Global Bifurcation of Fixed Points And the Poincaré Translation Operator on Manifolds

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ABSTRACT. Let M be a differentiable manifold and $\varphi\colon [0,\infty) \times M \to M$ be a C^1 map satisfying the condition $\varphi(0,p) = p$ for all $p \in M$. Among other results, we prove that when the degree (also called Hopf index or Euler characteristic) of the tangent vector field $w\colon M \to TM$, given by $w(p) = \frac{\partial\varphi}{\partial\lambda}(0,p)$, is well defined and nonzero, then the set (of nontrivial pairs)

$$S_+ = \{(\lambda,p) : \varphi(\lambda,p) = p, \quad \lambda > 0\}$$

admits a connected subset whose closure is not compact and meets the slice $\{0\} \times M$ of $[0,\infty) \times M$. This extends known results regarding the existence of harmonic solutions of periodic ordinary differential equations on manifolds.

0. Introduction

Let M be a boundaryless m-dimensional differentiable manifold in $\mathbf{R}^k$, and consider a smooth T-periodic tangent vector field $f\colon \mathbf{R} \times M \to \mathbf{R}^k$ on M (i.e. $f(t,p) \equiv f(t+T,p)$, and f(t,p) is tangent to M at p for all $(t,p) \in \mathbf{R} \times M$). In [FP1] we studied the one parameter family of periodic problems

$$\dot{x} = \lambda f(t,x), \quad \lambda \geq 0,\ x \in M, \tag{0.1}$$

$$x(0) = x(T), \tag{0.2}$$

and we proved a result (see Theorem 3.8 below) which, for simplicity, we state here in a reduced, self-contained version.

Theorem 0.1. *If the degree of the (autonomous) vector field $w\colon M \to \mathbf{R}^k$, given by*

$$w(p) = \frac{1}{T}\int_0^T f(t,p)dt,$$

is defined and nonzero, then there exists a connected subset Σ of $(0,\infty)\times M$ with the following properties:

(1) if $(\lambda,p)\in\Sigma$, then the solution $x(\cdot)$ of (0.1) satisfying $x(0)=p$ is T-periodic;

(2) the closure of Σ in $[0,\infty)\times M$ intersects $\{0\}\times M$ (i.e. contains a bifurcation point for the T-periodic problem (0.1)-(0.2));

(3) Σ is global, in the sense that is not contained in any compact subset of the open set $D=\{(\lambda,p)\in[0,\infty)\times M :$ the solution $x(\cdot)$ of (0.1) satisfying $x(0)=p$ is defined in the whole interval $[0,T]\}$.

We point out that in Theorem 0.1 the manifold M need not be compact; it could be, for example, any open subset of $\mathbf{R}^m$. However, in the particular case when M is compact, its Euler-Poincaré characteristic $\chi(M)$ is well defined and, by the well known Poincaré-Hopf theorem, coincides with the degree of any tangent vector field on M. Moreover, in this case, the open set D is precisely $[0,\infty)\times M$. Therefore, if $\chi(M)\neq 0$, then Theorem 0.1 implies the existence of an unbounded connected branch Σ satisfying (1) and (2).

Even though the proof of Theorem 0.1 given in [FP1] is too involved, the result seems to have some interesting consequences. One of these, for example, is the fact, proved in [FP2], that

any forced frictionless spherical pendulum admits harmonic oscillations.

In this paper we simplify the proof of Theorem 0.1, which is based on a formula for the computation of the fixed point index of the Poincaré T-translation operator associated with the equation (0.1). Namely, given a relatively compact open subset U of M, if there are no zeros of the vector field

$$w(p)=\frac{1}{T}\int_0^T f(t,p)dt$$

on the boundary of U, then for any $\lambda>0$ sufficiently small the fixed point index in U of the T-translation operator φ_λ is well defined and coincides with the degree of the vector field $-w$ in U. Therefore, one can write (see [FP1])

$$\lim_{\lambda\to 0^+}\operatorname{ind}(\varphi_\lambda,U)=\deg(-w,U). \tag{0.3}$$

We shall extend this formula to any C^1 map $\varphi\colon D\to M$ defined on an open subset D of $[0,\infty)\times M$ containing $\{0\}\times M$ and satisfying the assumption $\varphi(0,p)=p$ for all $p\in M$. In this extended context the tangent vector $w(p)$ at a given point $p\in M$ is just the derivative at $\lambda=0$ of the curve $\lambda\mapsto\varphi_\lambda(p)$.

Given $\varphi\colon D\to M$ as above, the fixed pairs of the map $(\lambda,p)\mapsto(\lambda,\varphi_\lambda(p))$ corresponding to $\lambda>0$ are called *nontrivial*, and an element $p_0\in M$ is a *bifurcation point* for the equation $\varphi_\lambda(p)=p$ if any neighborhood of $(0,p_0)$ contains a nontrivial fixed pair. As a consequence of (0.3) the assumption $\deg(-w, U) \neq 0$ implies the existence of fixed points for φ_λ in U, provided that $\lambda > 0$ is sufficiently small. Thus, since U is relatively compact in M, one gets the existence of a bifurcation point p_0 in the closure $\bar{U}$ of U. We shall see that a necessary condition for $p_0 \in M$ to be a bifurcation point is $w(p_0) = 0$, thus the assumption $w(p) \neq 0$ on the boundary ∂U of U implies $po \in U$.

Observe that the integer

$$\mathrm{i}(\varphi, U) = \lim_{\lambda \to 0^+} \mathrm{ind}(\varphi_\lambda, U)$$

may happen to be well defined even in the case when M is simply a metrizable Absolute Neighborhood Retract (ANR) and $\varphi \colon D \to M$ is a (continuous) locally compact map. This integer, when defined (in a sense to be specified later) will be called *bifurcation index* of φ in U. Inspired by the proof of Theorem 0.1, we shall show that, when $\mathrm{i}(\varphi, U)$ is well defined and different from zero, there exists a connected set Σ of nontrivial fixed pairs, whose closure meets $\{0\} \times U$ and which is not contained in any compact subset of D.

1. The Bifurcation Index

In this section we will introduce an index (in $\mathbf{Z}$) for the bifurcation of fixed points of a suitable one parameter family of locally compact maps $\{\varphi_\lambda\}$, $\lambda \geq 0$, defined on open subsets of a metrizable ANR. We shall prove that this integer has properties analogous to those of the fixed point index. For a definition of bifurcation index in a completely different context (multiparameter bifurcation), see for example [Ba] and references therein.

Regarding the space where the family $\{\varphi_\lambda\}$ is considered, the most interesting model we have in mind is a finite dimensional differentiable manifold, which is a very special ANR. Therefore, for the moment, we restrict our attention to the case when the space M is a locally compact metric ANR and $\varphi \colon D \to M$ is a continuous map defined on an open subset D of $[0, \infty) \times M$, that, for the sake of simplicity, we assume containing $\{0\} \times M$. In Remark 1.1 below we sketch how this index can be extended to the more general situation where M is a metrizable ANR and φ a locally compact map.

