FORCED OSCILLATIONS ON MANIFOLDS AND MULTIPLICITY RESULTS FOR PERIODICALLY PERTURBED AUTONOMOUS SYSTEMS

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1. Introduction and preliminaries

In this paper we obtain some qualitative results for forced oscillations on differentiable manifolds which cannot be deduced via variational or implicit function methods. Namely, given a boundaryless submanifold M of $\mathbf{R}^k$, we study the forced oscillations of a mass point constrained to M and acted on by two forces: an autonomous one and a T-periodic small perturbation of it. For this kind of problems we establish multiplicity results.

In fact, one can prove directly that, when the perturbing T-periodic force is small enough, to any non-T-resonant zero of the autonomous unperturbed force (see section 2 for a definition) there corresponds a T-periodic solution of the perturbed motion equation whose image is close to a neighborhood of such a zero (see e.g. [11]). In this paper we prove that, under reasonable topological conditions, there exists at least another T-periodic solution whose image is not necessarily near a zero of the unperturbed force.

The methods used can be traced back to [5], [6] and, mainly, [7], where a multiplicity result for the spherical pendulum was proved. The purpose of this work is to give a coherent and clear treatment of the techniques used, as well as to enlighten the interplay between the analytical and the topological viewpoint. In particular, we frame the core of our work in an abstract topological setting.

As a matter of fact, we establish a "topological" multiplicity result (Th. 3.3 below) and we use it to deduce an abstract theorem (Th. 3.7 below) concerning the multiplicity of forced oscillations for periodic perturbations of second order ODEs on boundaryless manifolds. Finally, as applications we provide more concrete multiplicity results such as, for instance, Theorem 4.3 (in presence of a frictional force) or Theorem 4.6 (possibly without friction) which includes, as particular cases, the multiplicity results for the ordinary and the spherical gravitational pendula contained in [6] and [7] respectively.

As remarked above, for the T-periodically perturbed gravitational pendulum (both ordinary and spherical) it is relatively easy to prove that, when the perturbation is small enough, there exists an harmonic solution near the north pole. As a consequence of Theorem 4.6 we will show that there exists at least another forced oscillation even in the resonant case.

In what follows, the standard inner product in $\mathbf{R}^k$ will be denoted by $\langle u, v\rangle$, with $u, v \in \mathbf{R}^k$, and the corresponding Euclidean norm by $|v| = \langle v, v\rangle^{1/2}$.

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If N is a differentiable manifold embedded in some $\mathbf{R}^l$, we will denote by $C^n_T(N)$, $n \geq 0$, the metric subspace of the Banach space $C^n_T(\mathbf{R}^l)$ of all the T-periodic C^n maps $x : \mathbf{R} \to N$ with the usual C^n norm $\|\cdot\|_n$ (when $n = 0$ we will simply write $C_T(N)$). Observe that $C^n_T(N)$ is not complete unless N is complete (i.e. closed in $\mathbf{R}^l$). Nevertheless, since N is locally compact, $C^n_T(N)$ is always locally complete.

Let us recall some basic facts about second order differential equations on manifolds.

Let M be a differentiable manifold in $\mathbf{R}^k$. Given $q \in M$, $T_qM \subset \mathbf{R}^k$ denotes the tangent space to M at q. By

$$TM = \left\{(q, v) \in \mathbf{R}^k \times \mathbf{R}^k : q \in M \ , \ v \in T_qM\right\}$$

we mean the tangent bundle of M.

Given a continuous map $\varphi : \mathbf{R} \times TM \to \mathbf{R}^k$ such that $\varphi(t, q, v) \in T_qM$ for all $(t, q, v) \in \mathbf{R} \times TM$, we will say that φ is *tangent to* M, though it is not a tangent vector field on M. The motion equation associated with the force φ can be written in the form

$$\ddot{x}_\pi = \varphi(t, x, \dot{x}). \tag{1}$$

A solution of (1) is a C^2 map $x : J \to M$, defined on a nontrivial interval J, such that $\ddot{x}_\pi(t) = \varphi\,(t, x(t), \dot{x}(t))$ for all $t \in J$, where $\ddot{x}_\pi(t)$ denotes the orthogonal projection of $\ddot{x}(t) \in \mathbf{R}^k$ onto $T_{x(t)}M$.

It is not difficult to prove that there exists a unique smooth map $r : TM \to \mathbf{R}^k$, with values in $(T_qM)^\perp$, which is quadratic in the second variable $v \in T_qM$, for any $q \in M$, and is such that, for any C^2 curve $x : J \to M$, $\ddot{x}_\nu(t) = r\big(x(t), \dot{x}(t)\big)$, where $\ddot{x}_\nu(t)$ denotes the orthogonal projection of $\ddot{x}(t)$ onto $(T_{x(t)}M)^\perp$. Hence, equation (1) can be equivalently written as

$$\ddot{x} = r(x, \dot{x}) + \varphi(t, x, \dot{x}).$$

The map r is strictly related to the second fundamental form on M and may be interpreted as the reactive force due to the constraint M. Actually, given $(q, v) \in TM$, $r(q, v)$ is the unique vector of $\mathbf{R}^k$ which makes $(v, r(q, v))$ tangent to TM at (q, v).

