Remarks on Global Branches of Harmonic Solutions to Periodic ODE's on Manifolds

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SUNTO. Sia $\dot{x} = \lambda f(t,x)$, $\lambda \geq 0$, una famiglia ad un parametro di equazioni differenziali su una varietà differenziabile M, dove f è un campo vettoriale continuo, T-periodico, tangente ad M. Fissato un aperto Ω dello spazio metrico $[0,\infty) \times C_T(M)$, sotto opportune ipotesi topologiche, si prova l'esistenza di un "ramo globale" di coppie $(\lambda, x) \in \Omega$, con $\lambda > 0$ e x soluzione T-periodica della suddetta equazione, la cui chiusura in Ω interseca $\{0\} \times C_T(M)$ in punti corrispondenti a soluzioni stazionarie. Questo risultato rappresenta l'analogo infinito-dimensionale di un precedente teorema degli autori, espresso in termini di punti iniziali di soluzioni T-periodiche.

0. – Introduction.

Consider the following one parameter family of differential equations:

$$\dot{x} = \lambda f(t,x), \quad \lambda \geq 0, \tag{0.1}$$

where $f \colon \mathbf{R} \times M \to \mathbf{R}^k$ is a T-periodic (continuous) vector field, tangent to a boundaryless smooth (not necessarily closed) submanifold of $\mathbf{R}^k$. Let $C_T(\mathbf{R}^k)$ be the Banach space of all continuous, T-periodic, $\mathbf{R}^k$-valued real maps, endowed with the standard norm of uniform convergence, and denote by $C_T(M)$ the metric subspace of $C_T(\mathbf{R}^k)$ consisting of those maps whose image lies in M. Given an arbitrary open subset Ω of the Cartesian product $[0,\infty) \times C_T(M)$, let $\Omega \cap M$ stand for the open subset of M of those points p such that the pair $(0, \hat{p})$ is in Ω, $\hat{p}$ being the constant map $\hat{p}(t) \equiv p$. Let $w \colon M \to \mathbf{R}^k$ be the average wind velocity associated with the map f; that is, the tangent vector field given by

$$w(p) = \frac{1}{T} \int_0^T f(t,p)dt.$$

In this paper, under the assumption that the Hopf index (also called Euler characteristic, or rotation, or degree) of w in $\Omega \cap M$ is well defined and nonzero, we prove the existence of a connected subset Γ of Ω satisfying the following properties:

- (a) any $(\lambda, x) \in \Gamma$ is a T-pair of (0.1), that is $\dot{x}(t) = \lambda f(t, x(t))$ identically on $\mathbf{R}$;
- (b) any $(\lambda, x) \in \Gamma$ is nontrivial, i.e. such that $\lambda > 0$;
- (c) the closure of Γ in Ω is noncompact and meets the set $\{(0, \hat{p}) \in \Omega : w(p) = 0\}$.

This represents the infinite-dimensional counterpart of a result in [FP3], where the existence of a global branch was expressed in terms of starting points; that is, of those elements $(\lambda, p) \in [0, \infty) \times M$ such that the equation (0.1) has (for such a λ) a T-periodic solution $x \colon \mathbf{R} \to M$ verifying the Cauchy condition $x(0) = p$.

In order to include the (very natural and important) case where M is an open subset of $\mathbf{R}^k$, we do not restrict our analysis to the class of closed submanifolds of $\mathbf{R}^k$. Consequently, the metric space $[0, \infty) \times C_T(M)$ need not be complete. The difficulties due to this fact make the use of infinite dimensional degree theories not easily applicable to our situation. Therefore, we avoid such theories and we develop a method which allows us to deduce the existence of the above "global bifurcating branch Γ of nontrivial solution pairs" directly from the finite dimensional result in [FP3] or (more conveniently) from a, still finite dimensional, result recently obtained in [FP4].

For related infinite dimensional results see [CZ], [C1], [C2] and references therein. We point out, however, that these results agree with our situation only in the case when M is a compact manifold. In fact, roughly speaking, we assume that our problem degenerates, for $\lambda = 0$, in a sort of resonant condition. We do not require, as in the above mentioned papers, the compactness in $C_T(M)$ of the solution set of the equation for $\lambda = 0$ (in fact, in our case, this set can be identified with M or, more generally, given Ω as above, with $\Omega \cap M$).