For any $\lambda \geq 0$ and any subset X of $[0, \infty) \times M$, denote by X_λ the slice of X at λ, i.e.

$$X_\lambda = \{p \in M : (\lambda, p) \in X\}.$$

By $\varphi_\lambda \colon D_\lambda \to M$ we mean the partial map $\varphi_\lambda(\cdot) = \varphi(\lambda, \cdot)$.

Consider the equation

$$\varphi(\lambda, p) = p. \tag{1.1}$$

An element $p_0 \in M$ is said to be a *bifurcation point* of (1.1) provided that in any neighborhood of $(0, p_0)$ there exists a solution (λ, p) of (1.1) with $\lambda > 0$. The set of bifurcation points of (1.1), which is clearly a closed subset of M, will be denoted by $B(\varphi)$. Since φ is continuous, $B(\varphi)$ is a subset of $\{p \in M : \varphi(0, p) = p\}$, called the set of *trivial fixed points* of (1.1).

Let U be a relatively compact open subset of M. Assume that φ is *strongly admissible* in U, that is $B(\varphi) \cap \partial U = \emptyset$. Then there exists $\varepsilon > 0$ such that, for any $0 < \lambda \leq \varepsilon$, one has:

i) $\bar{U} \subset D_\lambda$;

ii) $\varphi_\lambda(p) \neq p$ for any $p \in \partial U$.

Assertion i) is obvious, since $\bar{U}$ is compact and D is an open set containing $\{0\} \times \bar{U}$. As regards ii), suppose by contradiction that there exists a sequence $\{(\lambda_n, p_n)\}$ in D such that $\lambda_n \to 0$, $\lambda_n > 0$, $p_n \in \partial U$ and $\varphi(\lambda_n, p_n) = p_n$. Since ∂U is compact, without loss of generality we may assume $p_n \to p_0 \in \partial U$. Therefore, p_0 belongs to the set $B(\varphi) \cap \partial U$, which, by assumption, is empty.

Hence, i) and ii) imply that, for $0 < \lambda \leq \varepsilon$, the fixed point index $\operatorname{ind}(\varphi_\lambda, U)$ is well-defined and independent of λ (see e.g. [Br], [G], [N] and references therein). Thus, it makes sense to consider the integer

$$\lim_{\lambda \to 0^+} \operatorname{ind}(\varphi_\lambda, U).$$

This will be called *bifurcation index* of φ in U and denoted by $\mathrm{i}(\varphi, U)$.

The following properties of the bifurcation index are easy to check (the first two can be deduced from the third one).

Solution. *If* $\mathrm{i}(\varphi, U) \neq 0$, *then equation (1.1) has a bifurcation point in* U.

Excision. *Let* $V \subset U$ *be an open subset of* U *such that* $B(\varphi) \cap (\bar{U} \setminus V) = \emptyset$. *Then* $\mathrm{i}(\varphi, U) = \mathrm{i}(\varphi, V)$.

Additivity. *Let* U_i, $i = 1, 2$, *be relatively compact open subsets of* M *such that* $\partial U_i \cap B(\varphi) = \emptyset$, $i = 1, 2$. *Assume that* $U_1 \cap U_2$ *does not contain bifurcation points. Then* $\mathrm{i}(\varphi, U_1 \cup U_2)$ *is well defined and*

$$\mathrm{i}(\varphi, U_1 \cup U_2) = \mathrm{i}(\varphi, U_1) + \mathrm{i}(\varphi, U_2).$$

Homotopy invariance. *Let* $H\colon D \times [0,1] \to M$ *be continuous and denote by* $H_\mu\colon D \to M$ *the map* $H_\mu(\lambda, p) = H(\lambda, p, \mu)$. *Let* U *be a relatively compact open subset of* M *and assume that* $B(H_\mu) \cap \partial U = \emptyset$ *for any* $\mu \in [0,1]$. *Then* $\mathrm{i}(H_\mu, U)$ *is independent of* $\mu \in [0,1]$.

Let us prove the following additional property.

Normalization. *If* M *is a compact ANR and* φ_0 *is the identity on* M, *then the bifurcation index* $\mathrm{i}(\varphi, M)$ *coincides with* $\chi(M)$, *the Euler-Poincaré characteristic of* M.

Proof. Since M is compact and D is open in $[0, \infty) \times M$, we have $D_\lambda = M$ for λ in some interval $[0, \varepsilon)$. By the homotopy invariance of the fixed point index, it follows that

$\operatorname{ind}(\varphi_\lambda, M) = \operatorname{ind}(\varphi_0, M)$ for $0 \le \lambda < \varepsilon$. On the other hand, since φ_0 is the identity on M, it is well-known that $\operatorname{ind}(\varphi_0, M)$ equals $\chi(M)$. Thus,

$$\mathrm{i}(\varphi, M) = \lim_{\lambda \to 0^+} \operatorname{ind}(\varphi_\lambda, M) = \chi(M)$$

as claimed. □

Now, let U be an arbitrary open subset of M and suppose that φ is *admissible* in U, that is $B(\varphi) \cap U$ is compact. Then the notion of bifurcation index of φ in U still makes sense. In fact, according to the excision property quoted above, one can define $\mathrm{i}(\varphi, U)$ to be equal to $\mathrm{i}(\varphi, V)$, where V is any relatively compact open subset of U such that $\bar{V} \subset U$ and $B(\varphi) \cap U \subset V$.

Remark 1.1. We point out that the assumption that M is a locally compact space could be relaxed. In fact, assume that M is a metrizable ANR and $\varphi\colon D \to M$ is a locally compact map on an open subset D of $[0, \infty) \times M$ (not necessarily satisfying the condition $\{0\} \times M \subset D$). In this case, given U open in M, the fixed point index of φ_λ in U is well defined whenever $U \subset D_\lambda$ and $\{p \in U : \varphi_\lambda(p) = p\}$ is compact. In this more general situation one can still introduce the notion of bifurcation index, $\mathrm{i}(\varphi, U)$, provided that $B(\varphi) \cap U$ is compact and contained in D_0 (we say that φ is *admissible* in U). It is enough to restrict φ_λ to any open subset V of U containing $B(\varphi) \cap U$, with the property $[0, \delta) \times V \subset D$ and $\varphi([0, \delta) \times V)$ relatively compact in M for some $\delta > 0$. One can show, in fact, that $\operatorname{ind}(\varphi_\lambda, V)$ is well defined and independent of λ in a sufficiently small interval $(0, \varepsilon)$. Thus, as before, let

$$\mathrm{i}(\varphi, V) = \lim_{\lambda \to 0^+} \operatorname{ind}(\varphi_\lambda, V).$$

The excision property of the fixed point index allows us to define $\mathrm{i}(\varphi, U) = \mathrm{i}(\varphi, V)$, where V is any open set as above.

2. Global bifurcation

Our aim below is to give a global bifurcation result for the equation (1.1). To this end we need to introduce some terminology.