It is well known that (1) can be written, in an equivalent way, as a first order ODE on TM as follows:

$$\dot{\xi} = \widehat{\varphi}(t, \xi),$$

where $\widehat{\varphi}(t, q, v) = \big(v, r(q, v) + \varphi(t, q, v)\big)$. It can be shown that $\widehat{\varphi}$, called the *second order vector field associated to* φ, is a tangent vector field on TM. For a more extensive treatment of the subject of second order ODEs on manifolds from this embedded viewpoint see e.g. [1].

In what follows we deal with the following second order equation depending on a parameter:

$$\ddot{x}_\pi = h(x, \dot{x}) + \lambda f(t, x, \dot{x}), \quad \lambda \geq 0, \tag{2}$$

where $h : TM \to \mathbf{R}^k$ and $f : \mathbf{R} \times TM \to \mathbf{R}^k$ are assumed to be continuous maps such that $h(q, v)$ and $f(t, q, v)$ belong to T_qM for any $(t, q, v) \in \mathbf{R} \times TM$ (that is, with the terminology introduced above, f and h are tangent to M), and f is T-periodic with respect to the first variable. A pair $(\lambda, x) \in [0, \infty) \times C^1_T(M)$ is called a *T-pair for the second order equation* (2), if x is a solution of (2) corresponding to

λ. In particular we will say that (λ, x) is *trivial* if $\lambda = 0$ and x is constant. Notice that, in general, there may exist nontrivial T-pairs of (2) for $\lambda = 0$, as in the case of the inertial motion on a sphere. One can show that, no matter whether or not M is closed in $\mathbf{R}^k$, the set of T-pairs of (2) is always closed in $[0,\infty) \times C^1_T(M)$ and locally compact (see e.g. [2] or [7]). Moreover, if M is closed in $\mathbf{R}^k$, any bounded closed subset of T-pairs is compact.

As pointed out above, equation (2) can be written as

$$\ddot{x} = r(x,\dot{x}) + h(x,\dot{x}) + \lambda f(t,x,\dot{x}), \quad \lambda \geq 0, \tag{3}$$

or, equivalently, as

$$\dot{\xi} = \widehat{h}(\xi) + \lambda \bar{f}(t,\xi), \tag{4}$$

where

$$\widehat{h}(q,v) = (v, r(q,v) + h(q,v)) \ ,$$
$$\bar{f}(t,q,v) = (0, f(t,q,v)) \ .$$

It is readily verified that $\bar{f}$ is tangent to $TM \subset \mathbf{R}^{2k}$ (even if not a second order vector field); hence (4) is actually a first order equation on TM.

As in [8], we tacitly assume some natural identifications. For example we identify a point $q \in M$ with the constant function $\widehat{q} : t \mapsto q$ in $C^1_T(M)$, or a function $x \in C^1_T(M)$ with $(x,\dot{x}) \in C_T(TM)$. Also, we regard each of the above spaces as the zero-slice of the space obtained as the Cartesian product of $[0,\infty)$ and the space under consideration. In this manner, M becomes a subset of $[0,\infty) \times C^1_T(M)$ and of $[0,\infty) \times C_T(TM)$ as well, and so on. For instance, given an open set $\Omega \subset [0,\infty) \times C^1_T(M)$, by $\Omega \cap M$ we denote the set of all the points q of M which, seen as pairs $(0,\widehat{q})$, belong to Ω.

In the same spirit, by $h|_M : M \to \mathbf{R}^k$ we mean the function given by $h|_M(q) = h(q,0)$.

2. Global and local properties of the set of T-pairs

In order to obtain multiplicity results for equation (2), it is useful to point out some properties of the set of the T-pairs of this equation. In particular, our results will arise from a combination of local and global results on this set. This section is devoted to studying these two different aspects.

First we recall some basic facts about the topological degree of tangent vector fields on manifolds.

Let U be an open subset of a smooth, boundaryless manifold $N \subset \mathbf{R}^l$, and $v : N \to \mathbf{R}^l$ be a continuous tangent vector field which is admissible on U, i.e. such that the set $v^{-1}(0) \cap U$ is compact. Then, one can associate to the pair (v,U) an integer, called the degree of the vector field v in U and denoted by $\deg(v,U)$, which, roughly speaking, counts (algebraically) the number of zeros of v in U (see e.g. [12], and references therein).

In the flat case, namely if U is an open subset of $\mathbf{R}^l$, $\deg(v,U)$ is just the Brouwer degree (with respect to zero) of v in any bounded open set V containing $v^{-1}(0)$ and such that $\overline{V} \subset U$. One can see that all the standard properties of the Brouwer degree on open subsets of Euclidean spaces, such as homotopy invariance, excision, additivity, existence, etc. are still valid in the more general context of differentiable manifolds.

The celebrated Poincaré-Hopf theorem says that, if N is a compact manifold (possibly with boundary ∂N), and v is any tangent vector field which points outward along ∂N, then $\deg(v, N \setminus \partial N)$ equals the Euler-Poincaré characteristic $\chi(N)$ of N.

Given a compact relatively open subset Z of $v^{-1}(0)$, it is convenient to introduce the index $\mathrm{i}(v, Z)$ of v at Z as follows: $\mathrm{i}(v, Z) = \deg(v, U)$, where U is any open neighborhood of Z such that $Z = v^{-1}(0) \cap U$.