1. – Global branches of fixed points.

Let M be a boundaryless m-dimensional smooth manifold embedded in $\mathbf{R}^k$ and, for any $p \in M$, let $T_p(M) \subset \mathbf{R}^k$ denote the tangent space of M at p. Let $D \subset [0, \infty) \times M$ be an open subset containing $\{0\} \times M$ and $\varphi \colon D \to M$ be of class C^1. Consider the equation

$$\varphi(\lambda, p) = p \tag{1.1}$$

and assume that the fixed point problem degenerates for $\lambda = 0$; that is, $\varphi(0, p) = p$ for any $p \in M$. Let us associate to φ the continuous tangent vector field $v \colon M \to \mathbf{R}^k$ which assigns to any $p \in M$ the vector

$$v(p) = \frac{\partial \varphi}{\partial \lambda}(0, p) \in T_p(M).$$

For the applications we are going to discuss in the next section, we are interested in presenting conditions detecting those elements $p \in M$ which, loosely speaking, are emanating points of a branch of solutions (λ, q) of (1.1), with $\lambda > 0$. It is quite easy to show that a necessary condition is that the vector field v vanishes at p (see [FP4]). A sufficient condition can be obtained in terms of the index of v. Therefore, before stating precisely our result, we recall, for completeness, the notion and the basic properties of the index of a vector field.

Let $v\colon M \to \mathbf{R}^k$ be a continuous tangent vector field on M which is *admissible*, i.e. such that the set $\{p \in M : v(p) = 0\}$ is compact. Then, one can associate to v an integer $\chi(v)$, called the index (or Euler characteristic, or rotation, or degree) of v, which, roughly speaking, counts (algebraically) the number of zeros of v (see e.g. [GP], [H], [M], [T], andreferences therein). As a consequence of the Poincaré-Hopf theorem, when M is compact, this integer equals $\chi(M)$, the Euler-Poincaré characteristic of M. On the other hand, in the particular case when M is an open subset of $\mathbf{R}^m$, $\chi(v)$ is just the Brouwer degree (with respect to zero) of the map $v\colon M \to \mathbf{R}^m$. All the standard properties of the Brouwer degree on open subsets of Euclidean spaces, such as homotopy invariance, excision, additivity, existence, etc., are still valid in the more general context of differentiable manifolds. To see this, one can use an equivalent definition of index of a vector field based on fixed point index theory given in [FP3]. Let us point out that no orientability of M is required for the index of a tangent vector field to be defined.

In what follows, to emphasize that the index of a tangent vector field v on M, reduces, in the flat case, to the classical Brouwer degree (with respect to zero), $\chi(v)$ will be called the (global) degree of the vector field v and denoted by $\deg(v)$. Since any open subset U of a manifold M is still a manifold, the degree of the restriction of v to U makes sense provided that v is admissible on U, i.e. the set $\{p \in U : v(p) = 0\}$ is compact. This condition is clearly satisfied if U is a relatively compact open subset of M and $v(p) \neq 0$ for all $p \in \partial U$. The degree of the restriction of v to U, when defined, will be denoted by $\deg(v, U)$.

We are now in a position to state our sufficient condition for the existence of global branches of nontrivial fixed points. The proof is given in [FP4].

Theorem 1.1. *Let D be an open subset of $[0,\infty)\times M$ containing $\{0\}\times M$, $\varphi\colon D \to M$ be a C^1 map satisfying $\varphi(0,p) = p$ for all $p \in M$, and $v\colon M \to \mathbf{R}^k$ be the tangent vector field*

$$v(p) = \frac{\partial\varphi}{\partial\lambda}(0,p).$$

Given an open subset W of D, assume that v is admissible in the slice

$$W_0 = \{p \in M : (0,p) \in W\}$$

and that $\deg(v, W_0)$ is nonzero. Then, the equation (1.1) admits in W a connected set of solutions (λ, p), with $\lambda > 0$, whose closure (in $[0,\infty)\times M$) meets $\{0\}\times\{p \in W_0 : v(p) = 0\}$ and it is not contained in any compact subset of W.

