. When this holds, the pair (f,U) is said to be *admissible (for the fixed point index on* M*)*. Loosely speaking, $\operatorname{ind}(f,U)$ is an algebraic count of the elements of $\operatorname{fix}(f,U)$. The following is the precise definition.

Definition 2.20. Let M, U and f be as above. If the pair (f,U) is admissible, the *fixed point index* of f in U is the integer

$$\operatorname{ind}(f,U) = \deg_B\big(I - f\circ r, r^{-1}(U)\big), \tag{2.1}$$

where I is the identity in $\mathbb{R}^k$ and $r\colon W\to M$ is any retraction defined on an open neighborhood of M.

In the above definition, the existence of a retraction r is ensured by Theorem 2.18. Moreover, one can prove that $\deg_B\big(I - f\circ r, r^{-1}(U)\big)$ does not depend on the choice of r (see e.g. [DuGr, Gra, Nus]). Thus, $\operatorname{ind}(f,U)$ is well defined.

Note that, in particular, when $M=\mathbb{R}^k$, the fixed point index of a map $f\colon U\to\mathbb{R}^k$ coincides with the Brouwer degree of $I-f$ in U. Namely,

$$\operatorname{ind}(f,U) = \deg_B(I-f,U). \tag{2.2}$$

The fixed point index has a number of useful properties. Below we list some of the most important ones. For proofs and more details we refer to [Ama, DuGr, Gra, Nus].

Normalization. *Let* $f\colon M \to M$ *be constant. Then* $\operatorname{ind}(f, M) = 1$.

Additivity. *Given an admissible pair* (f, U)*, if* U_1 *and* U_2 *are two disjoint open subsets of* U *such that* $\operatorname{fix}(f, U) \subseteq U_1 \cup U_2$*, then*

$$\operatorname{ind}(f, U) = \operatorname{ind}(f|_{U_1}, U_1) + \operatorname{ind}(f|_{U_2}, U_2).$$

Homotopy invariance. *A map* $h\colon U \times [0,1] \to M$ *with the property that the set* $\{(x, \lambda) \in U \times [0,1] : x = h(x, \lambda)\}$ *is compact is called an* admissible homotopy*. In this case,*

$$\operatorname{ind}\big(h(\cdot, 0), U\big) = \operatorname{ind}\big(h(\cdot, 1), U\big).$$

Commutativity. *Let* U_1 *and* U_2 *be open subsets of two manifolds* M_1 *and* M_2*, respectively. Given* $f_1\colon U_1 \to M_2$ *and* $f_2\colon U_2 \to M_1$*, if one of the pairs* $\big(f_2 \circ f_1, f_1^{-1}(U_2)\big)$ *or* $\big(f_1 \circ f_2, f_2^{-1}(U_1)\big)$ *is admissible, then so is the other and*

$$\operatorname{ind}\big(f_2 \circ f_1, f_1^{-1}(U_2)\big) = \operatorname{ind}\big(f_1 \circ f_2, f_2^{-1}(U_1)\big).$$

Solution. *If* $\operatorname{ind}(f, U) \neq 0$*, then the fixed point equation* $f(x) = x$ *has a solution in* U*.*

Multiplicativity. *Let* U_1 *and* U_2 *be open subsets of two manifolds* M_1 *and* M_2*, respectively. Assume that* $\big(f_1, U_1\big)$ *and* $\big(f_2, U_2\big)$ *are admissible. Consider the map* $f_1 \times f_2\colon U_1 \times U_2 \to M_1 \times M_2$ *given by* $(x_1, x_2) \mapsto \big(f_1(x_1), f_2(x_2)\big)$*. Then the pair* $\big(f_1 \times f_2, U_1 \times U_2\big)$ *is admissible and*

$$\operatorname{ind}\big(f_1 \times f_2, U_1 \times U_2\big) = \operatorname{ind}(f_1, U_1) \cdot \operatorname{ind}(f_2, U_2).$$

Remark 2.21. Given any M-valued map f defined on a subset $\mathcal{D}(f)$ of M, the pair $(f, \emptyset)$ is admissible. This includes the case when $\mathcal{D}(f)$ is the empty set (it is coherent with the notion of a map as a triple (A, B, R) where A is the domain, B is the codomain and $R \subseteq A \times B$ is such that there exists exactly one $(a, b) \in R$ for any $a \in A$). A simple application of the additivity property shows that $\operatorname{ind}(f|_{\emptyset}, \emptyset) = 0$ and $\operatorname{ind}(f, \emptyset) = 0$.

As a consequence of the additivity property and of Remark 2.21 one easily gets the following often-neglected property, which shows that the index of an admissible pair (f, U) does not depend on the behavior of f outside U.

Localization. *If* (f, U) *is admissible, then* $\operatorname{ind}(f, U) = \operatorname{ind}(f|_U, U)$*.*

Another consequence of the additivity is the following important property.

Excision. *Given an admissible pair* (f, U) *and an open subset* U_1 *of* U *containing* $\operatorname{fix}(f, U)$*, one has* $\operatorname{ind}(f, U) = \operatorname{ind}(f, U_1)$*.*

A stronger form of the homotopy property is often useful (see e.g. [Ama, Nus]).

Generalized homotopy invariance. *Let* M *be a manifold and let* $W \subseteq M \times [0,1]$ *be open. Suppose* $h\colon W \to M$ *is such that the set*

$$\big\{(x, \lambda) \in W \times [0,1] : x = h(x, \lambda)\big\}$$

is compact. For any $\lambda \in [0,1]$*, put* $W_\lambda = \{x \in M : (x, \lambda) \in W\}$ *and* $h_\lambda = h(\cdot, \lambda)$*. Then* $\operatorname{ind}\big(h_\lambda, W_\lambda\big)$ *is well defined and independent of* $\lambda \in [0,1]$*.*

In the case when M is a compact manifold, it is well known that $\operatorname{ind}(f, M)$ coincides with the Lefschetz number $\Lambda(f)$ of f. This is often called the **strong normalization property** of the fixed point index. A discussion of the Lefschetz number, that would require homology theory, is beyond the scope of this chapter; the interested reader is referred to, e.g,, [DuGr, Spn].

It is well known that some of the above properties can be used as axioms for a fixed point index theory. For instance, it can be deduced from [Bro] that the first four determine uniquely the fixed point index. Actually the result of [Bro] is more general: it holds in the framework of metric ANRs. In this more general setting, other uniqueness results based on a stronger version of the normalization property are available for the (more restrictive) class of compact maps (see e.g. [DuGr, §16, Theorem 5.1]). Below, using the uniqueness result for the Brouwer degree given in [AmWa] we shall prove that the properties of normalization, additivity and homotopy invariance are sufficient to determine uniquely the fixed point index on manifolds.

An admissible pair for the index (f, U) will be called *regular* if f is smooth on U and any fixed point of f in U is nondegenerate; that is, 1 does not belong to the spectrum of the endomorphism $f'(x)\colon T_xM \to T_xM$ for any $x \in \operatorname{fix}(f, U)$. Note that, in this case, $\operatorname{fix}(f, U)$ is necessarily a discrete set, therefore finite, being compact.

The following remark shows that the computation of the fixed point index of an admissible pair can always be reduced to that of a regular pair.

Remark 2.22. Let (f, U) be admissible (for the index) and let U_1 be a relatively compact neighborhood of $\operatorname{fix}(f, U)$ such that $\overline{U}_1 \subseteq U$. By excision, $\operatorname{ind}(f, U_1)$ coincides with $\operatorname{ind}(f, U)$. It can be shown, via standard transversality arguments, that f is admissibly homotopic on U_1 to some smooth approximation φ of it such that (φ, U_1) is regular.

The following proposition shows that the properties of normalization, additivity and homotopy invariance enforce a formula for the computation of the fixed point index that is valid for any regular pair (f, U). Thus, by Remark 2.22 and by the homotopy invariance property, we see that there exists a unique integer-valued function on the set of admissible pairs that satisfies the normalization, additivity and homotopy invariance properties. In other words, *the fixed point index is uniquely determined by these three properties.*

Theorem 2.23 (Uniqueness of the fixed point index)**.** *Let $M \subseteq \mathbb{R}^k$ be an m-dimensional manifold and let "ind" be an integer-valued function on the set of admissible pairs satisfying the properties of normalization, additivity and homotopy invariance. Then, given any regular pair (f, U), one has*

$$\operatorname{ind}(f, U) = \sum_{x \in \operatorname{fix}(f,U)} \operatorname{sign}\Big(\det\big(I_x - f'(x)\big)\Big),$$

where I_x denotes the identity of T_xM.

Proof. Let W be an open subset of M which is diffeomorphic to the whole space $\mathbb{R}^m$ and let $\psi\colon W \to \mathbb{R}^m$ be any diffeomorphism (onto $\mathbb{R}^m$). Denote by $\mathcal{U}$ the set of all pairs (f, U) which are admissible for the fixed point index in M and such that $U \subseteq W$, $f(U) \subseteq W$. These pairs may be regarded as admissible for the fixed point index in W, and the restriction of the index function to $\mathcal{U}$ still satisfies the properties of normalization, additivity and homotopy invariance. We claim that for any $(f,U)\in\mathcal{U}$ one necessarily has

$$\operatorname{ind}(f,U)=\deg_B\big(I-\psi\circ f\circ\psi^{-1},\psi(U)\big), \tag{2.3}$$

where I is the identity in $\mathbb{R}^m$. To show this, denote by $\mathcal{V}$ the set of pairs (g,V) which are admissible for the degree in $\mathbb{R}^m$ and consider the one-to-one correspondence $\omega\colon\mathcal{U}\to\mathcal{V}$ defined by

$$\omega(f,U)=\big(I-\psi\circ f\circ\psi^{-1},\psi(U)\big).$$

We need to prove that $\operatorname{ind}=\deg_B\circ\,\omega$. Observe that

$$\omega^{-1}(g,V)=\big(\psi^{-1}\circ(I-g)\circ\psi,\psi^{-1}(V)\big),$$

and if two pairs $(f,U)\in\mathcal{U}$ and $(g,V)\in\mathcal{V}$ correspond under ω, then the sets $\operatorname{fix}(f,U)$ and $g^{-1}(0)\cap U$ correspond under ψ. It is also evident that the function $d\colon\mathcal{V}\to\mathbb{Z}$ defined by the composition $d=\operatorname{ind}\circ\,\omega^{-1}$ satisfies the Amann-Weiss axioms. Thus, $\deg_B$ and d coincide on $\mathcal{V}$, and this implies $\operatorname{ind}=\deg_B\circ\,\omega$, as claimed.