Theorem 2.1 ([8]). *Let $M \subset \mathbf{R}^k$ be a boundaryless manifold, and let $h : TM \to \mathbf{R}^k$ and $f : \mathbf{R} \times TM \to \mathbf{R}^k$ be tangent to M, with f T-periodic in the first variable. Given an open subset Ω of $[0, \infty) \times C^1_T(M)$, assume that $\deg(h|_M, \Omega \cap M)$ is well defined and nonzero. Then Ω contains a connected set Γ of nontrivial T-pairs for (2) whose closure meets $\Omega \cap (h|_M)^{-1}(0)$ and is not contained in any compact subset of Ω. In addition, if M is closed in $\mathbf{R}^k$, then Γ is not contained in any bounded and complete subset of Ω.*

The following consequence of the above theorem deserves to be mentioned.

Corollary 2.2. *Assume h is C^1 and $q \in (h|_M)^{-1}(0)$ is such that the Fréchet derivative $(h|_M)'(q) : T_qM \to \mathbf{R}^k$ of $h|_M$ at q is injective. Then (2) admits a connected set of nontrivial T-pairs whose closure meets q and is either noncompact or intersects $(h|_M)^{-1}(0) \setminus \{q\}$.*

Proof. Since q is a zero of the tangent vector field $h|_M$, the Fréchet derivative $(h|_M)'(q) : T_qM \to \mathbf{R}^k$ maps T_qM into itself (see e.g. [12]). Thus, $(h|_M)'(q)$ being injective, we get $\det(h|_M)'(q) \neq 0$. This implies $\mathrm{i}(h|_M, q) = \operatorname{sign}\det(h|_M)'(q) = \pm 1$. Now apply Theorem 2.1 taking $\Omega = \big([0, \infty) \times C^1_T(M)\big) \setminus \big((h|_M)^{-1}(0) \setminus \{q\}\big)$. Observe that Ω is open since the condition $\det(h|_M)'(q) \neq 0$ implies that q is an isolated zero of $h|_M$. ■

From Theorem 2.1 it follows that, if $q \in M$ is an isolated zero of $h|_M$ such that $\mathrm{i}(h|_M, q) \neq 0$, then q belongs to the closure of a connected set of nontrivial T-pairs of (2). However, it may happen that q is an accumulation point of nontrivial T-pairs all contained in the slice $\{0\} \times C^1_T(M)$, as in the harmonic oscillator equation (here $M = \mathbf{R}$ and $T = 2\pi$):

$$\ddot{x} = -x + \lambda \sin t\,, \quad \lambda \geq 0.$$

In order to find multiplicity results we need to avoid this "degenerate" situation. For this reason, given a point $q \in (h|_M)^{-1}(0)$, we will give conditions on h which ensure that q is isolated in the set of T-pairs corresponding to $\lambda = 0$ (and, consequently, isolated in $(h|_M)^{-1}(0)$) and has nonzero index.

From now on, unless differently stated, h will be assumed C^1. We say that a point $q \in (h|_M)^{-1}(0)$ is *T-resonant* for one of the two equivalent equations (2) or (3) if the linearized equation (on T_qM) of (3),

$$\ddot{x} = D_1h(q, 0)x + D_2h(q, 0)\dot{x}\,, \tag{5}$$

which corresponds to $\lambda = 0$, admits nonzero T-periodic solutions. Here $D_1h(q, 0)$ and $D_2h(q, 0)$ denote the partial derivatives at $(q, 0)$ of h with respect to the first and the second variable.

Remark 2.3. *To see that (5) is in fact an equation on T_qM we must prove that both $D_1h(q,0)$ and $D_2h(q,0)$ are endomorphisms of T_qM.*

Since q is a zero of the tangent vector field $h|_M$, the Fréchet derivative $(h|_M)'(q)$, which is the partial derivative $D_1h(q,0)$, maps T_qM into itself.

The fact that $D_2h(q,0)$ is an endomorphism of T_qM follows from $h(q,v) \in T_qM$ for any $v \in T_qM$.

Equivalently $q \in (h|_M)^{-1}(0)$ is T-resonant for (2) if the linearized problem (on $T_{(q,0)}TM$)

$$\begin{cases} \dot{\xi} = \widehat{h}'(q,0)\xi \\ \xi(0) = \xi(T) \end{cases}$$

has nonzero solutions. This is in turn equivalent to the fact that $\widehat{h}'(q,0)$ has at least an eigenvalue of the form $2l\pi i/T$ with $l \in \mathbf{Z}$.

It is convenient to express the T-resonance condition at a point $q \in \left(h|_M\right)^{-1}(0)$ in terms of the partial derivatives of h.