2. – Periodic orbits.

Let M be a boundaryless m-dimensional smooth manifold in $\mathbf{R}^k$. As in the introduction, let $C_T(M)$ denote the metric subspace of $C_T(\mathbf{R}^k)$ consisting of all T-periodic continuous maps $x\colon \mathbf{R} \to M$. Observe that this space is not necessarily complete, unless M is a closed submanifold of $\mathbf{R}^k$. However, due to the fact that M is locally compact, one can prove that $C_T(M)$ is always locally complete. We recall that a metric space Y is compact if and only if it is totally bounded (i.e. precompact) and complete. Therefore, as a consequence of Ascoli's theorem, a subset Y of $C_T(M)$ is totally bounded if (and only if) it is bounded and equicontinuous.

Consider in M the first order parametrized differential equation

$$\dot{x} = \lambda f(t, x), \quad \lambda \geq 0, \tag{2.1}$$

where $f\colon \mathbf{R} \times M \to \mathbf{R}^k$ is a T-periodic continuous tangent vector field (i.e. $f(t+T,p) = f(t,p) \in T_p(M)$ for all $(t,p) \in \mathbf{R} \times M$). An element $(\lambda, x) \in [0,\infty) \times C_T(M)$ will be called a *T-pair* (or a *solution pair*) of the equation (2.1) provided x is a (T-periodic) solution of (2.1). Denote by X the subset of $[0,\infty) \times C_T(M)$ of all the T-pairs of (2.1). It is not hard to show that X is closed in $[0,\infty) \times C_T(M)$ (and locally closed in $[0,\infty) \times C_T(\mathbf{R}^k)$). Consequently, X is locally complete, as a closed subset of a locally complete space. Therefore, since X is locally made up of equicontinuous functions from $[0,T]$ to $\mathbf{R} \times \mathbf{R}^k$, Ascoli's theorem implies that X is actually a locally compact space, and this fact will turn out to be useful in the sequel.

In what follows, it will be convenient to consider the commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & [0,\infty) \times M \\ \downarrow & & \downarrow \\ C_T(M) & \longrightarrow & [0,\infty) \times C_T(M), \end{array}$$

where the horizontal arrows are obtained by associating to $p \in M$ or, respectively, to $x \in C_T(M)$ the element $(0,p) \in [0,\infty) \times M$ or $(0,x) \in [0,\infty) \times C_T(M)$, and the vertical ones are defined by regarding any $p \in M$ as a constant map $\hat{p}(t) \equiv p$. With this identification, if A is any subset of $[0,\infty) \times C_T(M)$, $A \cap M$ will denote the set $\{p \in M : (0,\hat{p}) \in A\}$.

Since any element $p \in M$ may be viewed as a constant solution of (2.1) corresponding to the value $\lambda = 0$ of the parameter, the whole manifold M will be regarded as a subset of the set X of the T-pairs of (2.1). We point out that, despite the fact that $[0,\infty) \times C_T(M)$ may not be closed in $[0,\infty) \times C_T(\mathbf{R}^k)$, M is always closed in $[0,\infty) \times C_T(M)$, as well as in X. Any $p \in M$ (which we regard as $(0,\hat{p})$) will be called a trivial T-pair of (2.1) and, consequently, any $(\lambda, x) \in X \setminus M$, i.e. with $\lambda > 0$, will be a nontrivial T-pair. An element $p \in M$ will be called a *bifurcation point* of (2.1) if it lies in the closure of $X \setminus M$. Since X and M are locally closed in $[0,\infty) \times C_T(\mathbf{R}^k)$, this definition does not depend on whether the closure of $X \setminus M$ is taken in $[0,\infty) \times C_T(M)$ or in $[0,\infty) \times C_T(\mathbf{R}^k)$.