Let M be a metrizable ANR and $\varphi\colon D \to M$ a locally compact (continuous) map defined on an open subset D of $[0, \infty) \times M$, which for simplicity we assume contains $\{0\} \times M$. Let S denote the subset of D of all the solutions (λ, p) of equation (1.1). Observe that S is a (relatively) closed locally compact subset of D, since it coincides with the set of fixed points of the locally compact map $(\lambda, p) \mapsto (\lambda, \varphi(\lambda, p))$. For simplicity, we will regard M as a (closed) subset of $[0, \infty) \times M$, via the embedding $p \mapsto (0, p)$. In this context, the set S_0 of fixed points of $\varphi_0\colon M \to M$ will be identified with the set $\{0\} \times S_0$ of *trivial solutions* of (1.1). Consequently, we will refer to $S \setminus M = S \setminus S_0$ as the set of *nontrivial solutions*. According to this terminology, a point of M is a bifurcation point if and only if it lies in the closure of $S \setminus M$. Thus, since S is closed in D (and $M \subset D$), any bifurcation point is in S_0.

A *bifurcating branch* for (1.1) will be a (connected) component of $S \setminus M$ whose closure in S intersects M. A *global bifurcating branch* is a bifurcating branch whose closure in S is not compact. More generally, given an open subset W of D, the notions of *relative to W bifurcating branch* and *relative to W global bifurcating branch* are obtained just replacing (in the previous definition) S with $S \cap W$. To understand the meaning of this, observe that the closure in $S \cap W$ of a set $\Sigma \subset S \cap W$ is not compact if and only if its closure in $[0, \infty) \times M$ is not contained in any compact subset of W. Moreover, a bifurcation point $p \in M$ belongs to the closure in $S \cap W$ of Σ if and only if p is in the closure of Σ in $[0, \infty) \times M$ and $p \in M \cap W = W_0$.

The following global result is a consequence of the properties of the bifurcation index.

Theorem 2.1. *Let $\varphi \colon D \to M$ be as above and let W be an open subset of D. Assume that φ is admissible in the slice W_0 and that the bifurcation index $\mathrm{i}(\varphi, W_0)$ is nonzero. Then the equation (1.1) admits a relative to W global bifurcating branch; that is, a connected subset of $\{(\lambda, p) \in W : \lambda > 0,\ \varphi(\lambda, p) = p\}$ whose closure meets $\{0\} \times W_0$ and is not contained in any compact subset of W.*

The connectivity result stated below will be crucial in the proof of Theorem 2.1. For the sake of completeness we shall repeat here the simple proof given in [FP3].

Lemma 2.2. *Let Y be a locally compact Hausdorff space and Y_0 a compact subset of Y. Assume that any compact subset of Y containing Y_0 has nonempty boundary. Then $Y \setminus Y_0$ contains a connected set whose closure is not compact and intersects Y_0.*

Proof. Let us adjoin to Y a point ∞ and define a compact Hausdorff topology on $\hat{Y} = Y \cup \{\infty\}$ by taking the complements of the compact sets as open neighborhoods of ∞. Our assertion is now equivalent to proving the existence of a connected subset of $\hat{Y} \setminus (Y_0 \cup \{\infty\})$ whose closure contains ∞ and intersects Y_0. Suppose such a connected set does not exist. Then, since $\hat{Y}$ is a compact Hausdorff space, by a well-known point set topology result (see e.g. [A] and references therein), Y_0 and $\{\infty\}$ are separated in $\hat{Y}$, i.e. there exist two compact subsets C_0, C_∞ of $\hat{Y}$ such that $Y_0 \subset C_0$, $\infty \in C_\infty$, $C_0 \cap C_\infty = \emptyset$, $C_0 \cup C_\infty = \hat{Y}$. So, C_0 is a compact subset of Y containing Y_0 with empty boundary, a contradiction. Therefore, the existence of the required connected set is proved. □

Proof of Th. 2.1. Set $F = (S \setminus M) \cup B(\varphi)$ and denote $Y = F \cap W$. It is easy to see that the slice Y_0 of Y at $\lambda = 0$ coincides with the set of bifurcation points of φ lying in W_0, i.e. $Y_0 = B(\varphi) \cap W_0$. Thus, by the assumption $\mathrm{i}(\varphi, W_0) \neq 0$, we obtain that Y_0 is a nonempty (compact) set. Moreover, via the embedding $p \mapsto (0, p)$, we may regard Y_0 as a subset of Y. Observe that Y is locally compact. In fact, F is locally compact, as a closed subset of the locally compact set S; consequently, Y itself is locally compact, since it is open in F. Let us apply Lemma 2.2 to the pair (Y, Y_0) defined above. To this end, we have to show that any compact subset of Y containing Y_0 has nonempty boundary. By contradiction, suppose there exists in Y a compact open neighborhood C of Y_0. Then we can find in $[0, \infty) \times M$ an open set $\tilde{W} \subset W$ such that $\tilde{W} \cap Y = C$. The generalized homotopy property of the fixed point index implies that, for any $\lambda > 0$, $\operatorname{ind}(\varphi_\lambda, \tilde{W}_\lambda)$ is well-defined and independent of λ. Moreover, since the slice C_λ is empty for λ sufficiently large, the above index must be zero.

On the other hand, by the compactness of C, there exist an open neighborhood V of Y_0 in M and $\delta > 0$ such that $[0, \delta) \times V \subset \tilde{W}$, $\varphi([0, \delta) \times V)$ is relatively compact in M and $C_\lambda \subset V$ for all $\lambda \in [0, \delta)$. Thus, recalling the definition of bifurcation index given above and the excision property of the fixed point index, we obtain

$$\operatorname{ind}(\varphi_\lambda, \tilde{W}_\lambda) = \operatorname{ind}(\varphi_\lambda, V) = \mathrm{i}(\varphi, V) = \mathrm{i}(\varphi, W_0) \neq 0.$$

This contradiction shows that Lemma 2.2 applies to the pair (Y, Y_0). Consequently, $Y \setminus Y_0$ contains a component which is not relatively compact in Y and whose closure intersects Y_0. To conclude, it suffices to prove that the closure of such a component in Y coincides with its closure in $S \cap W$. This is a consequence of the fact that Y is closed in $S \cap W$. □

Corollary 2.3. *Let U be an open subset of M and let φ be admissible in U. Assume $\mathrm{i}(\varphi, U) \neq 0$. Then the equation (1.1) admits a connected set of nontrivial solutions whose closure in $[0, \infty) \times M$ contains a bifurcation point in U and is either not contained in any compact subset of the domain D of φ or intersects M in a bifurcation point outside U.*

Proof. As above, we will regard M as a (closed) subset of the domain D of φ. Then $M \setminus U$ is a closed subset of D. Hence, by applying Theorem 2.1 to the open set $\hat{U} = D \setminus (M \setminus U)$, we obtain a relative to $\hat{U}$ global bifurcating branch. Assume that the closure C in $[0, \infty) \times M$ of such a branch is a compact subset of D. By the definition of global branch, the set $C \cap \hat{U}$ is not contained in any compact subset of $\hat{U}$. This implies that C must intersect M outside U. □

Theorem 2.4. *Let Σ be a compact component of $(S \setminus M) \cup B(\varphi)$ and let Σ_0 denote the slice of Σ at $\lambda = 0$. Let U be an open subset of M such that $B(\varphi) \cap U = \Sigma_0$. Then $\mathrm{i}(\varphi, U) = 0$.*