Since

$$T_{(q,0)}TM = T_qM \times T_qM,$$

the linear operator $\widehat{h}'(q,0) : T_{(q,0)}TM \to T_{(q,0)}TM$ is represented by the block matrix

$$\begin{pmatrix} 0 & I \\ D_1h(q,0) & D_2h(q,0) \end{pmatrix},$$

where I is the identity on T_qM. Straightforward computations (see e.g. [11]) show that q is T-resonant if and only if

$$\det\left(D_1h(q,0) + \frac{2l\pi i}{T}\, D_2h(q,0) + \left(\frac{2l\pi}{T}\right)^2 I\right) = 0, \tag{6}$$

for some $l \in \mathbf{Z}$. Consequently, if $q \in \left(h|_M\right)^{-1}(0)$ is non-T-resonant, putting $l = 0$,

$$\det\left((h|_M)'(q)\right) = \det D_1h(q,0) \neq 0.$$

As a consequence of this fact and Corollary 2.2 we get the following

Corollary 2.4. *Assume $q \in (h|_M)^{-1}(0)$ is non-T-resonant. Then*

1. *the point q is isolated in the set of T-pairs corresponding to $\lambda = 0$;*
2. *there exists a connected set of nontrivial T-pairs of (2) whose closure meets q and is either noncompact or intersects $(h|_M)^{-1}(0) \setminus \{q\}$.*

Proof. If f is assumed C^1, the first assertion can be deduced from the Implicit Function Theorem. However, it remains true even though f is only continuous. In fact, since q is non-T-resonant, from Lemma 3.3 of [7] it follows that q is an isolated point of the set

$$\left\{x \in C^1_T(M) : (0,x) \text{ is a } T\text{-pair of (2) }\right\}.$$

The claim now follows from Corollary 2.2. ∎

In particular cases it is possible to obtain from (6) some simpler conditions for the T-resonance of $q \in \left(h|_M\right)^{-1}(0)$. For instance, if $D_2h(q,0) = 0$, then q is T-resonant if and only if $D_1h(q,0)$ has eigenvalues of the form $-(2l\pi/T)^2$, for $l \in \mathbf{Z}$.

Another interesting case is when h is the sum of a positional and a frictional force, namely $h(q,v) = g(q) - \eta v$, where $g : M \to \mathbf{R}^k$ is a C^1 tangent vector field and $\eta > 0$. We immediately get that $q \in g^{-1}(0)$ is T-resonant if and only if some $-(2l\pi/T)^2 + \eta 2l\pi i/T$ belong to the spectrum of $g'(q)$. In particular, if $g = \operatorname{grad} G$, with $G : M \to \mathbf{R}$ a C^2 function, then, by standard computations, using the fact that $\eta \neq 0$, one can show that any nondegenerate zero q of g (i.e. such that $g'(q) : T_qM \to T_qM$ is nonsingular) is non T-resonant for any $T > 0$.

3. Multiplicity results

This section is devoted to the problem of giving conditions ensuring the existence of multiple forced oscillations on compact manifolds. In order to do that, as a first step, we frame the problem in the 'abstract' setting of metric spaces, defining the notion of ejecting set (see below) and obtaining a purely topological multiplicity result. Then, we use the abstract result to deduce the desired condition.

Let Y be a metric space and X a subset of $[0,\infty) \times Y$. Given $\lambda \geq 0$, we denote by X_λ the slice $\{y \in Y : (\lambda, y) \in X\}$. Moreover, given a topological space S and two subsets A and B, with $A \subset B$, $\overline{A}^B$ and $\overline{A}$ will denote the closure of A in B and in S respectively. Analogously, by $\operatorname{Fr}_B(A)$ and by $\operatorname{Fr}(A)$ we refer to the boundary of A relative to B and to S respectively. Finally, given a set Z, by $\#Z$ we mean its cardinality.

Lemma 3.1. *Let Y be a metric space and let X be a locally compact subset of $[0,\infty) \times Y$. Assume K is a compact relatively open subset of the slice X_0. Then, for any sufficiently small open neighborhood U of K in Y, there exists a positive number δ such that*

$$X \cap \big([0,\delta] \times \operatorname{Fr}(U)\big) = \emptyset\,.$$

Proof. Since the compact set K is open in X_0, there exists an open neighborhood W of K in Y such that $X_0 \cap \overline{W} = K$. For the compactness of K and the local compactness of X, there exist an open neighborhood V of K in W and a positive number σ such that $X \cap \big([0,\sigma] \times \overline{V}\big)$ is compact. Let U be any open neighborhood of K contained in V. Let us show that for some $\delta \in (0,\sigma)$ one has

$$X \cap ([0,\delta] \times \operatorname{Fr}(U)) = \emptyset.$$

Assume by contradiction that there exists a sequence $\{(\delta_i, y_i)\}_{i \in \mathbf{N}}$ such that $\delta_i \to 0$ as $i \to \infty$, and

$$(\delta_i, y_i) \in X \cap ([0,\sigma] \times \operatorname{Fr}(U))\,.$$

For the compactness of $X \cap \big([0,\sigma] \times \overline{U}\big)$, we can assume that $(\delta_i, y_i) \to (0, y_0)$. Thus

$$y_0 \in X_0 \cap \operatorname{Fr}(U),$$

which is a contradiction since $X_0 \cap \big(\overline{W} \setminus U\big) = \emptyset$. ∎

Let X be a subset of $[0,\infty) \times Y$. We say that $A \subset X_0$ is an *ejecting set* (for X) if it is relatively open in X_0 and there exists a connected subset of X which meets A and is not contained in X_0.