A necessary condition for $p \in M$ to be of bifurcation is given by the following

Theorem 2.1. *Let $f: \mathbf{R} \times M \to \mathbf{R}^k$ be a continuous T-periodic tangent vector field and $w: M \to \mathbf{R}^k$ be the mean value autonomous vector field given by*

$$w(p) = \frac{1}{T} \int_0^T f(t,p)dt.$$

If p_0 is a bifurcation point for the equation (2.1), then $w(p_0) = 0$.

Proof. Let p_0 be a bifurcation point for the equation (2.1). Then, there exists a sequence $\{(\lambda_n, x_n)\}$ of nontrivial T-pairs of (2.1) such that $\lambda_n \to 0$ and $x_n(t) \to p_0$ uniformly.

Since we have assumed M to be a manifold in $\mathbf{R}^k$, it makes sense to integrate from 0 to T the equality

$$\dot{x}_n(t) = \lambda_n f(t, x_n(t)), \quad t \in \mathbf{R}.$$

Thus

$$0 = x_n(T) - x_n(0) = \lambda_n \int_0^T f(t, x_n(t))dt,$$

which implies

$$\int_0^T f(t, x_n(t))dt = 0.$$

Passing to the limit, we obtain

$$\int_0^T f(t, p_0)dt = 0,$$

which means $w(p_0) = 0$, as claimed. □

Remark 2.2. *Let $Z = \{p \in M : w(p) = 0\}$. As a consequence of the above necessary condition, the set $(X \setminus M) \cup Z$ is a closed subset of X. Therefore, since X is locally compact, $(X \setminus M) \cup Z$ is locally compact as well.*

In what follows, given an open subset Ω of $[0, \infty) \times C_T(M)$, by a *bifurcating branch* of (2.1) in Ω we mean a connected component of $\Omega \cap (X \setminus M)$, whose closure in X (or, equivalently, in $[0, \infty) \times C_T(M)$) intersects $\Omega \cap M$. A *global* bifurcating branch in Ω, is a bifurcating branch which is not relatively compact in $\Omega \cap X$. In particular, if M is closed in $\mathbf{R}^k$ and $\Omega = [0, \infty) \times C_T(M)$, as we shall see later any global bifurcating branch must be unbounded.

We can now state our main result on the existence of a global bifurcating branch.

Theorem 2.3. *Let $f: \mathbf{R} \times M \to \mathbf{R}^k$ be a continuous T-periodic tangent vector field and $w: M \to \mathbf{R}^k$ be the mean value autonomous vector field given by*

$$w(p) = \frac{1}{T} \int_0^T f(t,p)dt.$$

Let Ω be an open subset of $[0, \infty) \times C_T(M)$ and assume that $\deg(w, \Omega \cap M)$ is defined and nonzero. Then, the equation (2.1) admits in Ω a connected set Γ of nontrivial T-pairs whose closure in Ω is noncompact and meets $\Omega \cap M$ in $w^{-1}(0)$. In addition, if M is closed, then Γ cannot be contained in a bounded and complete subset of Ω.

The connectivity result stated below (see [FP4]) will be crucial in the proof of Theorem 2.3.

Lemma 2.4. *Let Y be a locally compact Hausdorff space and let Y_0 be a compact subset of Y. Assume that any compact subset of Y containing Y_0 has nonempty boundary. Then $Y \setminus Y_0$ contains a not relatively compact component whose closure intersects Y_0.*

Proof of Theorem 2.3. Assume first that f is smooth. Let us consider the set

$$D = \{(\lambda, p) \in [0, \infty) \times M : \text{ the solution } x(\cdot) \\ \text{of (2.1) satisfying } x(0) = p \text{ is defined in } [0, T]\}$$

and let $\varphi: D \to M$ be the operator which associates to any $(\lambda, p) \in D$ the value $x(T)$ of the solution $x(\cdot)$ of (2.1) with initial condition $x(0) = p$. It is a known fact that D is an open set (clearly containing $\{0\} \times M$) and that φ is smooth in D. Let us show that

$$\frac{\partial \varphi}{\partial \lambda}(0, p) = Tw(p).$$

In fact, given $(\lambda, p) \in D$, let $\psi(\lambda, p, t)$ denote the value at time $t \in [0, T]$ of the solution of (2.1) with initial condition p. As already observed, the integral

$$\varphi(\lambda, p) = p + \lambda \int_0^T f(t, \psi(\lambda, p, t))dt$$

makes sense. Thus

$$\frac{\varphi(\lambda, p) - \varphi(0, p)}{\lambda} = \frac{\varphi(\lambda, p) - p}{\lambda} = \int_0^T f(t, \psi(\lambda, p, t))dt.$$