Proof. For contradiction, assume $\mathrm{i}(\varphi, U) \neq 0$. As in Corollary 2.3, let us associate to U the open set $\hat{U} = D \setminus (M \setminus U)$. By Theorem 2.1, the equation (1.1) admits a relative to $\hat{U}$ global bifurcating branch whose closure C in $S \cap \hat{U}$ is clearly connected and contains a bifurcation point $p \in U$. By assumption, $B(\varphi) \cap U = \Sigma_0$. So $p \in \Sigma_0$. Since Σ is a component in $\hat{U}$, it turns out that C is a compact subset of Σ. This contradicts the notion of global branch. □

Let $p \in M$ be an isolated bifurcation point of equation (1.1). Then one can define the *bifurcation index of φ at p*, $\mathrm{i}(\varphi, p)$, to be the bifurcation index of φ in any open neighborhood U of p in M such that $U \cap B(\varphi) = \{p\}$. By making use of the above definition, one can state the following

Corollary 2.5. *Let Σ be as in Theorem 2.4 and assume $\Sigma_0 = \{p_1, p_2\}$. Then $\mathrm{i}(\varphi, p_1) = -\mathrm{i}(\varphi, p_2)$.*

3. Periodic Orbits on Manifolds

In this section, let us assume that M is a boundaryless m-dimensional smooth manifold embedded in $\mathbf{R}^k$ and, for any $p \in M$, let $T_p(M) \subset \mathbf{R}^k$ denote the tangent space of M at p. Let $T(M)$ denote the tangent bundle of M, i.e. the $2m$-differentiable submanifold

$$T(M) = \{(p, v) \in \mathbf{R}^k \times \mathbf{R}^k : p \in M,\ v \in T_p(M)\}$$

of $\mathbf{R}^k \times \mathbf{R}^k$, containing a natural copy M_0 of M, via the embedding $p \mapsto (p, 0)$.

If $p \in M$, by $N_p(M)$ we denote the orthogonal space $T_p(M)^\perp$ of $T_p(M)$ in $\mathbf{R}^k$. If Z is a submanifold of M and $p \in Z$, the orthogonal space of $T_p(Z)$ in $T_p(M)$ is denoted by $N_p(Z, M)$. The normal bundle of Z in $M \subset \mathbf{R}^k$ is the subset of $\mathbf{R}^k \times \mathbf{R}^k$ given by

$$N(Z, M) = \{(p, v) \in \mathbf{R}^k \times \mathbf{R}^k : p \in Z,\ v \in N_p(Z, M)\}.$$

As in Section 1, let $\varphi\colon D \to M \subset \mathbf{R}^k$ be a map defined on an open subset of $[0, \infty) \times M$ containing the section $\{0\} \times M$. Assume φ to be of class C^1 and such that the fixed point problem $\varphi(\lambda, p) = p$ degenerates for $\lambda = 0$, i.e. $\varphi(0, p) = p$, for any $p \in M$. Then one can consider on M the continuous tangent vector field w which associates to any $p \in M$ the vector

$$w(p) = \frac{\partial\varphi}{\partial\lambda}(0, p) \in T_p(M).$$

We have the following necessary condition for bifurcation.

Theorem 3.1. *Let φ and w be as above. Then a necessary condition for $p \in M$ to be a bifurcation point of the equation (1.1) is that $w(p) = 0$.*

Proof. Let $\{(\lambda_n, p_n)\}$ be a sequence in D such that $p_n \to p$, $\lambda_n \to 0$, $\lambda_n > 0$, and $\varphi(\lambda_n, p_n) = p_n$. Given $n \in \mathbf{N}$, define $\psi_n\colon [0, 1] \to \mathbf{R}^k$ by $\psi_n(s) = \varphi(s\lambda_n, p_n)$, and observe that $\psi_n(1) - \psi_n(0) = 0$. Then

$$0 = \frac{\psi_n(1) - \psi_n(0)}{\lambda_n} = \int_0^1 \frac{\partial\varphi}{\partial\lambda}(s\lambda_n, p_n)ds.$$

Since the right-hand side of the above equality converges to $\int_0^1 \frac{\partial\varphi}{\partial\lambda}(0, p)ds$, we obtain $w(p) = 0$. $\square$

Our aim below is to prove a sufficient condition for bifurcation, in terms of the associated vector field. To this end, we need to recall some preliminaries from both Degree Theory and Intersection Theory.

Let $w\colon M \to \mathbf{R}^k$ be a continuous tangent vector field on M which is *admissible,* i.e. such that the set $\{p \in M : w(p) = 0\}$ is compact. Then one can associate to w an integer $\chi(w)$, called the Euler characteristic (or Hopf index, or degree) of w, which, roughly speaking, counts (algebraically) the number of zeros of w (see e.g. [GP], [H], [M], [T], and references therein). As a consequence of the Poincaré-Hopf theorem, when M is compact, this integer equals $\chi(M)$, the Euler-Poincaré characteristic of M. On the other hand, in the particular case when M is an open subset of $\mathbf{R}^m$, $\chi(w)$ is just the Brouwer degree (with respect to zero) of the map $w\colon M \to \mathbf{R}^m$. Moreover, all standard properties of the Brouwer degree on open subsets of Euclidean spaces, such as homotopy invariance, excision, additivity, existence, etc., are still valid in the more general context of differentiable manifolds. To see this, one can use an equivalent definition of Euler characteristic of a vector field based on fixed point index theory given in [FP1]. Let us point out that no orientability of M is required for the Euler characteristic of a tangent vector field to be defined.

In what follows, to keep in mind that the Euler characteristic of a tangent vector field w on M, reduces, in the flat case, to the classical Brouwer degree (with respect to zero), $\chi(w)$ will be called the *(global) degree* of the vector field w and denoted by $\deg(w)$. Since any open subset U of a manifold M is still a manifold, the degree of the restriction of w to U makes sense provided that w is admissible on U, i.e. the set $\{p \in U : w(p) = 0\}$ is compact. The degree of such a restriction will be denoted by $\deg(w, U)$.

Let us now give a short summary of the notions of Intersection Theory that we are going to use in the sequel.

Let M and N be finite dimensional boundaryless manifolds, $f\colon M \to N$ a continuous map, and $Z \subset N$ a closed submanifold of N without boundary. Assume that the codimension of Z in N equals the dimension of M and that both M and the normal bundle $N(Z,N)$ of Z in N are oriented. If the set $f^{-1}(Z)$ is compact – we say that f is Z-admissible – then one can associate to the (admissible) pair (f,Z) an integer, $\mathrm{in}(f,Z)$, called the intersection number of f with Z, which, roughly speaking, counts algebraically the number of intersections of the image of f with Z (see e.g. [GP], [H]). This integer, in the particular case when Z reduces to a point $q \in N$ and both M and $T_q(N)$ are oriented (and have the same dimension), is just the Brouwer degree of f with respect to q (see e.g. [M]).