Remark 3.2. *An important example of ejecting set (or, rather, ejecting point) is provided by any non-T-resonant point of (2). In fact, as a consequence of Corollary 2.4, if X denotes the set of T-pairs of (2), any non-T-resonant $q \in h|_M^{-1}(0)$ for (2) turns out to be an isolated point of X_0 which is ejecting.*

Theorem 3.3. *Let Y be a metric space and let X be a locally compact subset of $[0,\infty)\times Y$. Assume that X_0 contains n pairwise disjoint ejecting subsets, $n-1$ of which are compact. Then there exists $\lambda_*>0$ such that $\#X_\lambda \geq n$ for any $\lambda\in[0,\lambda_*)$.*

Proof. If $n=1$, let A be an ejecting subset of X and Γ be a connected component of X which meets A and is not contained in X_0. Hence the projection of Γ on the first component of $[0,\infty)\times Y$ must contain a nontrivial interval of the form $[0,\lambda_*]$.

Assume now $n\geq 2$. Let $A_1,\dots,A_{n-1}$ be pairwise disjoint compact ejecting sets. For $i=1,\dots,n-1$, let Γ^i be a connected component of X which meets A_i and is not contained in X_0. By Lemma 3.1, given $i\in\{1\dots n-1\}$, there exists a neighborhood U_i of A_i in Y and a number $\delta_i>0$ such that

$$X\cap\big([0,\delta_i]\times \mathrm{Fr}(U_i)\big)=\emptyset\ . \tag{7}$$

We can assume $X_0\cap\overline{U_i}=A_i$ and $\overline{U_i}\cap\overline{U_j}=\emptyset$ for $i,j\in\{1,\dots,n-1\}$ with $i\neq j$. Furthermore, from the connectedness of the Γ^i's it follows that, reducing in case δ_i,

$$\Gamma^i_\lambda\cap U_i\neq\emptyset\ , \tag{8}$$

for $\lambda\in[0,\delta_i]$.

From (8) it follows $\#X_\lambda\geq n-1$, for any $\lambda\in[0,\delta_n]$, where $\delta_n=\min\{\delta_1,\dots,\delta_{n-1}\}$.

Since by assumption X_0 contains n distinct ejecting sets, there exists a connected set $\Gamma^n\subset X$ which meets $A_n=X_0\setminus\bigcup_{i=1}^{n-1}A_i$ and is not entirely contained in X_0. Hence the projection on $[0,\infty)$ of Γ^n must contain a nontrivial interval of the form $[0,\delta_*]$.

Take $\lambda_*=\min\{\delta_*,\delta_n\}$. Let us show that $\#X_\lambda\geq n$ for any $\lambda\in[0,\lambda_*)$. It is enough to prove that, for such λ's, the slice Γ^n_λ contains some points in the open neighborhood $V=Y\setminus\bigcup_{i=1}^{n-1}\overline{U_i}$ of A_n.

Assume by contradiction that there exists $\bar\lambda\in[0,\lambda_*)$ such that $\Gamma^n_{\bar\lambda}\cap V=\emptyset$. Consider the following closed subset of Γ^n:

$$\widetilde{\Gamma^n}=\Gamma^n\cap\big([0,\bar\lambda]\times\overline{V}\big)\,.$$

By (7), we have

$$X\cap([0,\delta_n]\times\mathrm{Fr}(V))=\bigcup_{i=1}^{n-1}\{X\cap([0,\delta_n]\times\mathrm{Fr}(U_i))\}=\emptyset.$$

Hence $\widetilde{\Gamma^n}$ coincides with $\Gamma^n\cap\big([0,\bar\lambda)\times V\big)$ and, consequently, it is open in Γ^n.

Obviously $\widetilde{\Gamma^n}$ is nonempty, and does not coincide with Γ^n, as $\bar\lambda<\delta_*$. This contradicts the connectedness of Γ^n. Hence $\Gamma^n_\lambda\cap V\neq\emptyset$ for any $\lambda\in[0,\lambda_*)$. ■

Notice that a crucial assumption in Theorem 3.3 is the local compactness of X, as shown by the following example.

Example 3.4. *Let $Y=\mathbf{R}$, and let X be the closure in $[0,\infty)\times Y$ of the graph of $(0,1]\ni t\mapsto\sin(1/t)$ devoid of $\{0\}\times(-1,1)$. The set X is connected but not locally compact, and $(0,1)$ and $(0,-1)$ are two disjoint compact ejecting subsets of X_0. Nevertheless $\#X_\lambda=1$ for any $\lambda\in(0,1]$.*

By means of two further examples, we will now show that the compactness assumption of the $n-1$ ejecting sets in Theorem 3.3 is sharp.

Example 3.5. *Let $Y = \mathbf{R}$, and let X be the closure in $[0,\infty) \times Y$ of the graph of $(0,1] \ni t \mapsto \sin(1/t)$ devoid of the point $(0,0)$. The set X is connected and locally compact, X_0 has two open (non-compact) disjoint ejecting subsets but $\#X_\lambda = 1$ for any $\lambda \in (0,1]$.*

The example below shows that even when X is closed, the compactness assumption cannot be removed.