Take any sequence $\lambda_n \to 0$, $n \in \mathbf{N}$. Then, the sequence of solutions $t \mapsto \psi(\lambda_n, p, t)$ tends uniformly in $[0, T]$ to the constant solution $\psi(0, p, t) \equiv p$. Consequently,

$$\frac{\partial \varphi}{\partial \lambda}(0, p) = \lim_{\lambda \to 0} \frac{\varphi(\lambda, p) - p}{\lambda} = \int_0^T f(t, p)dt = Tw(p),$$

as claimed.

Consider the set

$$S = \{(\lambda, p) \in D : \varphi(\lambda, p) = p\},$$

which is locally compact, since it is clearly closed in the locally compact set D. Moreover, the fact $\varphi(0, p) = p$ for any $p \in M$, implies that any pair $(0, p)$ with $p \in M$, belongs to S. Hence, by recalling the embedding $M \hookrightarrow [0, \infty) \times M$ of the above commutative diagram, we will regard M as a closed subset of S.

In the set X of all T-pairs of (2.1), let us consider the map $h: X \to S$ given by $h(\lambda, x) = (\lambda, x(0))$. Clearly h is continuous, onto and, since we are assuming f smooth, it is also one-to-one. Moreover, the continuous dependence property from the data of the solutions of differential equations ensures the continuity of its inverse $h^{-1}: S \to X$. Let Ω be the open subset of $[0, \infty) \times C_T(M)$ considered in the statement of the theorem. Set

$S_\Omega = h(\Omega \cap X)$. Clearly, S_Ω is open in S. Thus, there exists an open subset W of D such that $S_\Omega = W \cap S$. Our aim is to apply Theorem 1.1 to the equation $\varphi(\lambda, p) = p$ in W. To this end observe that, according to the embeddings $M \hookrightarrow X$ and $M \hookrightarrow S$, the restriction of the homeomorphism h to M turns out to be the identity. Hence, the set $\Omega \cap M$ and the slice W_0 can be identified, so that the assumption $\deg(w, \Omega \cap M) \neq 0$ is equivalent to

$$\deg(\frac{\partial \varphi}{\partial \lambda}(0, \cdot), W_0) \neq 0,$$

where, as proved above,

$$\frac{\partial \varphi}{\partial \lambda}(0, p) = Tw(p)$$

for all $p \in M$. Consequently, by Theorem 1.1, there exists in $W \cap (S \setminus M)$ a connected branch Σ whose closure in $[0, \infty) \times M$ meets $\{p \in W_0 : w(p) = 0\}$ and it is not contained in any compact subset of W. This means that the closure of Σ in the topological space S_Ω is not compact. Set $\Gamma = h^{-1}(\Sigma)$ and observe that Γ is a connected set of nontrivial T-pairs in Ω, whose closure (in Ω) is noncompact and meets $\Omega \cap M$ in $w^{-1}(0)$. Hence, the existence in Ω of a global branch of T-pairs possessing all the required properties is completely proved in the smooth case.