Assume now that (f,U) is a regular pair for the fixed point index in M. Let $\operatorname{fix}(f,U)=\{x_1,\dots,x_n\}$ and let $W_1,\dots,W_n$ be n pairwise disjoint open subsets of U such that $x_i\in W_i$, for $i=1,\dots,n$. Since M is locally diffeomorphic to the whole space $\mathbb{R}^m$, we may assume that each W_i is diffeomorphic to $\mathbb{R}^m$ under a diffeomorphism ψ_i. For any i, let U_i be an open subset of W_i such that $f(U_i)\subseteq W_i$. The additivity property yields

$$\operatorname{ind}(f,U)=\sum_{i=1}^n\operatorname{ind}(f,U_i),$$

and, by the previous argument, we get

$$\sum_{i=1}^n\operatorname{ind}(f,U_i)=\sum_{i=1}^n\deg_B\big(I-\psi_i\circ f\circ\psi_i^{-1},\psi_i(U_i)\big).$$

By Remark 2.19 and the chain property of the derivative, for any i one has

$$\deg_B\big(I-\psi_i\circ f\circ\psi_i^{-1},\psi_i(U_i)\big)=\operatorname{sign}\big(\det\big(I_{x_i}-f'(x_i)\big)\big).$$

Thus

$$\operatorname{ind}(f,U)=\sum_{i=1}^n\operatorname{sign}\big(\det\big(I_{x_i}-f'(x_i)\big)\big),$$

and this concludes the proof. □

2.4. **The degree of a tangent vector field.** Recall that, for the sake of simplicity, unless otherwise specified, all the manifolds are supposed smooth and all the maps are assumed continuous.

Let U be an open subset of a manifold $M\subseteq\mathbb{R}^k$ and let g be a tangent vector field defined at least on U. We say that (g,U) is an *admissible pair of M (for the degree of a vector field)* if $g^{-1}(0)\cap U$ is compact. When context precludes confusion about the universe M containing U, we will simply say that (g,U) is admissible (or, equivalently, that g is admissible on U). In this case (see e.g. [GuPo, Hir, Mil, Tro] and references therein) one can assign to g an integer, $\deg(g,U)$, called the degree (or index, or Euler characteristic, or rotation) of the tangent vector field g on U.

To avoid any possible confusion, we point out that in the literature there exists a different extension of the Brouwer degree to the context of differentiable manifolds (see e.g. [Mil] and references therein), called the Brouwer degree for maps between manifolds. This second extension, roughly speaking, counts the (algebraic) number of solutions of an equation of the form $h(x) = y$, where $h\colon M \to N$ is a map between oriented manifolds of the same dimension and $y \in N$ is such that $h^{-1}(y)$ is compact. This dichotomy of notions in the context of manifolds arises from the fact that counting the solutions of an equation of the form $h(x) = y$ cannot be reduced to the problem of counting the zeros of a vector field, as one can do in $\mathbb{R}^k$ by defining $g(x) = h(x) - y$. Therefore, from the point of view of global analysis, the degree of a vector field and the degree of a map are necessarily two separated notions. The first one, which we are interested in, does not require any orientability and is particularly important for differential equations, since a tangent vector field on a manifold can be regarded as an autonomous differential equation.

We give here a brief idea of how this degree can be defined. We need first the following result (see e.g. [Mil]).

Theorem 2.24. *Let $g\colon M \to \mathbb{R}^k$ be a C^1 tangent vector field on a manifold $M \subseteq \mathbb{R}^k$. If g is zero at some point $p \in M$, then the derivative $g'(p)\colon T_pM \to \mathbb{R}^k$ maps T_pM into itself. Therefore, $g'(p)$ can be regarded as an endomorphism of T_pM and, consequently, its determinant $\det(g'(p))$ is well defined.*

Proof. It suffices to show that $g'(p)v \in T_pM$ for any $v \in T_pM$ such that $|v| = 1$. Given such a vector v, consider a sequence in $M \setminus \{p\}$ such that $p_n \to p$ and $(p_n - p)/|p_n - p| \to v$. By Lemma 2.7 we have

$$g'(p)v = \lim_{n\to\infty} \frac{g(p_n) - g(p)}{|p_n - p|} = \lim_{n\to\infty} \frac{g(p_n)}{|p_n - p|}.$$

Observe that for all $n \in \mathbb{N}$, the vector $w_n = g(p_n)/|p_n - p|$ is tangent to M at p_n. Let us show that this implies that the limit $w = g'(p)v$ of $\{w_n\}$ is in T_pM. In fact, because of Theorem 2.14, we may assume that M (around p) is a regular level set of a smooth map $f\colon V \to \mathbb{R}^s$ defined on some open subset V of $\mathbb{R}^k$. Thus, by Theorem 2.13, $f'(p_n)w_n = 0$, and passing to the limit we get $f'(p)w = 0$, which means $w \in T_pM$, as claimed. □

Let $g\colon M \to \mathbb{R}^k$ be a C^1 tangent vector field on a manifold $M \subseteq \mathbb{R}^k$. A zero $p \in M$ of g is said to be *nondegenerate* if $g'(p)$, as a map from T_pM into itself, is an isomorphism. In this case, its *index* at p, $\mathrm{i}(g, p)$, is defined to be 1 or -1 according to the sign of the determinant $\det\big(g'(p)\big)$.

In the particular case when an admissible pair (g, U) is *regular* (i.e., g is smooth with only nondegenerate zeros), its degree, $\deg(g, U)$, is defined just summing up the indices at its zeros. This makes sense, since $g^{-1}(0) \cap U$ is compact (g being admissible in U) and discrete. Therefore, the sum is finite. Using transversality arguments (see e.g. [Hir]) one can show that if two such tangent vector fields can be joined by a smooth homotopy, then they have the same degree, provided that this homotopy is admissible (i.e., the set of zeros remains in a compact subset of U). Moreover, it is clear that given g as above, if V is any open subset of U containing $g^{-1}(0) \cap U$, then $\deg(g, U) = \deg(g, V)$.

The above "homotopy invariance property" for regular pairs gives an idea of how to proceed in the general case. If the pair (g, U) is admissible, consider any relatively compact open subset V of U containing $g^{-1}(0)\cap U$ and observe that, since the boundary ∂V of V (in M) is compact, we have $\min\{|g(x)| : x \in \partial V\} = \delta > 0$ (recall that M is embedded in $\mathbb{R}^k$). Let (g_1, V) be any regular pair with g defined (at least) on $\overline{V}$ such that

$$\max\left\{|g(x) - g_1(x)| : x \in \partial V\right\} < \delta.$$

Then $\deg(g, U)$ is defined as $\deg(g_1, V)$. To see that this definition does not depend on the approximating map, observe that if (g_2, V) is a different regular pair with g_2 satisfying the same inequality, we get $(1 - \lambda)g_1(x) + \lambda g_2(x) \neq 0$ for all $\lambda \in [0, 1]$ and $x \in \partial V$. Therefore $(x, \lambda) \mapsto (1 - \lambda)g_1(x) + \lambda g_2(x)$ is an admissible homotopy of tangent vector fields on V. This proves that $\deg(g_1, V) = \deg(g_2, V)$. The fact that this definition does not depend on the open set V containing $g^{-1}(0)$ is very easy to check and left to the reader.

The following are the main properties of the degree for admissible tangent vector fields on open subsets of a manifold.

Solution. *If $\deg(g, U) \neq 0$, then g has a zero in U.*

Additivity. *Let (g, U) be admissible. If U_1 and U_2 are two disjoint open subsets of U whose union contains $g^{-1}(0) \cap U$, then*

$$\deg(g, U) = \deg(g|_{U_1}, U_1) + \deg(g|_{U_2}, U_2).$$

Homotopy invariance. *Let $h\colon U \times [0, 1] \to \mathbb{R}^k$ be an admissible homotopy of tangent vector fields; that is, $h(x, \lambda) \in T_xM$ for all $(x, \lambda) \in U \times [0, 1]$ and $h^{-1}(0)$ is compact. Then $\deg\big(h(\cdot, \lambda), U\big)$ does not depend on $\lambda \in [0, 1]$.*

The above definition of degree implies that if two vector fields $g_1\colon M_1 \to \mathbb{R}^k$ and $g_2\colon M_2 \to \mathbb{R}^s$ correspond under a diffeomorphism $\psi\colon M_1 \to M_2$ (i.e. $\psi'(x)g_1(x) = g_2\big(\psi(x)\big)$ for any $x \in M_1$), then, if one is admissible, so is the other, and they have the same degree (on M_1 and M_2 respectively). More precisely, the following property holds.

Topological invariance. *Let ψ be a diffeomorphism of a manifold $M_1 \subseteq \mathbb{R}^k$ onto a manifold $M_2 \subseteq \mathbb{R}^s$. Let (g_1, U) be an admissible pair of M_1, and assume that $g_2\colon \psi(U) \to \mathbb{R}^s$ corresponds to g_1 under ψ, then $\big(g_2, \psi(U)\big)$ is an admissible pair of M_2 and*

$$\deg(g_1, U) = \deg\big(g_2, \psi(U)\big).$$

Note also that, in the particular case when U is an open subset of $M = \mathbb{R}^m$ and g is an admissible vector field on U (i.e. $g^{-1}(0) \cap U$ is compact), then $\deg(g, U)$ coincides with the Brouwer degree $\deg_B(g, U)$.

Another immediate consequence of the definition of degree is the relation between the degree of a vector field and that of its opposite. If (g, U) is an admissible pair of M, then

$$\deg(g, U) = (-1)^m \deg(-g, U), \tag{2.4}$$

where m is the dimension of M. More generally, for a given constant $\alpha \neq 0$, one has

$$\deg(g, U) = (\operatorname{sign} \alpha)^m \deg(\alpha g, U) \tag{2.5}$$

The additivity property, analogously to what happens for the fixed point index, implies the following property.

Excision. *Given an admissible pair* (g, U) *and an open subset* U_1 *of* U *containing* $g^{-1}(0) \cap U$*, one has* $\deg(g, U) = \deg(g|_{U_1}, U_1)$.

From the excision, taking $U_1 = U$, one gets the next property, which shows that the degree of an admissible pair (g, U) is not influenced by the behavior of g outside the open set U.

Localization. *If* (g, U) *is admissible, then so is* $(g|_U, U)$ *and*

$$\deg(g, U) = \deg(g|_U, U).$$

Given a relatively compact open set $U \subseteq M$, assume that $g_1 \colon M \to \mathbb{R}^k$ and $g_2 \colon M \to \mathbb{R}^k$ are tangent vector fields such that $g_1|_{\partial U} = g_2|_{\partial U}$. Clearly, (g_1, U) is admissible if and only if so is (g_2, U) and the map $(x, \lambda) \mapsto \lambda g_1(x) + (1 - \lambda) g_2(x)$ is an admissible homotopy in U. By the homotopy invariance property, one gets the following property.

Boundary dependence. *Assume the open set* $U \subseteq M$ *is relatively compact and that* (g_1, U) *or* (g_2, U) *is admissible. If* $g_1|_{\partial U} = g_2|_{\partial U}$*, then* $\deg(g_1, U) = \deg(g_2, U)$.

Actually, for a relatively compact open set $U \subseteq M$, in [Pug] there is given a formula, valid for "most" tangent vector fields, that relates $\deg(g, U)$ to the topology of U and the behavior of g along ∂U.

It is known that, unless the target manifold is flat, the boundary dependence property does not hold for the degree of maps between oriented manifolds.

Another consequence of the homotopy invariance property is that the degree of a tangent vector field g on a compact manifold $M \subseteq \mathbb{R}^k$ is independent of g. In fact, if g_1 and g_2 are two tangent vector fields on M, then $h \colon M \times [0, 1] \to \mathbb{R}^k$, given by

$$h(x, \lambda) = (1 - \lambda) g_1(x) + \lambda g_2(x),$$

is an admissible homotopy. This permits the assignment of an integer $\chi(M)$, called the *Euler-Poincaré characteristic of* M, by setting

$$\chi(M) = \deg(g, M), \tag{2.6}$$

where $g \colon M \to \mathbb{R}^k$ is any tangent vector field on M. Thus, if $\chi(M) \neq 0$, then any tangent vector field on M must vanish at some point. Moreover, the topological invariance implies that if two compact manifolds M and N are diffeomorphic, then $\chi(M) = \chi(N)$.