Example 3.6. *Take $Y = \mathbf{R}^2$, let X be the closure in $[0,\infty) \times Y$ of the curve*

$$t \mapsto \big(t\,;\ \cos(t^{-1}), t^{-1}\sin(t^{-1})\big)\ , \quad t \in (0,1]\ .$$

The set X is connected and locally compact. Moreover

$$X_0 = \big\{(x,y) \in \mathbf{R}^2 : x = \pm 1, y \in \mathbf{R}\big\}\,,$$

whereas, for any $\lambda \in (0,1]$, we have $\#X_\lambda = 1$.

We now want to apply to the study of forced oscillations on manifolds what we have proved above in an abstract setting.

Theorem 3.7. *Let $h : TM \to \mathbf{R}^k$ and $f : \mathbf{R} \times TM \to \mathbf{R}^k$ be tangent to a closed boundaryless submanifold M of $\mathbf{R}^k$. Assume that h is C^1, $h|_M^{-1}(0)$ is compact and f is T-periodic in t. Let $q_1, \dots, q_{n-1} \in h|_M^{-1}(0)$ be non-T-resonant and such that*

$$\sum_{i=1}^{n-1} \mathrm{i}\,(h|_M\,, q_i) \neq \deg\,(h|_M\,, M)\,. \tag{9}$$

Assume that the unperturbed equation

$$\ddot{x}_\pi = h(x,\dot{x}) \tag{10}$$

does not admit (in $C^1_T(M)$) unbounded connected sets of T-periodic solutions. Then for $\lambda > 0$ sufficiently small the equation (2) admits at least n T-periodic solutions.

Notice that, when M is a compact boundaryless manifold, by the Poincaré-Hopf theorem one gets $\deg(h|_M\,, M) = \chi(M)$, so that (9) becomes

$$\sum_{i=1}^{n-1} \mathrm{i}\,(h|_M\,, q_i) \neq \chi(M).$$

The most difficult assumption to verify in Theorem 3.7 is the non-existence of unbounded connected sets of T-periodic solutions of the unperturbed equation (10). Nevertheless, in many cases it is possible to show that this property holds. For instance, in the next section we will take into account two physically relevant situations: forced oscillations on a compact manifold with friction and the spherical pendulum (possibly without friction).

In the statement of Theorem 3.7 we assumed the nonexistence of unbounded connected sets of T-periodic solutions of (10). This has been done for the sake of simplicity since, as the following proof shows, such an hypothesis could be weakened by merely assuming that $(h|_M)^{-1}(0)$ does not meet unbounded connected sets of T-periodic solutions of (10).

Proof of Th. 3.7. For simplicity we will perform the proof in the case $n = 2$, as the general case can be proved in an analogous way. Put $q = q_1$ and $Z = h|_M^{-1}(0) \setminus \{q\}$.

Let X denote the set of T-pairs of (2). By Remark 3.2, q is an ejecting point for the set X of all the T-pairs of (2). We claim that also $X_0 \setminus \{q\}$ is ejecting. Thus the assertion follows straightforwardly from Theorem 3.3.

To prove our claim, let us define

$$\Omega = \Big([0,\infty) \times C^1_T(M)\Big) \setminus \{q\}.$$

Since q is an isolated zero of $h|_M^{-1}(0)$, the set Z is compact. Moreover

$$\deg\big(\, h|_M \,, \Omega \cap M\big) = \deg\,(h|_M\,, M) - \mathrm{i}\,(h|_M\,, q) \neq 0\ .$$

By Theorem 2.1 there exists a connected set Γ of nontrivial T-pairs for (2) whose closure $\overline{\Gamma}$ meets the subset Z of $X_0 \setminus \{q\}$ and is not contained in any bounded complete subset of Ω.

To show that $X_0 \setminus \{q\}$ is ejecting it is enough to prove that $\overline{\Gamma}$, which clearly is a connected subset of X, is not contained in X_0.

Assume the contrary, thus $\overline{\Gamma}$ is bounded. Moreover, as q is isolated in X_0, $\overline{\Gamma}$ is contained in Ω. Since $[0,\infty) \times C^1_T(M)$ is complete, $\overline{\Gamma}$ is complete as well. Hence, Γ is contained in the bounded complete subset $\overline{\Gamma}$ of Ω, which is a contradiction. ■

Remark 3.8. *We observe that the n T-periodic solutions ensured by the assertion of Theorem 3.7 have pairwise disjoint images, provided that $\lambda > 0$ is sufficiently small. This is due to the fact that, when λ is small, $n-1$ of them are close (in $C^1_T(M)$) to the constant solutions, $q_1, q_2, \ldots, q_{n-1}$, of the unperturbed equation, whereas the n-th solution is bounded away from those constants.*

4. Applications

In this section we show how Theorem 3.7 can be used to deduce some concrete multiplicity results for forced oscillations of constrained systems.

We will consider two remarkable cases of physical relevance:

(1) periodic perturbations of a positional force plus a friction on a compact boundaryless manifold,
(2) periodic perturbations of a bounded force (depending on both position and speed) tangent to the m-dimensional unit sphere S^m.

The first situation is perhaps the most physically meaningful, whereas the second one is more interesting from the mathematical viewpoint. In fact the nontrivial problem of the existence of forced oscillations for arbitrarily large periodic perturbations (independent of the speed) of a positional force, has been positively solved for even-dimensional spheres in [3] and [4]. However those results do not provide any information about the 'number' of forced oscillations.