Let us now remove the smoothness assumption on f. Let Y_0 denote the (compact) set of zeros of w in $\Omega \cap M$, i.e. $Y_0 = \Omega \cap Z$, with $Z = \{p \in M : w(p) = 0\}$. Set $Y = ((\Omega \cap X) \setminus M) \cup Y_0$ and observe that Y is locally compact, since it coincides with $\Omega \cap ((X \setminus M) \cup Z)$, which is the intersection of an open set with a locally compact set (recall Remark 2.2). Let us apply Lemma 2.4 to the pair (Y, Y_0). In order to verify all the assumptions of the lemma, we need only to show that any compact subset of Y containing Y_0 has nonempty boundary. Assume the contrary. Thus, there exists a relatively open compact subset C of Y containing Y_0. Consequently, one can find an open subset G of Ω such that $G \cap Y = C$, $\partial G \cap Y = \emptyset$ and the set $\{(\lambda, x(t)) \in [0, \infty) \times M : (\lambda, x) \in G, t \in [0, T]\}$ is contained in a compact subset K of $[0, \infty) \times M$. This implies that G is bounded with complete closure. Without loss of generality, we may also assume the closure of G contained in Ω. Hence, in particular, $G \cap M$ is relatively compact with closure contained in $\Omega \cap M$.

By a well-known approximation result on manifolds (see e.g. [H]), we may take a sequence $\{f_n\}$ of T-periodic smooth tangent vector fields uniformly converging to f in $\mathbf{R} \times M$. For any $n \in \mathbf{N}$, let

$$w_n(p) = \frac{1}{T} \int_0^T f_n(t, p) dt$$

be the mean value vector field associated to f_n. Clearly, the sequence $\{w_n\}$ converges uniformly to w on M. Moreover, since the zeros of w in $\Omega \cap M$ lie in a compact subset of $G \cap M$, it is easy to see that, for n large enough, the homotopy

$$(p, \tau) \mapsto \tau w_n(p) + (1 - \tau) w(p), \quad 0 \leq \tau \leq 1,$$

is admissible for the degree in $G \cap M$. Thus, $\deg(w_n, G \cap M)$ is well-defined and, by the homotopy invariance property of the degree, it is equal to $\deg(w, G \cap M)$, which, by excision, coincides with $\deg(w, \Omega \cap M)$ and, thus, it is nonzero. Therefore, by the first part of the proof, for n sufficiently large, any equation

$$\dot{x} = \lambda f_n(t, x),$$

has in Ω a connected set of nontrivial T-pairs Γ_n, whose closure in Ω is noncompact and meets $\Omega \cap M$ in $w^{-1}(0)$. Now, since the closure of G is a bounded and complete subset of Ω, any Γ_n must intersect the complement of G in Ω, which implies that, for any n, there exists a pair $(\lambda_n, x_n) \in \partial G \cap \Gamma_n$.

Now, by the definition of T-pair, any function x_n satisfies the condition

$$\dot{x}_n(t) = \lambda_n f_n(t, x_n(t)), \quad \text{for all } t \in [0, T].$$

Therefore, by the compactness of the set $K \subset [0, \infty) \times M$ introduced above, there exists a constant $L > 0$ such that $|\dot{x}_n(t)| \leq L$, for all $n \in \mathbf{N}$ and $t \in [0, T]$. Consequently, because of Ascoli's theorem, the sequence $\{x_n\}$ is totally bounded. Without loss of generality, we may assume $\{(\lambda_n, x_n)\}$ converging to $(\lambda_0, x_0) \in \partial G$. Hence, the sequence $\{\dot{x}_n(t)\}$ converges to the function $\lambda_0 f(t, x_0(t))$ uniformly in $\mathbf{R}$. This implies that x_0 is a T-periodic solution of the differential equation

$$\dot{x} = \lambda_0 f(t, x).$$

Thus, (λ_0, x_0) is a T-pair of (2.1) that, if $\lambda_0 > 0$, clearly belongs to Y. Otherwise, if $\lambda_0 = 0$, then x_0 is a constant function, say $x_0(t) \equiv p_0$. An argument similar to the one used in proving the necessary condition for bifurcation (given in Theorem 2.1) shows that $w(p_0) = 0$, i.e. $p_0 \in Y_0$. Therefore, in any case, $(\lambda_0, x_0) \in \partial G \cap Y$, which is a contradiction. Consequently, a straightforward application of Lemma 2.4 to the pair (Y, Y_0) implies the first part of our assertion.