Actually, there are other equivalent (and better) ways to define the Euler-Poincaré characteristic of a compact manifold. One of these is via homology theory, where $\chi(M)$ coincides with the Lefschetz number of the identity (see for example [DuGr, Spn]). The powerful homological method has the advantage of being applicable to a large class of topological spaces, which includes those of the same homotopy type as compact polyhedra (such as compact manifolds with boundary). The celebrated Poincaré-Hopf theorem asserts that *the above definition of the Euler-Poincaré characteristic coincides with the homological one* (see e.g. [DuGr, Hir, Mil]).

Observe that formulas (2.4) and (2.6) imply that if M is an odd dimensional compact manifold, then $\chi(M) = 0$.

2.5. **First order ordinary differential equations on manifolds.** An autonomous first order differential equation on a manifold $M \subseteq \mathbb{R}^k$ (or, more generally, on a subset of $\mathbb{R}^k$) is determined by a tangent vector field $g\colon M \to \mathbb{R}^k$ on M. The first order (autonomous) differential equation associated with g will be written in the form

$$\dot{x} = g(x), \quad x \in M. \tag{2.7}$$

However, the important fact about a differential equation is not the way this is written: what counts is the exact definition of what we mean by a solution (and this implicitly defines the notion of equation). By a solution of (2.7) we mean a C^1 curve $x\colon J \to \mathbb{R}^k$, defined on a (nontrivial) interval $J \subseteq \mathbb{R}$, which satisfies the conditions $x(t) \in M$ and $\dot{x}(t) = g\big(x(t)\big)$, identically on J. Thus, even if, according to Remark 2.2, the map g may be thought of as defined on an open set U containing M, a solution $x\colon J \to \mathbb{R}^k$ of

$$\dot{x} = g(x), \quad x \in U \tag{2.8}$$

is a solution of (2.7) if and only if its image lies in M. However, if M is closed in U, under the uniqueness assumption of the Cauchy problem for (2.8), one can check that any solution of (2.8) starting from a point of M must lie entirely in M.

If $\varphi\colon M \to N$ is a diffeomorphism between two manifolds and g is a tangent vector field on M, one gets a tangent vector field h on N by setting $h(z) = \varphi'\big(\varphi^{-1}(z)\big)g\big(\varphi^{-1}(z)\big)$. In this way, if $x \in M$ and $z \in N$ correspond under φ, the two vectors $h(z)$ and $g(x)$ correspond under the isomorphism $\varphi'(x)\colon T_xM \to T_zN$. For this reason we say that the two vector fields g and h correspond under φ (or they are φ-related). We observe that in this case, as an easy consequence of the chain rule for the derivative (and the definition of solution of a differential equation), equation (2.7) is equivalent to

$$\dot{z} = h(z), \quad x \in N, \tag{2.9}$$

in the sense that $x\colon J \to M$ is a solution of (2.7) if and only if the composition $z = \varphi \circ x$ is a solution of (2.9). That is, the solutions of (2.7) and (2.9) correspond under the diffeomorphism φ.

A non-autonomous first order differential equation on a manifold $M \subseteq \mathbb{R}^k$ is given by assigning, on an open subset V of $\mathbb{R} \times M$, a non-autonomous (continuous) vector field $f\colon V \to \mathbb{R}^k$ such that $f(t,x) \in T_xM$ for all $(t,x) \in V$. That is, for any $t \in \mathbb{R}$, the map $f_t\colon V_t \to \mathbb{R}^k$, given by $f_t(x) = f(t,x)$, is a tangent vector field on the (possibly empty) open subset $V_t = \{x \in M : (t,x) \in V\}$ of M. In other words, $f(t,x) \in T_xM$ for each $(t,x) \in V$.

The first order differential equation associated with f is denoted as follows:

$$\dot{x} = f(t,x), \quad (t,x) \in V. \tag{2.10}$$

A solution of (2.10) is a C^1 map $x\colon J \to M$, on an interval $J \subseteq \mathbb{R}$, such that, for all $t \in J$, $\big(t, x(t)\big) \in V$ and $\dot{x}(t) = f\big(t, x(t)\big)$.

We point out that (2.10) can be thought of as a special autonomous equation on the open submanifold V of $\mathbb{R} \times M \subseteq \mathbb{R}^{k+1}$. In fact (2.10) is clearly equivalent to the system

$$\begin{cases} \dot{t} = 1, \\ \dot{x} = f(t,x), \quad (t,x) \in V, \end{cases} \tag{2.11}$$

and the vector field $(t,x) \mapsto \big(1, f(t,x)\big)$ is tangent to V. By "equivalent" we mean that the solutions (2.10) and (2.11) are in a one-to-one correspondence.

As pointed out before, any differential equation on a manifold M is transformed into an equivalent one by a diffeomorphism $\varphi\colon M \to N$. Thus, since manifolds are locally diffeomorphic to open subsets of Euclidean spaces, the classical results about local existence and uniqueness for differential equations apply immediately to this more general context. Therefore, given $(t_0, x_0) \in V$, the continuity of the vector field $f\colon V \to \mathbb{R}^k$ is sufficient to ensure the existence, on an open interval J, of a solution $x\colon J \to M$ of (2.10) satisfying the Cauchy condition $x(t_0) = x_0$. If f is C^1 (or, more generally, locally Lipschitz), two solutions satisfying the same Cauchy condition coincide in their common domain. Moreover, by considering the partial ordering associated with graph inclusion, one gets that any solution of (2.10) can be extended to a maximal one (i.e., to a solution which is not the restriction of any different solution).

As in Euclidean spaces, one con prove that the domain of any maximal solution $x(\cdot)$ of (2.10) is an open interval (α, β), with $-\infty \le \alpha < \beta \le +\infty$. Moreover, given any $t_0 \in (\alpha, \beta)$ and any compact set K in the domain V of $f\colon V \to \mathbb{R}^k$, both the graphs of the restrictions of $x(\cdot)$ to $(\alpha, t_0]$ and to $[t_0, \beta)$ are not contained in K. This is referred as the Kamke property of the maximal solution (in a manifold). In particular, if M is a compact manifold and $V = \mathbb{R} \times M$, any maximal solution of (2.10) is defined on the whole real axis.

As in Euclidean spaces (see e.g. [Cop]), one has the following result regarding the continuous dependence on data.

Theorem 2.25. *Let $M \subseteq \mathbb{R}^k$ be a manifold and $\{f_n\}$ a sequence of C^1 non-autonomous tangent vector fields on M defined on an open subset V of $\mathbb{R} \times M$. Assume that f_n converges uniformly on compact sets to a C^1 tangent vector field f_0. Given $(\tau, p) \in V$, denote (when defined) by $x_n(t, \tau, p)$ the value at t of the maximal solution of*

$$\begin{cases} \dot{x} = f_n(t,x), \\ x(\tau) = p. \end{cases}$$

Let $\{(\tau_n, p_n)\}$ be a sequence in V converging to $(\tau_0, p_0) \in V$ and $[a,b]$ a compact interval contained in the domain of $x_0(\cdot, \tau_0, p_0)$. Then, for n sufficiently large, $x_n(\cdot, \tau_n, p_n)$ is defined on $[a,b]$ and

$$x_n(t, \tau_n, p_n) \to x_0(t, \tau_0, p_0)$$

uniformly on $[a,b]$. In particular, the set of all (t, τ, p) such that $x(t, \tau, p)$ is well defined is an open subset of $\mathbb{R} \times V$ (obviously containing any (τ, τ, p) with $(\tau, p) \in V$).

For autonomous tangent vector fields, the following consequence of the well-known Kupka-Smale theorem (see e.g. [Pei]) will be crucial in the sequel.

Proposition 2.26. *Let $g\colon M \to \mathbb{R}^k$ be a tangent vector field. Then, there exists a sequence $\{g_n\}$ of C^1 tangent vector fields on M, uniformly converging to g on compact sets and such that, for any $n \in \mathbb{N}$, any $T > 0$ and any compact set $C \subseteq M$, the equation $\dot{x} = g_n(x)$ admits finitely many periodic orbits contained in C and with period in $(0, T]$.*

3. The fixed point index of the Poincaré translation operator

3.1. **The autonomous case.** Let $g\colon M \to \mathbb{R}^k$ be a given C^1 vector field tangent to $M \subseteq \mathbb{R}^k$. For $p \in M$ and $t \in \mathbb{R}$, let $\Phi_t(p)$ be the value at t (if defined) of the maximal solution of (2.7) starting from p at time $t = 0$. We shall also use the (more cumbersome) notation $\Phi_t^g(p)$ whenever it will be necessary to emphasize the dependence on g. The map $p \mapsto \Phi_t(p)$, when (and where) defined, is called *flow operator at time t (associated with g)*. Obviously, if M is compact, $\Phi_t(p)$ is defined for all $(t,p) \in \mathbb{R} \times M$. Moreover, given a relatively compact subset U of M, $\Phi_t(p)$ is defined for any $p \in \overline{U}$ and $|t|$ small enough. In fact, Theorem 2.25 implies that the domain of the map $(t,p) \mapsto \Phi_t(p)$ is an open subset of $\mathbb{R} \times M$ containing the section $\{0\} \times M$.

Remark 3.1. Let ψ be a diffeomorphism from a manifold $M \subseteq \mathbb{R}^k$ onto a manifold $N \subseteq \mathbb{R}^s$, let $g\colon M \to \mathbb{R}^k$ be a C^1 tangent vector field on M, and let Φ_t be its associated flow operator at time t. Then, the composition $\psi \circ \Phi_t \circ \psi^{-1}$ coincides with the flow operator at time t associated with the tangent vector field $\hat{g}\colon N \to \mathbb{R}^s$ that corresponds to g under ψ.

Lemma 3.2. *Let $g\colon M \to \mathbb{R}^k$ be a C^1 tangent vector field. For any compact subset K of M such that $g(p) \neq 0$ for $p \in K$, there exists a positive constant $\tau = \tau(K)$ such that*

$$\Phi_t(p) \neq p, \quad \textit{for } 0 < |t| \leq \tau \textit{ and all } p \in K.$$

Proof. Assume, by contradiction, that there exist two sequences $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \neq 0$ and $t_n \to 0$, and $\{p_n\}_{n\in\mathbb{N}} \subseteq K$ such that $p_n = \Phi_{t_n}(p_n)$.

Without loss of generality we may assume $p_n \to p_0 \in K$. Denote by $x_n(\cdot)$ the solution of the initial value problem

$$\begin{cases} \dot{x}(t) = t_n g\big(x(t)\big), & t \in [0,1] \\ x(0) = p_n. \end{cases}$$

Clearly $x_n(1) = \Phi_{t_n}(p_n)$. As $M \subseteq \mathbb{R}^k$, the integrals $\int_0^1 g\big(x_n(t)\big)dt$ make sense. Hence,

$$0 = x_n(1) - x_n(0) = t_n \int_0^1 g\big(x_n(t)\big)dt,$$

so, as $t_n \neq 0$,

$$\int_0^1 g\big(x_n(t)\big)dt = 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

Using Theorem 2.25 and the fact that any $x_n(\cdot)$ is defined on the whole interval $[0,1]$ one can easily prove that $x_n(\cdot)$ converges uniformly to the (constant) solution $x_0(t) \equiv p_0$.