For the sake of completeness, we give here a brief idea of how to define the intersection number. Even if we do not assume that M is a compact manifold, as in [GP] and [H], the arguments that one can find in these nice textbooks may easily be adapted to this more general situation. In some cases we do not even require that M and $N(Z,N)$ are oriented (or orientable). As we shall see later, this requirement is not needed for instance when an orientation of M at any point $p \in f^{-1}(Z)$ determines uniquely an orientation of the vector bundle $N(Z,N)$ at $f(p)$.

To define $\mathrm{in}(f,Z)$, assume first that f is smooth and transverse to Z. That is, for any $p \in M$ such that $f(p) = q \in Z$, the following composite linear map is surjective (and hence, in our case, an isomorphism):

$$T_p(M) \xrightarrow{f'(p)} T_q(N) \xrightarrow{\pi_q} N_q(Z,N),$$

where π_q denotes the orthogonal projection of $T_q(N)$ onto the subspace $N_q(Z,N)$.

The inverse function theorem implies that the set $f^{-1}(Z)$ is discrete and, thus, necessarily finite, since it is compact by assumption. In this case, the intersection number is just the sum

$$\operatorname{in}(f, Z) = \sum_{p \in f^{-1}(Z)} \operatorname{sign} \pi_{f(p)} f'(p),$$

where $\operatorname{sign}(\pi_{f(p)} f'(p))$ is $+1$ or -1 according to whether or not the composite isomorphism $\pi_{f(p)} f'(p)$ preserves or inverts the orientations. We observe that, when the two spaces $T_p(M)$ and $N_{f(p)}(Z, N)$ can be canonically identified, no orientation is required in order to define the integer $\operatorname{sign}(\pi_{f(p)} f'(p))$. As we shall see later, this happens for the definitions, based on intersection theory, of the Euler-Poincaré characteristic of a (not necessarily orientable) compact manifold, of the Lefschetz number of a self mapping, of the fixed point index, and of the degree of a tangent vector field.

The above defined integer turns out to be invariant under smooth Z-admissible homotopies. That is, if $H: M \times [0,1] \to N$ is smooth and $H^{-1}(Z)$ is compact, then

$$\operatorname{in}(H(\cdot, 0), Z) = \operatorname{in}(H(\cdot, 1), Z),$$

provided that both $H(\cdot, 0)$ and $H(\cdot, 1)$ are transverse to Z. Moreover, if U is any open subset of M containing $f^{-1}(Z)$, one has $\operatorname{in}(f, Z) = \operatorname{in}(f|_U, Z)$, where $f|_U$ denotes the restriction of f to U.

The general case when f is a continuous Z-admissible map is carried out as follows. Recall first that any continuous map $f: M \to N$ can be uniformly approximated by smooth maps which are transverse to Z (again, see e.g. [GP], [H]). Take any relatively compact open subset U of M containing $f^{-1}(Z)$ and define $\operatorname{in}(f, Z)$ to be the intersection number $\operatorname{in}(g|_U, Z)$ of a "sufficiently close" smooth approximation $g: \bar{U} \to N$ of f, which is transverse to f. How close g must be to f depends on the distance (in the space $\mathbf{R}^s$ containing N) between $f(\partial U)$ and Z. We observe that this distance turns out to be positive, since $f(\partial U)$ is compact and Z is closed. To convince oneself that this is a good definition, we recall that any smooth manifold N in $\mathbf{R}^s$ is a smooth neighborhood retract in $\mathbf{R}^s$, and this implies that, given two sufficiently close smooth maps $g_1, g_2: \bar{U} \to N$, one can join them by a smooth homotopy which turns out to be Z-admissible in U. Moreover, this definition does not depend on the choice of the open relatively compact set U containing $f^{-1}(Z)$. In fact, if U_1 and U_2 are relatively compact open subsets of M containing $f^{-1}(Z)$, and $g: \bar{U}_1 \cup \bar{U}_2 \to N$ is sufficiently close to f, one has $g^{-1}(Z) \subset U_1 \cap U_2$. This is a consequence of the fact that $f(\bar{U}_1 \cup \bar{U}_2 \setminus U_1 \cap U_2)$ is a compact set which does not intersect Z.

The following are the main properties of the intersection number.

Solution. *If* $\operatorname{in}(f, Z) \neq 0$ *then* $f^{-1}(Z) \neq \emptyset$.

Excision. *If* U *is an open subset of* M *containing* $f^{-1}(Z)$, *then* $\operatorname{in}(f, Z) = \operatorname{in}(f|_U, Z)$.

Additivity. *If* U_1 *and* U_2 *are open in* M, $U_1 \cap f^{-1}(Z)$ *and* $U_2 \cap f^{-1}(Z)$ *are compact, and* $U_1 \cap U_2 \cap f^{-1}(Z)$ *is empty, then* $\operatorname{in}(f|_{U_1 \cup U_2}, Z) = \operatorname{in}(f|_{U_1}, Z) + \operatorname{in}(f|_{U_2}, Z)$.

Homotopy invariance. *If* $H: M \times [0,1] \to N$ *is continuous and* Z*-admissible (i.e.* $H^{-1}(Z)$ *is compact), then* $\operatorname{in}(H(\cdot, \mu), Z)$ *does not depend on* $\mu \in [0,1]$.

Lemma 3.2 below will be used in the proof of Theorem 3.3.

Lemma 3.2. *Let U and N_1 be finite dimensional boundaryless manifolds, $f\colon U \to N_1$ a continuous map and Z_1 a closed submanifold of N_1 without boundary. Assume that the codimension of Z_1 in N_1 equals the dimension of U and that both U and the normal bundle $N(Z_1, N_1)$ are oriented. Let $h\colon N_1 \to N_2$ be a smooth map transverse to a closed boundaryless submanifold Z_2 of N_2 such that $Z_1 = h^{-1}(Z_2)$. If $N(Z_2, N_2)$ is oriented according to the bundle isomorphism $N(Z_1, N_1) \to N(Z_2, N_2)$ induced by the derivative of h, then the intersection numbers* $\mathrm{in}(f, Z_1)$ *and* $\mathrm{in}(hf, Z_2)$ *are equal, provided that they are defined (i.e., $f^{-1}(Z_1) = (hf)^{-1}(Z_2)$ is compact).*

Proof. Without loss of generality, we may assume f smooth and transverse to Z_1. Therefore, since f is Z_1-admissible, the set $f^{-1}(Z_1)$ is finite. By the definition of intersection number, it suffices to show that, given $p \in f^{-1}(Z_1) = (hf)^{-1}(Z_2)$, one has $\mathrm{sign}(\pi_1 f'(p)) = \mathrm{sign}(\pi_2 h'(f(p))f'(p))$, where π_1 and π_2 denote the orthogonal projections of $T_{f(p)}(N_1)$ and $T_{h(f(p))}(N_2)$ onto $N_{f(p)}(Z_1, N_1)$ and $N_{h(f(p))}(Z_2, N_2)$ respectively. This is a straightforward consequence of the choice of orientation on $N(Z_2, N_2)$. $\square$

Let us now go back to the situation considered at the beginning of this section. The next result provides a relationship between bifurcation index and degree and turns out to be useful in obtaining the sufficient condition for global bifurcation stated in the sequel (Theorem 3.6).