Assume now that the manifold M is closed and let $\Gamma \subset \Omega$ be the global branch obtained above. Suppose Γ bounded. We need to show that its closure in Ω is not complete. In fact, since Γ is bounded and M is closed, the set $\{(\lambda, x(t)) \in [0, \infty) \times M : (\lambda, x) \in \Gamma,\ t \in [0, T]\}$ is contained in a compact subset of $[0, \infty) \times M$. Hence, because of Ascoli's theorem, Γ is totally bounded. Consequently, the closure of Γ in Ω is not complete since, otherwise, it would be compact. $\square$

We give below some corollaries illustrating the global bifurcation result of Theorem 2.3. We begin with the following Continuation Principle, which is an extension of Theorem 2.4 in [FP3], where the open set Ω_0 has the special form $\Omega_0 = \{x \in C_T(M) : x(t) \in V \text{ for all } t \in [0, T]\}$, with V a relatively compact open subset of M. An extension of the same result in a different direction (i.e. in the case where M is a closed Euclidean Neighborhood Retract in $\mathbf{R}^k$) has been obtained in [CZ], Corollary 2.

Corollary 2.5. *Let f and w be as in Theorem 2.3. Let Ω_0 be a bounded open subset of $C_T(M)$ with complete closure and such that $\{f(t, x(t)) \in \mathbf{R}^k : x \in \Omega_0,\ t \in [0, T]\}$ is bounded. Assume that $\deg(w, \Omega_0 \cap M)$ is defined and nonzero. Then, the equation (2.1) has in $[0, \infty) \times \Omega_0$ a connected branch of nontrivial T-pairs whose closure* *in $[0,\infty) \times C_T(M)$ meets $\Omega_0 \cap M$ in $w^{-1}(0)$ and it is either unbounded with respect to λ or intersects $[0,\infty) \times \partial\Omega_0$. In particular, the equation*

$$\dot{x} = f(t,x)$$

has a T-periodic solution in Ω_0, provided that in addition $w(p) \neq 0$ for all $p \in M \cap \partial\Omega_0$ and the following a priori bound is satisfied:

- *(i) if (λ, x) is a T-pair of (2.1) in $(0,1] \times \bar{\Omega}_0$, then $x \notin \partial\Omega_0$.*

Proof. Apply Theorem 2.3 to the open set $\Omega = [0,\infty) \times \Omega_0$. Then, there exists in Ω a connected bifurcating branch Γ of nontrivial T-pairs whose closure in Ω is noncompact. Suppose Γ bounded with respect to λ. Hence, as in the last part of the proof of Theorem 2.3, Γ turns out to be totally bounded. Consequently, since Ω_0 has complete closure, the closure $\bar{\Gamma}$ of Γ in $[0,\infty) \times C_T(M)$ must be compact. On the other hand, since the closure of Γ in Ω is not compact, if Γ is bounded with respect to λ, its closure in Ω must contain a pair $(\lambda, x) \in [0,\infty) \times \partial\Omega_0$, as claimed.

Assume now that $\bar{\Gamma}$ does not meet $(0,1] \times \partial\Omega_0$. This means that $\bar{\Gamma}$ must intersect either $\{0\} \times \partial\Omega_0$ or $\{1\} \times \Omega_0$. Because of Theorem 2.1, the assumption $w(p) \neq 0$ for all $p \in M \cap \partial\Omega_0$ implies that the first situation does not occur. □

Corollary 2.6. *Let f and w be as in Theorem 2.3. Let M be a closed manifold and U an open subset of M. Assume that* $\deg(w, U)$ *is defined and nonzero. Then, the equation (2.1) admits in $[0,\infty) \times C_T(M)$ a connected branch of nontrivial T-pairs whose closure meets U in $w^{-1}(0)$ and which satisfies at least one of the following properties:*

- *(i) it is unbounded;*
- *(ii) it contains a bifurcation point in $M \setminus U$.*

Proof. Consider the following open subset of $[0,\infty) \times C_T(M)$:

$$\Omega = ([0,\infty) \times C_T(M)) \setminus (M \setminus U).$$

Since $\Omega \cap M = U$, by applying Theorem 2.3 to Ω and by recalling that we have assumed M closed in $\mathbf{R}^k$, we obtain the existence of a bifurcating branch of T-pairs whose closure in Ω is not both bounded and complete. Now, since in this case $[0,\infty) \times C_T(M)$ is a complete metric space, the closure of the branch in $[0,\infty) \times C_T(M)$ is complete. Thus, if bounded, it must contain a bifurcation point in $M \setminus U$. □

In the flat case, i.e. when M is an open subset of $\mathbf{R}^m$, a result which is in the spirit of the above Corollary has been obtained in [FP1] as an application of some abstracts results involving nonlinear compact perturbations of linear Fredholm operators of index zero.