Taking the limit in (3.1), we get $0 = \int_0^1 g\big(x_0(t)\big)dt = g(p_0)$. This is a contradiction. □

In the "flat case", i.e. when M is an open subset of $\mathbb{R}^k$, there exists a simple relation between the degree of a vector field and the fixed point index of the associated flow operator. Namely, the following result holds.

Proposition 3.3. *Let U be a relatively compact open subset of $\mathbb{R}^k$ and let g be a C^1 vector field in $\mathbb{R}^k$ defined at least on $\overline{U}$ and such that $g^{-1}(0) \cap \partial U = \emptyset$. Then,*

$$\deg(g, U) = \lim_{t \to 0^-} \deg(I - \Phi_t^g, U),$$

where I denotes the identity in $\mathbb{R}^k$ and Φ_t^g is the flow operator associated with g.

Proof. From the proof of Lemma 6.1 §2 in [Kra] (or from Corollary 3.4 in [FuPe4]) it follows that

$$\deg(-g, U) = \deg(I - \Phi_t^g, U)$$

when $t > 0$ is sufficiently small. We see that if $\Phi_t^{-g}(p)$ is defined, so is $\Phi_{-t}^g(p)$ and $\Phi_t^{-g}(p) = \Phi_{-t}^g(p)$. Thus the assertion follows. □

A similar result holds on manifolds.

Theorem 3.4. *Let $g\colon M \to \mathbb{R}^k$ be a C^1 vector field tangent to a manifold $M \subseteq \mathbb{R}^k$. Let U be a relatively compact, open subset of M, and assume $g^{-1}(0) \cap \partial U = \emptyset$. Then*

$$\deg(g, U) = \lim_{t \to 0^-} \operatorname{ind}(\Phi_t^g, U),$$

where Φ_t^g denotes the flow operator associated with g.

Proof. By Lemma 3.2, (Φ_t^g, U) is admissible for $|t| > 0$ sufficiently small. By standard approximation results on manifolds, one can find a smooth approximation γ of g with the following properties:

1. γ has only nondegenerate zeros;
2. for $|t| > 0$ small enough, the flow operator Φ_t^γ associated with γ is admissibly homotopic to Φ_t^g in U.

By the homotopy invariance property of both the degree and the fixed point index, it is not restrictive to assume that properties 1 and 2 hold true for g.

Since the zeros of g are nondegenerate, $g^{-1}(0)$ is a finite set, say $\{p_1, \ldots, p_n\}$. Let $V_1, \ldots, V_n$ be pairwise disjoint open neighborhoods in U of $x_1, \ldots, x_n$, respectively.

By Lemma 3.2 one can take $\tau > 0$ so small that Φ_t^g has no fixed points in the compact set $\overline{U} \setminus \bigcup_{i=1}^n V_i$ for $0 < |t| \le \tau$. Therefore, by the additivity property (both of the degree and of the index), it is enough to prove the assertion for the particular case when $g^{-1}(0) \cap U$ consists of just one nondegenerate zero p and $\overline{U}$ is contained in an open set W diffeomorphic to $\mathbb{R}^m$.

Let ψ be the diffeomorphism of W onto $\mathbb{R}^m$, and let $\hat{g}$ be the vector field in $\mathbb{R}^m$ corresponding to g under ψ. By Remark 3.1 and Proposition 3.3, for $t \neq 0$ sufficiently small, one has

$$\deg\big(\hat{g}, \psi(U)\big) = \operatorname{ind}\big(\psi \circ \Phi_t^{\hat{g}} \circ \psi^{-1}, \psi(U)\big).$$

The assertion follows from the topological invariance property of the degree and from the commutativity property of the fixed point index. □

Remark 3.5. Let U be a relatively compact open subset of M and let $\Phi\colon \mathbb{R} \times M \to M$ be a dynamical system with no *rest points* (i.e., points $x \in M$ with the property that $\Phi(t, x) = x$ for all $t \in \mathbb{R}$) on ∂U. In this case the *rest point index*

$$\mathrm{I}(\Phi, U) = \lim_{t \to 0^+} \operatorname{ind}\big(\Phi(t, \cdot), U\big)$$

introduced in [Srz1] in the more general settings of ENRs is well defined (because of the homotopy invariance property of the fixed point index). Assume that (2.7) induces a dynamical system $\Phi^g\colon \mathbb{R} \times M \to M$. Thus, taking into account that $\Phi_t^{-g} = \Phi_{-t}^g$, from Theorem 3.4 and formula (2.4) one gets the following relation between the degree of a vector field and the rest point index of the associated dynamical system:

$$\mathrm{I}(\Phi^g, U) = (-1)^m \deg(g, U),$$

where m is the dimension of M.

Below, following [FuSp1], we shall extend to the manifold setting a formula proved in [Maw, CaMaZa] for the computation of the fixed point index of the flow operator associated with autonomous differential equations in Euclidean spaces (Theorem 3.8 below). A similar formula had been previously proved by Krasnosel'skii [Kra], still in the "flat context", in the (more general) non-autonomous case under an additional T-irreversibility assumption (see below).

Below, by an *orbit* we mean the image of a periodic solution of (2.7). Given $T > 0$, by A_T we denote the union of all τ-periodic orbits with $0 < \tau \leq T$. Note that $g^{-1}(0) \subseteq A_T$ for all $T > 0$.

Lemma 3.6. *Given $T > 0$, let $O \subseteq M$ be a nontrivial isolated orbit of (2.7) in A_T. Then, there exists an open neighborhood W of O such that, for all $0 < \tau \leq T$, Φ_τ is defined on $\overline{W}$, admissible on W, and $\mathrm{ind}(\Phi_\tau, W) = 0$.*

Proof. Since O is a periodic orbit, Φ_t is defined on O for all $t \in \mathbb{R}$. Thus, O being compact, Φ_T is defined on some relatively compact, open neighborhood W of O. Observe that, since O is isolated in A_T, one can choose W such that Φ_τ is fixed point free on ∂W for all $\tau \in (0, T]$. This implies, by the homotopy invariance property, that $\mathrm{ind}(\Phi_\tau, W)$ is independent of $\tau \in (0, T]$. Moreover, by the nontriviality of O, there exists a positive minimal period σ of O. Thus $\mathrm{ind}(\Phi_{\sigma/2}, W) = 0$ since $\Phi_{\sigma/2}$ is fixed point free on $\overline{W}$. □

Lemma 3.7. *Assume that Φ_T is defined on a relatively compact open subset U of M. Suppose there exist only finitely many orbits with period in $(0, T]$ which meet $\overline{U}$. Then, given $\tau, \sigma \in (0, T]$ such that Φ_τ and Φ_σ are fixed point free on ∂U, we have*

$$\mathrm{ind}(\Phi_\tau, U) = \mathrm{ind}(\Phi_\sigma, U).$$

Proof. If all the orbits with period in $(0, T]$ that meet $\overline{U}$ are trivial, then Φ_τ and Φ_σ are admissibly homotopic on U. Otherwise, let $O_1, \ldots, O_n$ be all the nontrivial ones. Lemma 3.6 implies the existence of n open subsets of M, $W_1, \ldots, W_n$, such that $O_i \subseteq W_i$ and

$$\mathrm{ind}(\Phi_\tau, W_i) = \mathrm{ind}(\Phi_\sigma, W_i) = 0,$$

for all $i = 1 \ldots n$. We can assume $\overline{W}_i \cap \overline{W}_j = \emptyset$ when $i \neq j$. Define

$$U_1 = U \setminus \bigcup_{i=1}^{n} \overline{W}_i \ .$$

By the additivity and the excision properties,

$$\mathrm{ind}(\Phi_\tau, U) = \mathrm{ind}(\Phi_\tau, U_1),$$

and

$$\mathrm{ind}(\Phi_\sigma, U) = \mathrm{ind}(\Phi_\sigma, U_1).$$

Using the homotopy invariance, we can write

$$\operatorname{ind}(\Phi_\tau, U_1) = \operatorname{ind}(\Phi_\sigma, U_1),$$

and the claim follows. □

Theorem 3.8. *Let $g\colon M \to \mathbb{R}^k$ be a C^1 tangent vector field on a manifold $M \subseteq \mathbb{R}^k$ and let U be a relatively compact open subset of M. Let $T > 0$ and assume that, for any $p \in \overline{U}$, the (maximal) solution of the Cauchy problem*

$$\begin{cases} \dot{x} = g(x) \\ x(0) = p \end{cases}$$

is defined on $[0, T]$. If Φ_T^g is fixed point free on ∂U, then

$$\operatorname{ind}(\Phi_T^g, U) = \deg(-g, U).$$

Proof. Consider a sequence $\{g_n\}$ of C^1 tangent vector fields as in Proposition 2.26. As usual, for $n \in \mathbb{N}$, we will denote by $\{\Phi_t^{g_n}\}_{t\in\mathbb{R}}$ the local flow associated with the equation $\dot{x} = g_n(x)$. Since the flow is a continuous map of the twofold variable $(t, x) \in \mathbb{R} \times M$, the "attainable set" $\widehat{U}_T = \Phi_{[0,T]}(\overline{U})$ is a compact subset of M. Let B be a relatively compact open set containing $\widehat{U}_T$. Let c be the distance (in $\mathbb{R}^k$) between $\widehat{U}_T$ and ∂B. One can choose a sufficiently large $\bar{n}$ such that $\|\Phi_t(x) - \Phi_t^{g_n}(x)\| \leq c/2$ for all $x \in \overline{U}$, $t \in [0, T]$ and $n > \bar{n}$. This implies that, if $n > \bar{n}$, any solution of $\dot{x} = g_n(x)$ which meets $\overline{U}$ is contained in $\overline{B}$. By the choice of the sequence $\{g_n\}$, $\overline{B}$ contains only finitely many periodic orbits of $\dot{x} = g_n(x)$ with period in $(0, T]$.

Since $\overline{U}$ is compact and $g^{-1}(0) \cap \partial U = \emptyset$, by Theorem 3.4 there exists $\varepsilon > 0$ such that

$$\operatorname{ind}(\Phi_{-\varepsilon}^g, U) = \deg(g, U).$$

Since $\Phi_\varepsilon^{-g} = \Phi_{-\varepsilon}^g$, one has

$$\operatorname{ind}(\Phi_\varepsilon^g, U) = \deg(-g, U). \tag{3.2}$$

Using the continuous dependence on data and the compactness of ∂U we can assume $\Phi_T^{g_n}(x) \neq x$ and $\Phi_\varepsilon^{g_n}(x) \neq x$ for all $x \in \partial U$. Moreover, by the homotopy invariance property of the index, we get

$$\operatorname{ind}(\Phi_T^{g_n}, U) = \operatorname{ind}(\Phi_T^g, U), \tag{3.3}$$

$$\operatorname{ind}(\Phi_\varepsilon^{g_n}, U) = \operatorname{ind}(\Phi_\varepsilon^g, U), \tag{3.4}$$

provided that n is large enough. Applying Lemma 3.7, we have

$$\operatorname{ind}(\Phi_T^{g_n}, U) = \operatorname{ind}(\Phi_\varepsilon^{g_n}, U).$$

Thus, by (3.2), (3.3) and (3.4), we obtain

$$\operatorname{ind}(\Phi_T^g, U) = \operatorname{ind}(\Phi_\varepsilon^g, U) = \deg(-g, U),$$

and this completes the proof. □

In spite of the fact that the restriction to ∂U of the flow operator Φ_T may be strongly influenced by the behavior of g outside ∂U, this is not so for its fixed point index. In fact, in some sense, the fixed point index of the flow depends only on how points are shot along the boundary of U. More precisely, we have the following result.