Let us start with the first case. Assume M to be a compact boundaryless submanifold of $\mathbf{R}^k$, and consider the equation

$$\ddot{x}_\pi = g(x) - \rho\dot{x}, \tag{11}$$

where $g : M \to \mathbf{R}^k$ is a C^1 tangent vector field on M and ρ is a positive constant.

Lemma 4.1. *Let g and ρ be as above. Then, the set of T-periodic solutions in $C^1_T(M)$ of the second order equation (11) is a priori bounded.*

Proof. Let us show that for any T-periodic solution x of (11) one has $\|\dot{x}\|_0 \le G/\rho$, where $G = \max\{g(q) : q \in M\}$. Define $\vartheta(t) = |\dot{x}(t)|^2$ and let $\tau \in \mathbf{R}$ be such that $\vartheta(\tau) = \max\{\vartheta(t) : t \in \mathbf{R}\}$. We get

$$\begin{aligned} 0 = \dot{\vartheta}(\tau) &= 2\,\langle \dot{x}(\tau), \ddot{x}(\tau)\rangle = 2\langle \dot{x}(\tau), \ddot{x}_\pi(\tau)\rangle \\ &= 2\,\langle \dot{x}(\tau), g\,(x(\tau))\rangle - 2\rho\,|\dot{x}(\tau)|^2 \\ &\le 2\,|\dot{x}(\tau)|\,G - 2\rho\,|\dot{x}(\tau)|^2 . \end{aligned}$$

Hence $\|\dot{x}\|_0 \le G/\rho$, as claimed. ∎

Remark 4.2. *With only minor changes in the proof, one can show that the same assertion of Lemma 4.1 holds also for the slightly more general equation*

$$\ddot{x}_\pi = b(q,v) - \rho\,(|v|)\,v, \tag{12}$$

where $b : TM \to \mathbf{R}^k$ is continuous and tangent to M, $\rho : [0,+\infty) \to \mathbf{R}$ is continuous, and there exist constants $\alpha > 0$ and $\beta \ge 0$ such that

$$\begin{aligned} |b(q,v)| &\le \alpha + \beta\,|v|\,, \\ \beta &< \liminf_{z\to+\infty} \rho(z). \end{aligned}$$

Combining Theorem 3.7 with Remark 3.8 and Lemma 4.1 we get the following multiplicity result.

Theorem 4.3. *Let $M \subset \mathbf{R}^k$ be a compact boundaryless manifold. Consider the equation*

$$\ddot{x}_\pi = g(x) - \rho\dot{x} + \lambda f(t,x,\dot{x}), \tag{13}$$

where g and ρ are as above, and $f : \mathbf{R}\times TM \to \mathbf{R}^k$ is T-periodic in the first variable and tangent to M. Assume $q_1, \dots, q_{n-1}$ are non-T-resonant zeros such that

$$\sum_{i=1}^{n-1} \mathrm{i}\,(g, q_i) \ne \chi(M).$$

Then, for $\lambda > 0$ sufficiently small, the equation (2) has at least n T-periodic solutions with pairwise different images.

Proof. By Poincaré-Hopf theorem $\deg\,(g\,, M) = \chi(M)$. The assertion now follows from Theorem 3.7, Remark 3.8 and Lemma 4.1. ∎

The above theorem has the following consequence.

Corollary 4.4. *Let $M \subset \mathbf{R}^k$ be a compact boundaryless manifold with $|\chi(M)| \ne 1$ and let g and ρ be as in Theorem 4.3. If there exists a non-T-resonant zero, then for $\lambda > 0$ sufficiently small equation (13) has at least two T-periodic solutions with disjoint images.*

Proof. The assertion follows from the theorem above, recalling that the index of a non-T-resonant zero is ± 1. ∎

Another remarkable case in which it is possible to prove the boundedness of the connected sets of T-periodic solutions of equation (10) is when h is bounded and M is the m dimensional unit sphere S in $\mathbf{R}^{m+1}$, although the whole set of T-periodic solutions may be unbounded, as in the inertial case $h = 0$.

Lemma 4.5. *Let $h : TS \to \mathbf{R}^{m+1}$ be C^1, bounded and tangent to S. Then any connected set of T-periodic solutions in $C^1_T(S)$ of (10) is bounded.*

Since for the proof we need the rotation index with respect to the origin of an admissible closed curve in a not necessarily two-dimensional space, we give a brief description of this notion. For further details and a more general treatment of this topic we refer to [4].

Given a continuous T-periodic curve $\sigma : \mathbf{R} \to \mathbf{R}^2 \setminus \{0\}$, the winding number of σ with respect to 0 is the algebraic count of the turns of σ around 0 in a period, and it is given by $\mathcal{W}_0(\sigma) = \deg_B(\hat{\sigma})$, where $\hat{\sigma} : S^1 \to S^1$ is defined by $\hat{\sigma}(\theta) = \sigma(\theta T/2\pi)/\left|\sigma(\theta T/2\pi)\right|$ and $\deg_B$ stands for the Brouwer degree of maps between manifolds.