In Theorem 2.1, we have proved that a necessary condition for $p \in M$ to be a bifurcation point is that the mean value vector field w vanishes at p. The following consequence of Corollary 2.6 provides a sufficient condition for a point $p \in M$ to be of bifurcation.

Corollary 2.7. *Let p be a zero of the mean value vector field w. Assume that w is differentiable at p and that $w'(p): T_p(M) \to \mathbf{R}^k$ is one-to-one. Then p is a bifurcation point of the equation (2.1).*

Proof. The assumption $w(p) = 0$ implies that $w'(p)$ maps $T_p(M)$ into itself (see e.g.[M]). Consequently, $w'(p)$ is an isomorphism and p an isolated zero. Thus, there exists an open neighborhood U of p in M such that $\deg(w, U) = \operatorname{sign}\det w'(p) \neq 0$. □

Corollary 2.8 below extends to the continuous case a global result obtained in [FP2] and it represents the infinite dimensional counterpart of Corollary 2.1 in [FP3]. An analogous result has been recently obtained in [C2], with different methods, in the case when f is locally Lipschitz and M is a closed Euclidean Neighborhood Retract in $\mathbf{R}^k$.

Let us recall that, if M is a compact manifold with boundary and $v: M \to \mathbf{R}^k$ is a continuous tangent vector field on M satisfying $v(p) \neq 0$ for all $p \in \partial M$, then the degree of v in M still makes sense. In fact, it suffices to observe that, in this case, v is admissible in the boundaryless manifold $\mathring{M} = M \setminus \partial M$. Hence, one can define $\deg(v, M)$ as the degree of the restriction of v to the interior $\mathring{M}$ of M. The Poincaré-Hopf theorem asserts that this degree equals the Euler-Poincaré characteristic of M, provided v points outward along ∂M.

Corollary 2.8. *Let M be compact and possibly with boundary. Assume that the Euler-Poincaré characteristic $\chi(M)$ of M is nonzero and that f points outward (or inward) along ∂M for any $t \in \mathbf{R}$. Then, the equation (2.1) has an unbounded bifurcating branch whose closure intersects M in $w^{-1}(0)$. In particular, since $C_T(M)$ is bounded, (2.1) has a solution for any $\lambda \geq 0$.*

Proof. If M is boundaryless, then $\deg(w) = \chi(M) \neq 0$, where w is the mean value tangent vector field associated with f. If $\partial M \neq \emptyset$ and f points outward along ∂M for each t, then w points outward as well, so that again one has $\deg(w) = \chi(M) \neq 0$. If f and, thus, w are inward, then the vector field $-w$ is outward. Therefore, by recalling that $\deg(-w) = (-1)^m \deg(w)$, where $m = \dim M$, one can deduce that, still in this case, $\deg(w)$ is defined and nonzero. Hence, Theorem 2.3 applies in the boundaryless manifold $\mathring{M}$ yielding the existence in $[0, \infty) \times C_T(\mathring{M})$ of a global bifurcating branch whose closure meets M in $w^{-1}(0)$. Now, if $\partial M = \emptyset$, to get the assertion it suffices to observe that, since M is closed, the branch must be unbounded. Otherwise, if $\partial M \neq \emptyset$, the fact that the vector field f is never tangent to ∂M implies that there are no T-periodic orbits of (2.1) which may hit ∂M. Therefore, the closure in $[0, \infty) \times C_T(\mathring{M})$ of the obtained branch coincides with its closure in $[0, \infty) \times C_T(M)$ and thus, it is complete. Consequently, also in this case the bifurcating branch turns out to be unbounded. □

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