Theorem 3.3. *Let $\varphi\colon D \subset [0, \infty) \times M \to M$ be a C^1 map satisfying $\varphi(0, p) = p$ and let U be an open subset of M. Assume that the vector field $w\colon M \to \mathbf{R}^k$ given by*

$$w(p) = \frac{\partial \varphi}{\partial \lambda}(0, p)$$

is admissible in U. Then φ is admissible in U and $\mathrm{i}(\varphi, U) = \deg(-w, U)$.

Proof. By Theorem 3.1, w admissible in U implies φ admissible in U. Without loss of generality we may assume U to be relatively compact and such that $w(p) \neq 0$ for $p \in \partial U$. Thus, $B(\varphi) \cap \partial U = \emptyset$. Hence, if $\varphi_\lambda\colon \bar{U} \to M$ denotes the map $\varphi_\lambda(\cdot) = \varphi(\lambda, \cdot)$, one obtains, as already observed at the very beginning of Section 1, $\varphi_\lambda(p) \neq p$ for all $p \in \partial U$ and λ sufficiently small, say $0 < \lambda \leq \varepsilon$.

By making use of Lemma 3.2, let us show that the intersection numbers of two convenient maps are equal (equality (3.1) below). To this end assume first M oriented. Consider the product manifold $M \times M$ and its diagonal $\Delta \subset M \times M$. Orient the normal bundle of Δ as follows. For any $(p, p) \in \Delta$ assign to the subspace $E_p = \{(u, 0) \in \mathbf{R}^k \times \mathbf{R}^k : u \in T_p(M)\}$ of $T_{(p,p)}(M \times M)$ the orientation inherited by $T_p(M)$ via the isomorphism $u \mapsto (u, 0)$. Since E_p is a complement of $T_{(p,p)}(\Delta)$ in $T_{(p,p)}(M \times M)$, it is canonically isomorphic to the normal space $N_{(p,p)}(\Delta, M \times M)$. Consequently, the given orientation in E_p induces an orientation on $N_{(p,p)}(\Delta, M \times M)$. Let us point out that, with this choice, given any constant map $p \mapsto p_0$ on M, the intersection number of the associated graph map $p \mapsto (p, p_0)$ with the diagonal turns out to be 1. Now, for $0 < \lambda \leq \varepsilon$, consider the graph map associated to φ_λ, that is the map $f_\lambda : U \to M \times M$ given by $f_\lambda(p) = (p, \varphi_\lambda(p))$. Since the set $f_\lambda^{-1}(\Delta)$ coincides with the fixed point set $\{p \in \bar{U} : p = \varphi_\lambda(p)\}$, the assumption $\varphi_\lambda(p) \neq p$ on ∂U yields that f_λ is Δ-admissible. Therefore, the intersection number $\mathrm{in}(f_\lambda, \Delta)$ is well-defined.

Our aim now is to construct the framework we will need to apply Lemma 3.2. To this end, consider the null section M_0 of $T(M)$ and recall that M can be identified with M_0 via the embedding $p \mapsto (p, 0)$. Clearly, given $p \in M$, one has

$$T_{(p,0)}(T(M)) = T_p(M) \times T_p(M) \subset \mathbf{R}^k \times \mathbf{R}^k$$

and

$$T_{(p,0)}(M_0) = T_p(M) \times \{0\}.$$

Hence, the orthogonal space

$$N_{(p,0)}(M_0, T(M))$$

of $T_{(p,0)}(M_0)$ in $T_{(p,0)}(T(M))$ is the space $\{0\} \times T_p(M)$, which is a natural copy of $T_p(M)$. This implies that the orientation on M determines uniquely an orientation of the normal bundle $N(M_0, T(M))$.

Now, let $h : M \times M \to T(M)$ be the map given by

$$h(p, q) = (p, \pi_p(p - q)),$$

where $\pi_p : \mathbf{R}^k \to T_p(M)$ is the orthogonal projection. It is not hard to show that, in a suitable neighborhood N_Δ of Δ in $M \times M$, the map h is transverse to M_0 and satisfies $h^{-1}(M_0) \cap N_\Delta = \Delta$. Moreover, by restricting U if necessary, we may also assume $f_\lambda(U)$ to be contained into N_Δ. Let us apply Lemma 3.2 with $N_1 = N_\Delta$, $Z_1 = \Delta$, $N_2 = T(M)$, $Z_2 = M_0$, $f = f_\lambda$. As stated in the assumptions of the Lemma, $N(M_0, T(M))$ must be oriented according to the bundle isomorphism $N(\Delta, M \times M) \to N(M_0, T(M))$ induced by the derivative of h. Observe that, the orientation of $N(M_0, T(M))$ considered above satisfies exactly this requirement. In fact, given $p \in M$, by computing the derivative

$$h'(p, p) : T_{(p,p)}(M \times M) \to T_{(p,0)}(T(M)),$$

we obtain

$$h'(p, p)(u, v) = (u, u - v)$$

(recall that both $T_{(p,p)}(M \times M)$ and $T_{(p,0)}(T(M))$ coincide with the subspace $T_p(M) \times T_p(M)$ of $\mathbf{R}^k \times \mathbf{R}^k$). Hence, $h'(p, p)$ sends the oriented space

$$E_p = \{(u, 0) \in \mathbf{R}^k \times \mathbf{R}^k : u \in T_p(M)\}$$

isomorphically onto the subspace

$$\{(u, u) \in \mathbf{R}^k \times \mathbf{R}^k : u \in T_p(M)\}$$

which, being a complement of $T_{(p,0)}(M_0)$ in $T_{(p,0)}(T(M)$, is canonically isomorphic to $N_{(p,0)}(M_0, T(M)) = \{0\} \times T_p(M)$. Thus, by considering the composite isomorphism from

E_p into $N_p(M_0, T(M))$ one obtains that any element $(u, 0)$ is mapped in $(0, u)$. This shows that the chosen orientation on $N_p(M_0, T(M))$ coincides with the one induced by $h'(p, p)$. Consequently, by Lemma 3.2,

$$\mathrm{in}(f_\lambda, \Delta) = \mathrm{in}(hf_\lambda, M_0), \quad 0 < \lambda \leq \varepsilon. \tag{3.1}$$

Let us point out that, by inverting the orientation on M, the above intersection numbers do not change, since the induced normal orientations change accordingly. Since any finite dimensional manifold is locally orientable, the above argument shows that M need not be orientable in order to define $\mathrm{in}(f_\lambda, \Delta)$ and $\mathrm{in}(hf_\lambda, M_0)$. Hence, equality (3.1) is still valid even without any assumption on the orientability of M.