Corollary 3.9. *Let M, g, U and T be as in the Theorem 3.8. Let $h\colon M \to \mathbb{R}^k$ be a C^1 tangent vector field and Φ_t^h its associated flow operator. If $g|_{\partial U} = h|_{\partial U}$, then*

$$\operatorname{ind}(\Phi_T^g, U) = \operatorname{ind}(\Phi_T^h, U),$$

provided that they are both fixed point free on the boundary of U.

Proof. The assertion follows immediately from Theorem 3.8 and the boundary dependence property of the degree. □

Note also that Theorem 3.8 is not a trivial consequence of the homotopy property because, in general, the map $(p,t) \mapsto \Phi_t^g(p)$ is not an admissible homotopy on U. For example, consider in $M = \mathbb{R}^2$ the differential equation

$$(\dot{x}, \dot{y}) = (y, -x)$$

and let U be the unit open disk in $\mathbb{R}^2$. A direct computation shows that $\operatorname{ind}(\Phi_t, U)$ is well defined and equal to 1 for any $t \neq 2k\pi$, and it is not defined for $t = 2k\pi$ ($k \in \mathbb{Z}$). Therefore if t is considered in an interval containing one of these values, the flow does not give an admissible homotopy.

Let g, U and T satisfy the assumptions of Theorem 3.8. Consider the following differential equation depending on a parameter $\lambda \geq 0$:

$$\dot{x} = \lambda g(x). \tag{3.5}$$

Clearly, if $\lambda > 0$, any T-periodic solution of $\dot{x} = g(x)$ corresponds to a (T/λ)-periodic one of (3.5). Denote by $\Phi_t^{\lambda g}$ the flow operator associated with this equation. Observe that, for $\lambda \in [0,1]$ and $p \in \overline{U}$, $\Phi_T^{\lambda g}(p)$ and $\Phi_{\lambda T}^g(p)$ are both defined on $\overline{U}$ and

$$\Phi_T^{\lambda g}(p) = \Phi_{\lambda T}^g(p). \tag{3.6}$$

Given $\lambda_1, \lambda_2 \in (0,1]$, assume that $\Phi_T^{\lambda_1 g}$ and $\Phi_T^{\lambda_2 g}$ are fixed point free on ∂U. Then, by Theorem 3.8, we have

$$\operatorname{ind}(\Phi_{\lambda_1 T}^g, U) = \operatorname{ind}(\Phi_{\lambda_2 T}^g, U).$$

Therefore, by (3.6), we get

$$\operatorname{ind}(\Phi_T^{\lambda_1 g}, U) = \operatorname{ind}(\Phi_T^{\lambda_2 g}, U), \tag{3.7}$$

and this equality holds true no matter whether or not the homotopy

$$H : \overline{U} \times [\lambda_1, \lambda_2] \to M$$

given by $H(x,s) = \Phi_T^{sg}(x)$ is admissible.

3.2. **The periodic case.** Let $f\colon \mathbb{R} \times M \to \mathbb{R}^k$ be a T-periodic C^1 tangent vector field on M. For $p \in M$, let $P_T^f(p)$ be the value at T (if defined) of the maximal solution of the Cauchy problem

$$\dot{x} = f(t,x), \tag{3.8a}$$

$$x(0) = p. \tag{3.8b}$$

By Theorem 2.25, the domain of the map $p \mapsto P_T^f(p)$ is an open subset of M. This map is called the *(Poincaré) T-translation operator* associated with (3.8a). Observe that for an autonomous equation $\dot{x} = g(x)$ one has $\Phi_T^g = P_T^g$.

Let U be a relatively compact open subset of M. Assume that P_T^f is defined on $\overline{U}$. One could ask if a formula like (3.7) is still valid replacing the flow with the Poincaré operator.

Consider the differential equation

$$\dot{x} = \lambda f(t,x), \quad \lambda \geq 0. \tag{3.9}$$

Given $\lambda_1, \lambda_2 \in (0,1]$, assume that $P_T^{\lambda_1 f}$ and $P_T^{\lambda_2 f}$ are fixed point free on ∂U. The question is whether the following equality holds:

$$\operatorname{ind}\big(P_T^{\lambda_1 f}, U\big) = \operatorname{ind}\big(P_T^{\lambda_2 f}, U\big). \tag{3.10}$$

The answer is affirmative in the case when $U = M$ is a compact manifold (this is an easy consequence of the homotopy invariance property of the fixed point index), but it is false in general. To see this, let U be the open unit disk in $M = \mathbb{R}^2$, and consider the following system:

$$\begin{cases} \dot{x}_1 = \lambda x_2 \\ \dot{x}_2 = -\lambda x_1 + \lambda \sin t \end{cases} \tag{3.11}$$

This is a differential equation of the form (3.9) with $x = (x_1, x_2)$ and

$$f(t,x) = (x_2, -x_1 + \sin t).$$

For $\lambda = 1$, (3.11) does not admit 2π-periodic solution. Thus,

$$\operatorname{ind}(P_{2\pi}^f, U) = 0.$$

On the other hand, for λ sufficiently small, from Theorem 3.11 below (see also [FuPe3]) it follows that

$$\operatorname{ind}(P_{2\pi}^{\lambda f}, U) = \deg(-w, U) = 1,$$

where $w \colon \overline{U} \to \mathbb{R}^2$ is defined by

$$w(x_1, x_2) = \frac{1}{2\pi} \int_0^{2\pi} (x_2, -x_1 + \sin t)dt = (x_2, -x_1),$$

contradicting (3.10).

Let f and T be as above. Following Krasnosel'skii (see [Kra]), a point $p \in M$ is said to be of T-*irreversibility* for equation (3.8a) if the (maximal) solution $x(\cdot, p)$ of (3.8) is defined on $[0,T]$ and $x(t,p) \neq p$ for any $t \in (0,T]$. Using the homotopy property of the degree, Krasnosel'skii proves a formula for computing the fixed point index of the operator of translation along trajectories of a non-autonomous differential equation. His result (reformulated in the framework of differentiable manifolds) is the following.

Theorem 3.10. *Let $f \colon \mathbb{R} \times M \to \mathbb{R}^k$ be a C^1 tangent vector field that is T-periodic in the first variable. Let U be a relatively compact open subset of M. Assume that P_T^f is defined on $\overline{U}$. Suppose that all points of ∂U are of T-irreversibility for* (3.8a) *and that $f(0,x) \neq 0$ on ∂U. Then*

$$\operatorname{ind}(P_T^f, U) = \deg\big(-f(0,\cdot), U\big).$$

Theorem 3.8 shows that, at least in the case of autonomous differential equations, the assumption of T-irreversibility can be removed: the essential fact is the absence of fixed points for P_T^f on ∂U (i.e. the admissibility on U of the Poincaré T-translation operator). Now, the question is if one can eliminate the T-irreversibility hypothesis also for the non-autonomous case. Equation (3.11), with $\lambda = 1$, shows that this is not possible. In fact, let U be the open unit disk in $\mathbb{R}^2$. A direct computation gives $\deg\big(-f(0,\cdot), U\big) = 1$ and $\mathrm{ind}\big(P^f_{2\pi}, U\big) = 0$ (since (3.11) has no 2π-periodic orbits for $\lambda = 1$).

Despite this limitation, it is possible to give a formula for the fixed point index of the T-translation operator associated with the equation

$$\dot{x} = \lambda f(t,x), \quad \lambda \geq 0, \tag{3.12}$$

where $f\colon \mathbb{R}\times M \to \mathbb{R}^k$ is a C^1 tangent vector field, T-periodic in the first variable.

Define the *average wind* vector field

$$w_f(p) = \frac{1}{T}\int_0^T f(t,p)dt,$$

which is clearly tangent to M.

The following theorem (compare the proof of Theorem 2.1 in [FuPe3]) provides a formula for the computation of the fixed point index of the T-translation operator $P_T^{\lambda f}$ associated with equation (3.12) for λ sufficiently small.

Theorem 3.11. *Let $f\colon \mathbb{R}\times M \to \mathbb{R}^k$ be C^1 tangent vector field that is T-periodic in the first variable. Consider a relatively compact open subset U of the manifold $M \subseteq \mathbb{R}^k$. Assume that (w_f, U) is admissible for the degree. Then, there exists $\lambda_0 > 0$ such that, for $0 < \lambda \leq \lambda_0$, $P_T^{\lambda f}$ is defined on $\overline{U}$, fixed point free on ∂U and*

$$\mathrm{ind}(P_T^{\lambda f}, U) = \deg(-w_f, U).$$

Proof. Consider the equation

$$\dot{x} = \lambda\big(\mu f(t,x) + (1-\mu)w_f(x)\big), \quad \lambda \geq 0,\ \mu\in[0,1]. \tag{3.13}$$

Denote by H_T the translation operator that associates to any (λ, p, μ) the value at time T (if defined) of the solution of (3.13) starting from p at time 0. One can show that for $\lambda \geq 0$ small enough $H_T(\lambda,p,\mu)$ is defined for $p\in\overline{U}$ and $\mu\in[0,1]$.

We claim that there exists $\lambda_0 > 0$ such that $H_T(\lambda,p,\mu) \neq p$ for $0 < \lambda \leq \lambda_0$, $p \in \partial U$ and $\mu\in[0,1]$. Assume this is not the case. Thus there exist sequences $\lambda_n \to 0$, $\mu_n \in [0,1]$ and $p_n \in \partial U$ such that $\lambda_n > 0$ and

$$\begin{aligned} 0 &= H_T(\lambda_n, p_n, \mu_n) - p_n \\ &= \lambda_n \int_0^T \Big(\mu_n f\big(t, x_n(t)\big) + (1-\mu_n) w_f\big(x_n(t)\big)\Big)dt, \end{aligned}$$

where x_n denotes the solution of

$$\begin{cases} \dot{x} = \lambda_n\big(\mu_n f(t,x) + (1-\mu_n)w_f(x)\big), \\ x(0) = p_n. \end{cases}$$

Since $\lambda_n > 0$, one has

$$0 = \int_0^T \Big(\mu_n f\big(t, x_n(t)\big) + (1-\mu_n) w_f\big(x_n(t)\big)\Big)dt. \tag{3.14}$$

Without loss of generality we can assume $\mu_n \to \mu_0 \in [0,1]$ and $p_n \to p_0 \in \partial U$. Thus, by Theorem 2.25, x_n converges uniformly on $[0,T]$ to a (necessarily constant) solution $x_0(t) \equiv p_0$. Hence, passing to the limit in (3.14), we get

$$0 = \int_0^T \big(\mu_0 f(t,p_0) + (1-\mu_0)w_f(p_0)\big)dt,$$

that implies $w_f(p_0) = 0$. This contradicts the assumption.