Let E be an oriented Euclidean space and α be an oriented axis in E, i.e. an oriented 2-codimensional, possibly trivial, subspace of E (recall that the trivial space, as any other finite dimensional space, has two orientations, conventionally denoted by $+1$ and -1). We orient the quotient E/α in such a way that the resulting orientation of $(E/\alpha) \times \alpha$ (as a product) coincides with the one induced by the canonical identification $E \simeq (E/\alpha) \times \alpha$. Given $\sigma : \mathbf{R} \to E \setminus \{\alpha\}$ continuous and T-periodic, we define

$$w(\sigma, \alpha) = \mathcal{W}_0(\pi \circ \sigma),$$

where $\pi : E \to E/\alpha$ is the canonical projection.

Let $\gamma : \mathbf{R} \to E$ be C^1, T-periodic and such that, for every $t \in \mathbf{R}$, the vectors $\gamma(t)$ and $\dot{\gamma}(t)$ are linearly independent. Given $\tau \in \mathbf{R}$, we denote by α_τ the oriented axis through the origin, orthogonal to the plane P_τ spanned and oriented by the ordered pair $(\gamma(\tau), \dot{\gamma}(\tau))$. Here the orientation of α_τ is chosen accordingly to the one of E in the identification $E \simeq \alpha_\tau \times P_\tau$. We say that γ is *admissible* if $\gamma(t) \notin \alpha_\tau$ for any $t, \tau \in \mathbf{R}$. If γ is a T-periodic, admissible curve, by the homotopy property of the Brouwer degree, $w(\gamma, \alpha_\tau)$ is independent of the choice of $\tau \in \mathbf{R}$. Thus, we can define the rotation index of γ with respect to the origin, as follows:

$$\mathfrak{I}(\gamma) = w(\gamma, \alpha_\tau).$$

Notice that, due to the chosen orientations, although the winding number can be an arbitrary integer, the rotation index of an admissible curve is necessarily non-negative.

It is important to observe that the integer valued function $\mathfrak{I}$ is defined on the open set

$$\Omega = \left\{\gamma \in C^1_T(E) : \gamma \text{ is admissible}\right\},$$

and is continuous.

Proof of Lemma 4.5. Put $H = \sup_{(q,v)\in TS} |h(q, v)|$ and let $M > 2TH$. Following the proof of Lemma 1 in [3] and of Lemmas 2.2 and 2.3 in [4] we obtain that any T-periodic solution x of (10) such that $\|\dot{x}\|_0 \geq M$ is admissible with respect to the origin and its rotation index satisfies

$$\mathfrak{I}(x) \geq \frac{T}{2\pi}\left(\|\dot{x}\|_0 - TH\right). \tag{14}$$

Assume by contradiction that there exists a connected unbounded set G of non-trivial T-periodic solutions for (10) in $C^1_T(S)$. Observe that the closure $\overline{G}$ of G is a connected and unbounded set of (possibly trivial) T-periodic solutions. By the

Tietze extension theorem there exists a continuous function $\omega : \overline{G} \to \mathbf{R}$ such that $\omega(x) = \Im(x)$ if $\|x\|_1 = \|x\|_0 + \|\dot{x}\|_0 \geq 1 + M$. The inequality (14) shows that the image of ω is unbounded. Moreover, since $\overline{G}$ is connected, this image must actually be an unbounded interval. This is impossible because ω takes integer values outside the set

$$K = \left\{ x \in C^1_T(S) : \|x\|_1 \leq 1 + M \right\},$$

which, by Ascoli's theorem, is compact. ∎

From Theorem 3.7, Remark 3.8 and the above lemma we get the following

Theorem 4.6. *Let $h : TS \to \mathbf{R}^{m+1}$ and $f : \mathbf{R} \times TS \to \mathbf{R}^{m+1}$ be tangent to the m-dimensional unit sphere $S \subset \mathbf{R}^{m+1}$ and let h be C^1 and bounded. Assume that f is T-periodic in the first variable and let $q_1, \dots, q_{n-1} \in h|_S^{-1}(0)$ be non-T-resonant and such that*

$$\sum_{i=1}^{n-1} \mathrm{i}\left(h|_S , q_i \right) \neq \begin{cases} 2 & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

Then, for $\lambda > 0$ sufficiently small the equation (2) admits at least n T-periodic solutions with pairwise different images.

Proof. It is enough to observe that by the Poincaré-Hopf theorem one has

$$\deg\left(h|_S , S\right) = \chi(S) = \begin{cases} 2 & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

The assertion follows from Theorem 3.7, Remark 3.8 and Lemma 4.5. ∎

As an immediate consequence of Theorem 4.6 we have that, whatever the dimension m is, if h has a non-T-resonant zero, then (2) admits at least two T-periodic solutions for small λ. In fact, the index of non-T-resonant zero is either $+1$ or -1 which, in any case, is neither 0 nor 2.

Two particular cases, which have been studied in [6] and [7], are the gravitational pendula, either ordinary ($M = S^1$) or spherical ($M = S^2$). In both cases the unperturbed equation can be written in the form

$$\ddot{x}_\pi = h_g(x),$$

where h_g represents the tangential (to M) component of the gravitational force. Observe that, given any $T > 0$, the north pole is a non-T-resonant zero of h_g.

Summarizing, Theorem 4.6 implies the following proposition (see also [6] and [7]):

> Given any $T > 0$, the T-periodically perturbed gravitational pendulum (both ordinary and spherical) has at least two T-periodic solutions for small perturbations.

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