At the beginning of the proof we assumed $\varphi_\lambda(p) \neq p$ for all $p \in \partial U$ and $0 < \lambda \leq \varepsilon$. Therefore, for $0 < \lambda \leq \varepsilon$, the fixed point index $\mathrm{ind}(\varphi_\lambda, U)$ is well-defined and independent of λ. Let us relate now $\mathrm{in}(f_\lambda, \Delta)$ and $\mathrm{in}(hf_\lambda, M_0)$ with $\mathrm{ind}(\varphi_\lambda, U)$ and $\deg(-w, U)$, respectively. It can be shown that $\mathrm{ind}(\varphi_\lambda, U)$ may be defined in terms of intersection number by setting

$$\mathrm{ind}(\varphi_\lambda, U) = \mathrm{in}(f_\lambda, \Delta), \quad 0 < \lambda \leq \varepsilon$$

(see e.g. [GP] and [H] where this differentiable viewpoint is treated for the Lefschetz number, which is just the fixed point index of a self-mapping). Let us explicitly point out that, with the above definition, the index is equal to 1 when φ_λ is a constant map (this is not so in some textbooks in differential topology).

As regards the degree, consider the tangent vector field $v_\lambda \colon M \to \mathbf{R}^k$ given by $v_\lambda(p) = \pi_p(p - \varphi_\lambda(p))$. Clearly, $v_\lambda(p) \neq 0$ for any $p \in \partial U$ and $0 < \lambda \leq \varepsilon$, so that the degree $\deg(v_\lambda, U)$ is defined and independent of λ. As is well-known (again see e.g. [GP] and [H]), the degree of v_λ in U can also be viewed as the intersection number of the graph map $p \in U \mapsto (p, v_\lambda(p)) = hf_\lambda(p)$ with the null section M_0 of $T(M)$, provided that the orientations (or, local orientations) on M and $N(M_0, T(M))$ are related each other as above. Hence, for any $0 < \lambda \leq \varepsilon$, equality (3.1) implies

$$\mathrm{ind}(\varphi_\lambda, U) = \mathrm{in}(f_\lambda, \Delta) = \mathrm{in}(hf_\lambda, M_0) = \deg(v_\lambda, U). \tag{3.2}$$

Now, in the open set U, the vector field v_λ is clearly homotopic, in an admissible way for the degree, to the map

$$p \longmapsto \pi_p \left(\frac{p - \varphi_\lambda(p)}{\lambda} \right),$$

which, for $\lambda \to 0^+$, tends to $\pi_p(-w(p)) = -w(p)$ (observe that the restriction of π_p to $T_p(M)$ is the identity). Therefore, using the homotopy invariance of the degree, it follows that

$$\deg(v_\lambda, U) = \deg(-w, U),$$

which implies, together with (3.2),

$$\mathrm{ind}(\varphi_\lambda, U) = \deg(-w, U), \quad 0 < \lambda \leq \varepsilon.$$

Now, to complete the proof of the theorem, it suffices to observe that the assertion $\mathrm{i}(\varphi, U) = \deg(-w, U)$ follows immediately from the equality $\mathrm{ind}(\varphi_\lambda, U) = \deg(-w, U)$, by recalling our definition of bifurcation index. $\square$

Let us now give two applications of Theorem 3.3 to classical results. In such applications, the map φ of our abstract results has a concrete meaning: it is just the flow associated to a given tangent vector field.

Corollary 3.4. (see e.g. [K]). *Let U be an open subset of $\mathbf{R}^m$ and $w\colon U \to \mathbf{R}^m$ a C^1 map such that $w^{-1}(0)$ is compact. Let V be a relatively compact open subset of U such that $\bar{V} \subset U$ and $w(p) \neq 0$ on ∂V. Then there exists $\varepsilon > 0$ such that, for $0 < t \leq \varepsilon$, the flow Φ_t associated to w is defined in $\bar{V}$ and $\Phi_t(p) \neq p$ for $p \in \partial V$. Moreover,*

$$\deg(I - \Phi_t, V) = \deg(-w, V),$$

where I denotes the identity in $\mathbf{R}^m$.

Proof. Let

$$D = \{(t, p) \in [0, \infty) \times U : \text{ the solution of } \dot{x} = w(x) \text{ with } x(0) = p \text{ is defined in } [0, t]\}$$

and let $\varphi\colon D \to \mathbf{R}^m$ be defined by $\varphi(t, p) = \Phi_t(p)$. The set D is open, contains $\{0\} \times U$, and one clearly has

$$w(p) = \frac{\partial \varphi}{\partial t}(0, p).$$

Since, by assumption, the zeros of w form a compact subset of V, Theorem 3.1 implies that φ is admissible in V. Consequently, there exists $\varepsilon > 0$ such that $\Phi_t(p) \neq p$ for $0 < t \leq \varepsilon$ and $p \in \partial V$. Hence, as in the proof of Theorem 3.3, $\mathrm{ind}(\Phi_t, V) = \deg(-w, V)$ for $0 < t \leq \varepsilon$. To conclude, recall that, in the open subset V of the finite dimensional space $\mathbf{R}^m$, the fixed point index of the map Φ_t is nothing else but the degree of the map $I - \Phi_t$. $\square$

We will show below how the classical Poincaré-Hopf Theorem (see e.g. [M]) can be deduced from our Theorem 3.3. Let us recall that, if N is a compact manifold with boundary and $v\colon N \to \mathbf{R}^k$ is a continuous tangent vector field on N satisfying $v(p) \neq 0$ for all $p \in \partial N$, then the degree of v in N still makes sense. In fact, it suffices to observe that, in this case, v is admissible in the boundaryless manifold $M = N \setminus \partial N$. Hence, one can define $\deg(v, N)$ as the degree of the restriction of v to the interior M of N. In particular, if v is (strictly) outward along the boundary, then v is admissible and $\deg(v, N)$ is well defined. Observe also that if v_1 and v_2 are two continuous tangent vector fields on N, both outward along the boundary, then the homotopy

$$h(p, s) = (1 - s)v_1(p) + sv_2(p),$$

does not vanish on ∂N. Therefore, $h^{-1}(0)$ is a compact subset of $M \times [0, 1]$, and this implies $\deg(v_1, N) = \deg(v_2, N)$. This shows that the common degree of all continuous tangent vector fields on N pointing outward along ∂N is a well defined integer associated with N. The following famous result shows that this integer is just the Euler-Poincaré characteristic of N.

Corollary 3.5. *(Poincaré-Hopf Theorem). Let N be a finite dimensional compact manifold with boundary and let v be a continuous tangent vector field on N pointing outward along ∂N. Then* $\deg(v, N) = \chi(N)$.

Proof. Observe first that any sufficiently close approximation of v still points outward on ∂N. Therefore, without loss of generality we may assume v to be smooth. Consider the vector field $w = -v$, which clearly points inward along ∂N. Thus, the flow Φ_t associated with w is defined in N for any $t \geq 0$. Consequently, the map $\varphi(t, p) = \Phi_t(p)$ is defined in $[0, \infty) \times N$ and satisfies $\varphi(0, p) = p$ for all $p \in N$. Hence, by the normalization property of the bifurcation index,

$$\mathrm{i}(\varphi, N) = \chi(N).$$

Now, the fact that w is inward implies that there are no bifurcation points of the equation $\varphi(t, p) = p$ on ∂N and that $\varphi(t, p)$ belongs to the boundaryless manifold $M = N \setminus \partial N$ for any $(t, p) \in [0, \infty) \times M$. Thus, by using the excision property of the bifurcation index and by applying Theorem 3.3, we obtain

$$\mathrm{i}(\varphi, N) = \mathrm{i}(\varphi, M) = \deg(-\frac{\partial \varphi}{\partial t}(0, \cdot), M) = \deg(-w, M) = \deg(v, M) = \deg(v, N). \quad \square$$

The following sufficient conditions for global bifurcation are straightforward consequences of Theorems 2.1 and 3.3, and of Corollary 2.3.