Thus, there exists $\lambda_0 > 0$ such that, when $0 < \lambda \leq \lambda_0$, the map

$$H_T(\lambda, \cdot, \cdot)\colon \overline{U} \times [0,1] \to M$$

given by $(p, \mu) \mapsto H_T(\lambda, p, \mu)$ is an admissible homotopy. The homotopy invariance property of the fixed point index shows that for such λ's

$$\operatorname{ind}(\Phi_T^{\lambda w_f}, U) = \operatorname{ind}(P_T^{\lambda f}, U). \tag{3.15}$$

By Theorem 3.8 and formula (2.5), one has

$$\operatorname{ind}(\Phi_T^{\lambda w_f}, U) = \deg(-\lambda w_f, U) = \deg(-w_f, U). \tag{3.16}$$

The assertion follows from equations (3.15) and (3.16). □

The following result is a well known consequence of the homotopy invariance property of the Lefschetz number (see e.g. [DuGr]) since, when the manifold M is compact, P_T^f is admissibly homotopic to the identity. We give here a different proof based on Theorem 3.11.

Corollary 3.12. *Let $M \subseteq \mathbb{R}^k$ be a compact manifold. Consider a C^1 tangent vector field $f\colon \mathbb{R} \times M \to \mathbb{R}^k$, T-periodic in the first variable. Then $\operatorname{ind}(P_T^f, M)$ is well defined and*

$$\operatorname{ind}(P_T^f, M) = \chi(M).$$

Proof. By Theorem 3.11, taking $U = M$ one can find $\lambda > 0$ such that $\operatorname{ind}(P_T^{\lambda f}, M) = \deg(-w_f, M)$. Moreover, the Poincaré-Hopf theorem implies

$$\deg(-w_f, M) = \chi(M).$$

Since M is compact, $P_T^{\lambda f}$ is admissibly homotopic to P_T^f. The assertion follows from the homotopy invariance property of the fixed point index. □

When the manifold M is not compact, Theorem 3.11 allows the computation of the fixed point index of P_T^f only for small values of λ. There is a case, however, when this limitation is not necessary: that is, when $f(t,x) = a(t)h(x)$.

Namely, consider the equation

$$\dot{x} = \lambda a(t)h(x), \quad \lambda \geq 0, \tag{3.17}$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is a T-periodic continuous function and $h\colon M \to \mathbb{R}^k$ is a C^1 tangent vector field. Assume that the average

$$\bar{a} = \frac{1}{T}\int_0^T a(t)dt \neq 0.$$

As in the autonomous case, the fixed point index of the translation operator $P_T^{\lambda a h}$, when defined, does not depend on λ. In fact, the following result holds (compare [Spa3]).

Theorem 3.13. *Let $h\colon M \to \mathbb{R}^k$ be a C^1 tangent vector field, and let $a\colon \mathbb{R} \to \mathbb{R}$ be continuous, T-periodic, with*

$$\bar{a} = \frac{1}{T}\int_0^T a(t)dt \neq 0.$$

Given a relatively compact open subset U of M, assume that $P_T^{\lambda a h}$ is defined on $\overline{U}$ and fixed point free on ∂U. Then

$$\operatorname{ind}(P_T^{\lambda a h}, U) = (\operatorname{sign} \bar{a})^m \deg(-h, U),$$

where m is the dimension of M.

Proof. Without loss of generality, we may assume $\lambda = 1$ and $\bar{a} = 1$. Take any $p \in M$ and consider the Cauchy problems

$$\dot{x} = h(x), \quad x(0) = p; \tag{3.18a}$$

$$\dot{x} = a(t)h(x), \quad x(0) = p\,. \tag{3.18b}$$

Denote by $x\colon I \to M$ and $\xi\colon J \to M$ the (unique) maximal solutions of (3.18a) and of (3.18b), respectively. Clearly, if $\int_0^\tau a(s)ds \in I$ for all $\tau \in [0,t]$, then

$$\xi(t) = x\Big(\int_0^t a(s)ds\Big).$$

Hence $t \in J$. Moreover, by a standard maximality argument, one can prove that $t \in J$ implies $\int_0^t a(s)ds \in I$. In particular, if $T \in J$, then $\int_0^T a(s)ds = T \in I$. When this happens, one has $\xi(T) = x(T)$. In other words, if $P_T^{ah}(p)$ is defined, then so is $P_T^h(p)$, and $P_T^h(p) = P_T^{ah}(p)$. Theorem 3.8 implies

$$\operatorname{ind}(P_T^{ah}(p), U) = \operatorname{ind}(P_T^h(p), U) = \deg(-h, U).$$

This proves the assertion. □

Let us now consider the parametrized equation

$$\dot{x} = g(x) + \lambda f(t,x), \quad \lambda \geq 0, \tag{3.19}$$

where $f\colon \mathbb{R} \times M \to \mathbb{R}^k$ and $g\colon M \to \mathbb{R}^k$ are C^1 tangent vector fields on $M \subseteq \mathbb{R}^k$, and f is T-periodic in the first variable. As before $P_T^{g+\lambda f}$ denotes the Poincaré T-translation operator associated with (3.19). Note that for $\lambda = 0$ one has $P_T^g = \Phi_T^g$.

By a *starting point* of (3.19) we mean a pair $(\lambda, p) \in [0,\infty) \times M$ such that $P_T^{g+\lambda f}(p) = p$. Clearly, (λ, p) is a starting point of (3.19) if and only if the unique solution of $\dot{x} = g(x) + \lambda f(t,x)$ starting at p for $t = 0$ is T-periodic. Observe that $p \in M$ belongs to a T-periodic orbit of (3.19) for $\lambda = 0$ if and only if $(0,p)$ is a starting point. In particular, the set $\{0\} \times g^{-1}(0)$ is made up of starting points, and will be referred to as the set of *trivial starting points* (of (3.19)). Of course there may exist starting points $(0,p)$ which are nontrivial (this happens when p belongs to a nontrivial T-periodic orbit of (3.19) for $\lambda = 0$).

The following results holds.

Theorem 3.14. *Let $f\colon \mathbb{R} \times M \to \mathbb{R}^k$ and $g\colon M \to \mathbb{R}^k$ be C^1 tangent vector fields on a manifold $M \subseteq \mathbb{R}^k$, and let f be T-periodic in the first variable. Let U be a relatively compact open subset of M such that $P_T^{g+\lambda f}$ is defined on $\overline{U}$ for any $\lambda \in [0,1]$, and assume there are no starting points on $[0,1] \times \partial U$ of (3.19). Then*

$$\operatorname{ind}(P_T^{g+f}, U) = \deg(-g, U).$$

Proof. Since there are no fixed points of $P_T^g = \Phi_T^g$ on ∂U, (Φ_T^g, U) is admissible for the fixed point index. Thus, by Theorem 3.8,

$$\operatorname{ind}(P_T^g, U) = \operatorname{ind}(\Phi_T^g, U) = \deg(-g, U). \tag{3.20}$$

By assumption, there are no fixed points of $P_T^{g+\lambda f}$ on ∂U for any $\lambda \in [0,1]$. Consequently, the map

$$(p,\lambda) \mapsto P_T^{g+\lambda f}(p)$$

is an admissible homotopy. The homotopy invariance property of the fixed point index yields

$$\operatorname{ind}(P_T^g, U) = \operatorname{ind}(P_T^{g+f}, U). \tag{3.21}$$

The assertion follows from (3.20) and (3.21). □

Theorem 3.15. *Let $f\colon \mathbb{R} \times M \to \mathbb{R}^k$ and $g\colon M \to \mathbb{R}^k$ be C^1 tangent vector fields on a manifold $M \subseteq \mathbb{R}^k$, and let f be T-periodic in the first variable. Let U be a relatively compact open subset of M such that Φ_T^g is defined on $\overline{U}$ and fixed point free on ∂U. Then, there exists $\lambda_0 > 0$ such that $\operatorname{ind}(P_T^{g+\lambda f}, U)$ is well defined for $0 \leq \lambda \leq \lambda_0$, and*

$$\operatorname{ind}(P_T^{g+\lambda f}, U) = \deg(-g, U).$$

Proof. Because of Theorem 2.25, the set

$$\big\{(\lambda, p) \in [0,\infty) \times M : P_T^{g+\lambda f}(p) \text{ is defined}\big\}$$

is open in $[0,\infty) \times M$. Consequently, since by assumption $P_T^g = \Phi_T^g$ is defined for any p in the compact set $\overline{U}$, so is the operator $P_T^{g+\lambda f}$ for small values of λ. As the set of starting points is closed and P_T^g is fixed point free on the compact set ∂U, there are no starting points on $[0,\lambda_0] \times \partial U$ for some $\lambda_0 > 0$. The assertion now follows from Theorem 3.14 replacing f by f/λ_0. □

Since in the above result we have assumed that (Φ_T^g, U) is fixed point free on ∂U, as a consequence one has $g(p) \neq 0$ for all $p \in \partial U$. Thus Theorem 3.15 cannot be seen as an extension of Theorem 3.11. As far as we know a result which includes both these cases is still unknown.

An argument as in the proof of Theorem 3.15 yields a similar result for the parametrized equation

$$\dot{x} = a(t)g(x) + \lambda f(t,x), \quad \lambda \geq 0, \tag{3.22}$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is a T-periodic continuous function, $g\colon M \to \mathbb{R}^k$ and $f\colon \mathbb{R} \times M \to \mathbb{R}^k$ are C^1 tangent vector fields, and f is T-periodic in the first variable. As before, a pair $(\lambda, p) \in [0,\infty) \times M$ is a *starting point for* (3.22) if p is a fixed point of $P_T^{ag+\lambda f}$.

Theorem 3.16. *Assume that $a\colon \mathbb{R} \to \mathbb{R}$ is a T-periodic continuous function with nonzero average $\bar{a}$, $g\colon M \to \mathbb{R}^k$ and $f\colon \mathbb{R} \times M \to \mathbb{R}^k$ are C^1 tangent vector fields on a manifold $M \subseteq \mathbb{R}^k$, and f is T-periodic in the first variable. Let U be a relatively compact open subset of M such that P_T^{ag} is defined on $\overline{U}$ and fixed point free on ∂U. Then, there exists $\lambda_0 > 0$ such that $\operatorname{ind}(P_T^{ag+\lambda f}, U)$ is well defined for $0 \leq \lambda \leq \lambda_0$, and*

$$\operatorname{ind}(P_T^{ag+\lambda f}, U) = (\operatorname{sign} \bar{a})^m \deg(-g, U),$$

where m is the dimension of the manifold M.

The C^1 assumption on the vector fields f and g made throughout this section can be clearly relaxed. The important requirement is the uniqueness of the solutions of the initial value problems. Actually, using the techniques described in [FuPe5, Spa2], the results presented in this section could be extended to the Carathéodory case. Namely, when g is locally Lipschitz and f is assumed to satisfy the following hypotheses:

(C1) for each $p \in M$, the map $t \mapsto f(t,p)$ is Lebesgue measurable on $\mathbb{R}$;
(C2) for a.a. $t \in \mathbb{R}$, the map $p \mapsto f(t,p)$ is continuous on M;
(C3) for any compact set $K \subseteq M$, there exists $\gamma_K \in L^1([0,T])$ such that $|f(t,p)| \leq \gamma_K(t)$ for a.a. $t \in [0,T]$ and all $p \in K$;
(C4) for any $p \in M$, $f(t+T,p) = f(t,p) \in T_pM$ a.e. in $\mathbb{R}$;
(C5) for any compact subset K of M, there exists $\alpha_K \in L^1([0,T])$ such that
$$|f(t,p_1) - f(t,p_2)| \leq \alpha_K(t)\,|p_2 - p_1|\,,$$
for a.a. $t \in \mathbb{R}$ and for any $p_1, p_2 \in K$.

Conditions (C1) – (C3) are the so-called Carathéodory type assumptions while (C4) says that f is a time-dependent T-periodic tangent vector field on M. The assumption (C5) ensures the uniqueness of the solutions of the initial value problems (compare, e.g., [CoLe]).

In this framework, a solution to (3.19) is an absolutely continuous function $x\colon J \to M \subseteq \mathbb{R}^k$ defined on a (nontrivial) interval and satisfying the condition
$$\dot{x}(t) = g\big(x(t)\big) + \lambda f\big(t, x(t)\big), \quad \text{for a.a. } t \in J.$$

4. Applications and examples

Below, we shall use the results of the previous section to investigate the structure of the set of T-periodic solutions of equations (3.12) and (3.19). The possible field of application ranges from continuation theorems for periodic solutions to multiplicity results. However, we shall confine ourselves to the simplest applications and only present those that seem most appropriate to shed light on the results discussed in the previous section.

We will need the following global connectivity result.

Lemma 4.1 ([FuPe6])**.** *Let X be a locally compact metric space and let $K \subseteq X$ be nonempty and compact. Assume that any compact subset of X containing K has nonempty boundary. Then $X \setminus K$ contains a connected set whose closure intersects K and is not compact.*

By Theorem 2.25, the set $\Omega \subseteq [0,\infty) \times M$ given by
$$\Omega = \left\{(\lambda,p) : \begin{array}{l}\text{the solution } x(\cdot) \text{ of (3.19) satisfying}\\ x(0) = p \text{ is defined on } [0,T]\end{array}\right\},$$
is open. Thus it is locally compact. Clearly Ω contains the set S of all starting points of (3.19). Observe that S is closed in Ω, though not necessarily closed in $[0,\infty) \times M$. Therefore it is locally compact.

In the sequel, given any subset A of $[0,\infty) \times M$ and $\lambda \geq 0$, the symbol A_λ denotes the slice $\{x \in M : (\lambda,x) \in A\}$ of A.

Theorem 4.2 ([FuSp1])**.** *Let $f\colon \mathbb{R} \times M \to \mathbb{R}^k$ and $g\colon M \to \mathbb{R}^k$ be two C^1 tangent vector fields on a manifold $M \subseteq \mathbb{R}^k$, and let f be T-periodic in the first variable. Denote by S the set of starting points of (3.19) and let V be an open subset of Ω. If $g^{-1}(0) \cap V_0$ is compact and $\deg(g, V_0)$ is nonzero, then the set $(S \cap V) \setminus \big(\{0\} \times g^{-1}(0)\big)$ of nontrivial starting points in V admits a connected subset whose closure in V meets $\{0\} \times g^{-1}(0)$ and is not compact.*

Proof. To prove the assertion it is enough to show that the topological pair

$$(X,K) = \big(S \cap V, \{0\} \times (g^{-1}(0) \cap V_0)\big)$$

satisfies the assumptions of Lemma 4.1. Since the set $S \cap V$ is open in the locally compact set S, it is locally compact as well. Moreover, as $\deg(g, V_0)$ is nonzero, the compact set $\{0\} \times (g^{-1}(0) \cap V_0)$ is nonempty. Assume, by contradiction, that there exists a compact subset C of $S \cap V$ containing $\{0\} \times \big(g^{-1}(0) \cap V_0\big)$ and with empty boundary in the space $S \cap V$. Thus C is open in $S \cap V$ (in fact it is clopen). As V is open in $[0,\infty) \times M$, C is actually open as a subspace of S. Thus there exists an open subset W of $[0,\infty) \times M$ such that $S \cap W = C$. Because of the compactness of the slice C_0 of C, we may choose W is such a way that the neighborhood W_0 of C_0 turns out to be relatively compact in M. Moreover, without loss of generality, we may assume that the boundary of W_0 in M does not contain points of S_0 (i.e. fixed points of $P_T^g = \Phi_T^g$). Thus, applying the excision property of the degree, Theorem 3.8 and formula (2.4), one gets

$$\begin{aligned} \operatorname{ind}(P_T^g, W_0) &= \deg(-g, W_0) = (-1)^m \deg(g, W_0) \\ &= (-1)^m \deg(g, V_0) \neq 0, \end{aligned}$$

where m is the dimension of M. As C is compact, there exists $\mu > 0$ such that the Poincaré operator $P_T^{g+\mu f}$ is fixed point free on the slice W_μ. Then, from the generalized homotopy property and the solution property of the index, we get

$$\operatorname{ind}\big(P_T^g, W_0\big) = \operatorname{ind}\big(P_T^{g+\mu f}, W_\mu\big) = 0,$$

and this is a contradiction. □

A similar (but more general) result can be proved for equation (3.22). The proof is analogous to that of Theorem 4.2 and, therefore, it will be omitted.

Theorem 4.3 ([Spa3])**.** *Let $a\colon \mathbb{R} \to \mathbb{R}$ be a continuous function, and let $f\colon \mathbb{R} \times M \to \mathbb{R}^k$ and $g\colon M \to \mathbb{R}^k$ be two C^1 tangent vector fields on $M \subseteq \mathbb{R}^k$. Assume also that f and a are T-periodic, and that the average of a is nonzero. Denote by S the set of starting points of* (3.22) *and let V be an open subset of $[0,\infty) \times M$ such that $P_T^{ag+\lambda f}(p)$ is defined for any $(\lambda, p) \in V$. Assume that $\deg(g, V_0)$ is well defined and nonzero. Then the set $(S \cap V) \setminus \big(\{0\} \times g^{-1}(0)\big)$ of nontrivial starting points (in V) of* (3.22) *admits a connected subset whose closure in V meets $\{0\} \times g^{-1}(0)$ and is not compact.*

The following result regarding equation (3.12) can be proved similarly to Theorem 4.2.

Theorem 4.4 ([FuPe3])**.** *Let $f\colon \mathbb{R} \times M \to \mathbb{R}^k$ be a C^1 tangent vector field which is T-periodic in the first variable. Denote by S the set of starting points of* (3.12) *and let $V \subseteq [0,\infty) \times M$ be open and such that $P_T^{\lambda f}(p)$ is defined for any $(\lambda, p) \in V$. Assume that $\deg(w_f, V_0) \neq 0$. Then the set $(S \cap V) \setminus \big(\{0\} \times M\big)$ of nontrivial starting points (in V) of* (3.12) *admits a connected subset whose closure in V meets $\{0\} \times w_f^{-1}(0)$ and is not compact.*

Below, we give some simple consequences of Theorems 4.2, 4.3 and 4.4 which illustrate their usefulness in describing the structure of the starting point sets.

Corollary 4.5. *Let M, a, g and f be as in Theorem 4.3. Assume that M is closed as a subset of $\mathbb{R}^k$ and*

$$\|f(t,x)\| \le \alpha + \beta\|x\|, \quad \|g(x)\| \le \alpha + \beta\|x\|$$

for some $\alpha, \beta \ge 0$ and all $(t,x) \in \mathbb{R} \times M$. If $g^{-1}(0)$ is compact and $\deg(g, M) \neq 0$, then there exists an unbounded connected set of starting points for (3.22) which meets $\{0\} \times g^{-1}(0)$.

Proof. Since M is closed in $\mathbb{R}^k$, the assumptions on f and g imply that any (maximal) solution of (3.22) is defined on the whole real line. Thus, taking $V = [0,\infty) \times M$, Theorem 4.3 implies the existence a connected set Σ of starting points for the equation (3.22) whose closure (in $[0,\infty) \times M$ or, equivalently, in $[0,\infty) \times \mathbb{R}^k$) is not compact and meets $\{0\} \times g^{-1}(0)$. This implies that Σ is unbounded. □

We point out that the unbounded set of starting points ensured by Corollary 4.5 may be "completely vertical"; that is, contained in the slice $\{0\} \times M$ of $[0,\infty) \times M$. Of course this may happen only if M is not compact. The following example with $M = \mathbb{R}^2$ and $T = 2\pi$ illustrates this phenomenon:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + \lambda \sin t \end{cases}$$

Corollary 4.6. *Assume that M is a compact manifold with $\chi(M) \neq 0$. Let a, f and g be as in Theorem 4.3. Then, there exists a connected set of starting points Σ for (3.22) which meets $\{0\} \times g^{-1}(0)$ and such that $\pi_1(\Sigma) = [0,\infty)$, where π_1 denotes the projection on the first factor of $[0,\infty) \times M$.*

Proof. By the compactness of M, any solution of (3.22) is globally defined. We apply Theorem 4.3 to the open set $V = [0,\infty) \times M$. By the Poincaré-Hopf Theorem, we have $\deg(g, V_0) = \chi(M) \neq 0$. Therefore there exists a connected set Σ of starting points of (3.22) which meets $\{0\} \times g^{-1}(0)$ and is not contained in any compact subset of V. This implies that Σ is unbounded and, M being compact, its projection on $[0,\infty)$ must be unbounded, connected and containing 0. □

An analogous argument proves (see e.g. [FuPe3]) the following consequence of Theorem 4.4.

Corollary 4.7. *Assume that M is a compact manifold with $\chi(M) \neq 0$. Let f be as in Theorem 4.4. Then there exists an unbounded connected set of starting points which meets $\{0\} \times w_f^{-1}(0)$. In particular the equation $\dot{x} = f(t,x)$ admits a T-periodic solution.*

The fact that the global branch ensured by Theorem 4.2 emanates from the set of zeros of g, and not merely from the set of all T-periodic orbits of $\dot{x} = g(x)$, allows us to obtain information about the starting point set of equation (3.22) also in the case of a compact manifold with zero Euler-Poincaré characteristic, as in the following multiplicity result. Here the *index* at an isolated zero z of a tangent vector field $g \colon M \to \mathbb{R}^k$ is defined as $\deg(g, U)$, where U is an isolating neighborhood of z. This makes sense because of the excision property of the degree of a vector field. In particular, when z is a nondegenerate zero, its index is either $+1$ or -1.

Corollary 4.8. *Let $M \subseteq \mathbb{R}^k$ be a compact manifold. Assume that f and g are as in Theorem 4.2 and, in addition, that g has exactly two distinct zeros z_1 and z_2 with nonzero index. Denote by S_1 and S_2 the connected components of the set of* *starting points of* (3.19) *which contain respectively* z_1 *and* z_2*. Then just one of the following two possibilities holds:*

(1) $S_1 = S_2$,
(2) S_1 *and* S_2 *are disjoint and both unbounded (in* $[0,\infty) \times M$*).*

In particular, if (2) holds, there exist at least two distinct T*-periodic solutions of* (3.22) *for each* $\lambda \in [0,\infty)$.