Theorem 3.6. *Let $\varphi \colon D \subset [0, \infty) \times M \to M$ be a C^1 map satisfying $\varphi(0, p) = p$ for any $p \in M$, and $w \colon M \to \mathbf{R}^k$ be the tangent vector field*

$$w(p) = \frac{\partial \varphi}{\partial \lambda}(0, p).$$

Let W be an open subset of D. Assume that w is admissible in the slice W_0 of W and that the degree $\deg(w, W_0)$ is nonzero. Then the equation (1.1) admits a relative to W global bifurcating branch.

Corollary 3.7. *Let φ and w be as in Theorem 3.6 and let U be an open subset of M. Assume w admissible in U and $\deg(w, U) \neq 0$. Then the equation (1.1) admits a connected set of nontrivial solutions whose closure in $[0, \infty) \times M$ contains a zero of w in U and is either not contained in any compact subset of the domain D of φ or intersects M in a bifurcation point outside U.*

We close the paper with an application of Theorem 3.6 to ordinary differential equations on manifolds. In the example we are going to illustrate, the map φ of Theorem 3.6 is the Poincaré T-translation operator associated to a first order T-periodic differential equation depending on a real parameter. In this case, the fixed points of the map

$(\lambda, p) \mapsto (\lambda, \varphi(\lambda, p))$ turn out to be the starting points of the T-periodic solutions to the considered problem. A more complete investigation concerning the existence of global branches of periodic orbits to ODE's on manifolds will appear in a forthcoming paper.

Let M be a boundaryless m-dimensional manifold in $\mathbf{R}^k$ and consider in M the first order parametrized differential equation

$$\dot{x} = \lambda f(t, x), \quad \lambda \geq 0, \tag{3.3}$$

where $f \colon \mathbf{R} \times M \to \mathbf{R}^k$ is a T-periodic smooth tangent vector field, i.e. $f(t+T, p) = f(t, p)$ and $f(t, p) \in T_p(M)$ for all $(t, p) \in \mathbf{R} \times M$.

A pair $(\lambda, p) \in [0, \infty) \times M$ will be called a *starting point* of equation (3.3) if there exists a T-periodic solution $x \colon \mathbf{R} \to M$ of (3.3) corresponding to the value λ of the parameter and satisfying the initial condition $x(0) = p$. By means of the embedding $p \mapsto (0, p)$, any element $p \in M$ will be regarded as a trivial starting point of (3.3), that is, a starting point corresponding to the constant solution $x(t) \equiv p$ of the equation $\dot{x} = 0$. Consequently, we will refer to any starting point (λ, p) with $\lambda > 0$ as to a *nontrivial starting point*.

Let W be an open subset of $[0, \infty) \times M$. For any $(\lambda, p) \in W$, assume the global existence on the interval $[0, T]$ of the maximal solution of the Cauchy problem

$$\begin{cases} \dot{x} = \lambda f(t, x) \\ x(0) = p. \end{cases}$$

Our aim is to present a sufficient condition for the existence in W of a branch of nontrivial starting points of equation (3.3). To this end, let us associate to the time-dependent vector field f the "average wind" vector field

$$w(p) = \frac{1}{T} \int_0^T f(t, p) dt.$$

We can state the following (see also [FP1]).

Theorem 3.8. *Let f, w and the open set W be as above. Assume that w is admissible in the slice W_0 of W and that the degree $\deg(w, W_0)$ is nonzero. Then the equation (3.3) admits in W a connected set of nontrivial starting points whose closure in $[0, \infty) \times M$ intersects W_0 in the set $w^{-1}(0)$ and is not contained in any compact subset of W.*

Proof. Let us consider the set

$$D = \{(\lambda, p) \in [0, \infty) \times M : \text{ the solution } x(\cdot) \text{ of (3.3) satisfying } x(0) = p \text{ is defined in} [0, T]\}$$

and let $\varphi \colon D \to M$ be the operator which associates to any $(\lambda, p) \in D$ the value $x(T)$ of the solution $x(\cdot)$ of (3.3) with initial condition $x(0) = p$. It can be shown (see e.g. [L]) that D is an open set (clearly containing $\{0\} \times M$) and that φ is smooth in D. Let us show that

$$\frac{\partial \varphi}{\partial \lambda}(0, p) = T w(p).$$

In fact, given $(\lambda, p) \in D$ one has

$$\varphi(\lambda, p) = p + \lambda \int_0^T f(t, \psi(\lambda, p, t))dt,$$

where $\psi(\lambda, p, t)$ denotes the value at time $t \in [0, T]$ of the solution of (3.3) corresponding to λ and with initial condition p; thus $\varphi(\lambda, p) = \psi(\lambda, p, T)$. Recall that we have assumed the manifold M to be contained in $\mathbf{R}^k$; thus, the above integral makes sense. Then

$$\frac{\varphi(\lambda, p) - \varphi(0, p)}{\lambda} = \frac{\varphi(\lambda, p) - p}{\lambda} = \int_0^T f(t, \psi(\lambda, p, t))dt.$$

Take any sequence $\lambda_n \to 0$. Then the sequence of solutions $t \mapsto \psi(\lambda_n, p, t)$ tends uniformly in $[0, T]$ to the constant solution $\psi(0, p, t) = p$. Consequently,

$$\frac{\partial \varphi}{\partial \lambda}(0, p) = \lim_{\lambda \to 0^+} \frac{\varphi(\lambda, p) - p}{\lambda} = Tw(p),$$

as claimed.

As in Section 2, let us denote by S the set of fixed pairs of φ, i.e.

$$S = \{(\lambda, p) \in D : \varphi(\lambda, p) = p\}.$$

Again, we will regard M itself as a subset of S. Clearly, S coincides with the set of starting points of (3.3), and the trivial solutions of $\varphi(\lambda, p) = p$ are exactly the trivial starting points of (3.3). Moreover, because of the global continuation assumption, the set W is an open subset of the domain D of φ. Therefore, by Theorem 3.6, the equation $\varphi(\lambda, p) = p$ has a relative to W global bifurcating branch. This means that the equation (3.3) has in W a connected branch of nontrivial starting points whose closure in W is noncompact and intersects the slice W_0 in a bifurcation point of $\varphi(\lambda, p) = p$.

Now, to complete the proof, observe that, by the necessary condition of Theorem 3.1, any bifurcation point of $\varphi(\lambda, p) = p$ is a zero of the tangent vector field

$$\frac{\partial \varphi}{\partial \lambda}(0, \cdot) = Tw.$$

Consequently, the closure of the obtained global branch intersects W_0 in the set $w^{-1}(0)$, as claimed. □

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