Proof. Since M is compact, any solution of (3.19) is globally defined. Take

$$V_1 = [0,\infty) \times M \setminus \{(0, z_2)\},$$
$$V_2 = [0,\infty) \times M \setminus \{(0, z_1)\}.$$

Obviously $(0, z_i) \in V_i$ and, by the excision property, $\deg(g, V_i) \neq 0$ for $i \in 1, 2$. We may assume $S_1 \neq S_2$. In this case S_1 and S_2, being connected components, are clearly disjoint and, consequently, $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$. Because of Theorem 4.2, S_1 and S_2 are not contained in any compact subset of V_1 and V_2 respectively and, in particular, they are not compact. Now S_1 and S_2 are closed in the set S of all starting points of (3.19). Since S is closed in $V = [0,\infty) \times M$, which is closed in $\mathbb{R}^{k+1}$, the two components S_1 and S_2 must be unbounded. □

A remarkable consequence of Theorem 4.4 is the following continuation result regarding T-periodic solutions (see [FuPe3] for a more general version). Observe that in Theorems 4.2, 4.3 and 4.4 we have assumed the global existence in $[0,T]$ of the solutions of the initial value problem. This hypothesis is not explicitly stated in Theorem 4.9 below, since it is ensured by suitable a priori bounds on the T-periodic orbits of the equation (see e.g. [CaMaZa, Kra, Maw]).

Theorem 4.9. *Let* $f\colon \mathbb{R} \times M \to \mathbb{R}^k$ *be a* C^1 *tangent vector field which is* T*-periodic in the first variable. Assume that:*

(1) $\deg(w_f, M)$ *is well defined and nonzero;*
(2) *the* T*-periodic orbits of* (3.12) *for* $\lambda \in (0,1]$ *lie in a compact subset of* M*.*

Then, the equation

$$\dot{x} = f(t,x) \tag{4.1}$$

has a T*-periodic solution.*

Proof. Let K be a compact subset of M containing all the zeros of w_f and all the T-periodic orbits of (3.12) for $\lambda \in (0,1]$, and let $U \subseteq M$ be a relatively compact open set containing K. Let $\sigma\colon M \to [0,1]$ be a C^1 function with compact support in M and such that $\sigma(p) = 1$ for each $p \in U$. Since the tangent vector field σf has compact support, for each $(\lambda, p) \in [0,\infty) \times M$ the solution of the initial value problem

$$\begin{cases} \dot{x} = \lambda\sigma(x) f(t,x) \\ x(0) = p \end{cases} \tag{4.2}$$

is globally defined on $\mathbb{R}$. We have $\sigma w_f = w_{\sigma f}$. Thus, from the excision property it follows that

$$\deg(w_{\sigma f}, U) = \deg(w_f, U) = \deg(w_f, M) \neq 0.$$

Hence, taking $V = [0,1) \times U$, Theorem 4.4 implies the existence of a connected set $\Sigma \subseteq V$ of starting points of (4.2) which is closed in V, meets $w_{\sigma f}^{-1}(0) \cap U = w_f^{-1}(0)$ and is not compact. Consider the following subset of Σ:

$$\tilde{\Sigma} = \left\{ (\lambda, p) \in \Sigma : \begin{array}{l} \text{the solution } x \text{ of (4.2) corresponding to} \\ (\lambda, p) \text{ is such that } x(t) \in U \text{ for all } t \end{array} \right\}.$$

As Σ meets $w_f^{-1}(0)$, $\tilde{\Sigma}$ is nonempty. Moreover, it is easy to check that $\tilde{\Sigma}$ is open and closed in Σ. Thus, $\tilde{\Sigma} = \Sigma$. Since $\sigma(p) \equiv 1$ in U, this implies that any $(\lambda, p) \in \Sigma$ is, in fact, a starting point of (3.12).

Observe that, Σ being noncompact and closed in $[0,1) \times U$, the closure of Σ in the compact set $[0,1] \times \overline{U}$ must intersect the boundary of $[0,1) \times U$ in $[0,\infty) \times M$, which is the union of $[0,1) \times \partial U$ and $\{1\} \times \overline{U}$.

By the continuous dependence on data of the solutions, if (λ, p) is in the closure of Σ, then it is a starting point. Therefore, by the choice of the compact set K, one has $p \in K$. As $K \subseteq U$, the closure of Σ does not intersect the set $[0,1) \times \partial U$. Thus, there exists a starting point of (3.12) of the form $(1, p)$, and this proves the assertion. □

In this section we have introduced only a small number of consequences of the formulas obtained in Section 3. Among the applications to T-periodic solutions of differential equations not presented here, we mention multiplicity results, guiding function-like existence results and other continuation results (see e.g. [FuPe2, Spa1, Spa3]). It should also be remarked that the results of this section, mainly Theorems 4.2, 4.3 and 4.4, can be reformulated somewhat more elegantly in an infinite-dimensional framework (see e.g. [FuPe1, FuSp2] and also [FuPe5, Spa2] for the Carathéodory case).

Since second order ordinary differential equations on a manifold M can be written as first order equations on its tangent bundle TM (see e.g. [Fur]), one can apply the results of Section 3 to this kind of equation (see e.g. [FuPe6, FuSp3], see also [FuPeSp1, FuPeSp2] and references therein for a discussion of the multiplicity results obtainable using this methods).

References

[Ama] Amann H., *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Review **18** n.4 (1976), 620–709.

[AmWa] Amann H., Weiss S., *On the uniqueness of the topological degree*, Math. Z. **130** (1973), 37-54.

[Bou] Bouligand G., *Introduction à la géométrie infinitesimale directe*, Gauthier-Villard, Paris, 1932.

[Bro] Brown R.F., *An elementary proof of the uniqueness of the fixed point index*, Pacific J. Math. **35** (1970), 549-558.

[CaMaZa] Capietto A., Mawhin J., Zanolin F., *Continuation theorems for periodic perturbations of autonomous systems*, Trans. Amer. Math. Soc. **329** (1992), 41-72.

[CoLe] Coddington E.A., Levinson N., *Theory of Ordinary Differential Equation*, McGraw-Hill Book Company Inc., New York 1955.

[Cop] Coppel W.A., *Stability and asymptotic behavior of differential equations*, Heath Math. Monograph, Boston, 1965.

[DuGr] Dugundji J., Granas A., *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[Fur] Furi M., *Second order differential equations on manifolds and forced oscillations*, Topological Methods in Differential Equations and Inclusions, A. Granas and M. Frigon Eds., Kluwer Acad. Publ. series C, vol. 472, 1995.

[FuPe1] Furi M., Pera M.P., *Global branches of harmonic solutions to periodic ODEs on manifolds*, Boll. Un. Mat. Ital. **11-A** (1997), 709–722.

[FuPe2] Furi M., Pera M.P., *Global branches of periodic solutions for forced differential equations on nonzero Euler characteristic manifolds*, Boll. Un. Mat. Ital. VI Ser., C, Anal. Funz. Appl. **3** (1984), 157-170.

[FuPe3] Furi M., Pera M.P., *A continuation principle for forced oscillations on differentiable manifolds*, Pacific J. Math. **121** (1986), 321–338.

[FuPe4] Furi M., Pera M.P., *Global bifurcation of fixed points and the Poincaré translation operator on manifolds*, Ann. Mat. Pura ed Appl. **173** (1997), 313–331.

[FuPe5] Furi M., Pera M.P., *Carathéodory periodic perturbations of the zero vector field on manifolds*, Topol. Meth. Nonlin. Analysis **10** (1997), 79–92.

[FuPe6] Furi M., Pera M.P., *A continuation principle for periodic solutions of forced motion equations on manifolds and application to bifurcation theory*, Pacific J. Math. **160** (1993), 219–244.

[FuPeSp1] Furi M., Pera M.P., Spadini M., *Forced oscillations on manifolds and multiplicity results for periodically perturbed autonomous systems*, J. Comp. App. Math, **113** (2000), 241–254.

[FuPeSp2] Furi M., Pera M.P., Spadini M., *Multiplicity of forced oscillations on manifolds and applications to motion problems with one-dimensional constraints*, Set-Valued Analysis **9** (2001), 67–73.

[FuSp1] Furi M., Spadini M., *On the fixed point index of the flow and applications to periodic solutions of differential equations on manifolds*, Boll. Un. Mat. Ital. (7) 10-A (1996), 333–346.

[FuSp2] Furi M., Spadini M., *On the set of harmonic solutions to periodically perturbed autonomous differential equations on manifolds*, Nonlin. Analysis TMA **29** (1997), 963–470.

[FuSp3] Furi M., Spadini M., *Branches of forced oscillations for periodically perturbed second order autonomous ODEs on manifolds*, J. Diff. Equations **154** (1999), 96–106.

[Gra] Granas A., *The Leray-Schauder index and the fixed point theory for arbitrary ANRs*, Bull. Soc. Math. France **100** (1972), 209–228.

[GuPo] Guillemin V., Pollack A., *Differential Topology*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1974.

[Hir] Hirsch M.W., *Differential Topology*, Graduate Texts in Math. **33**, Springer-Verlag, Berlin 1976.

[Kra] Krasnosel'skii M.A., *The Operator of Translation along the Trajectories of Differential Equations*, Transl. Math. Monographs, Vol. 19, Amer. Math. Soc., Providence, Rode Island (1968).

[Ler] Leray J., *Thèorie des points fixes: indice total et nombre de Lefschetz*, Bull. Soc. Mat. France **87** (1959), 221-233.

[Llo] Lloyd N. G., *Degree Theory*, Cambridge Univ. Press 73, 1978.

[Maw] Mawhin J., *Topological degree and boundary value problems for nonlinear differential equations*, C.I.M.E. course on Topological Methods for Ordinary Differential Equations, M. Furi and P. Zecca eds., Lecture notes in Math. 1537, Springer Verlag 1991, 74-142.

[Mil] Milnor J.W., *Topology from the Differentiable Viewpoint*, Univ. Press of Virginia 1965.

[Nir] Nirenberg L., *Topics in Nonlinear Functional Analysis*, Courant Institute of Mathematical Sciences, New York, 1974.

[Nus] Nussbaum R. D., *The fixed point index and fixed points theorems*, C.I.M.E. course on Topological Methods for Ordinary Differential Equations, M. Furi and P. Zecca eds., Lecture notes in Math. 1537, Springer Verlag 1991, 143-205.

[Pei] Peixoto M.M., *On an approximation theorem of Kupka and Smale*, J. of Diff. Equations **3** (1966), 214–227.

[Pug] Pugh C.C., *A generalized Poincaré index formula*, Topology **7** (1968), 217–226.

[Sev] Severi F., *Conferenze di geometria algebrica*, Zanichelli, Bologna, 1927.

[Spa1] Spadini M., *Perturbazioni periodiche di equazioni differenziali ordinarie su varietà differenziabili*, Doctoral dissertation, University of Florence, Florence, 1997.

[Spa2] Spadini M., *Harmonic solutions of periodic Carathéodory perturbations of autonomous ODEs on manifolds*, Nonlin. Analysis TMA, **41a** (2000), 477-487.

[Spa3] Spadini M., *Harmonic solutions to perturbations of periodic separated variables ODEs on manifolds*, Electronic J. Diff. Equations, 2003.

[Spn] Spanier E.H., *Algebraic Topology*, McGraw-Hill Series in Higher Math., McGraw-Hill, New York, 1966.

[Srz1] Srzednicki R., *On rest points of dynamical systems*, Fund. Math. **126** (1985), 69–81.

[Srz2] Srzednicki R., *Generalized Lefschetz theorem and a fixed point index formula*. Topology App. **81** (1997), 207–224.

[Srz3] Srzednicki R., *On periodic solutions inside isolating chains*. J. Diff. Equations **165** (2000), 42–60.

[Tro] Tromba A.J., *The Euler characteristic of vector fields on Banach manifolds and a globalization of Leray-Schauder degree*, Adv. in Math. **28** (1978), 148-173.

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