Carathéodory Periodic Perturbations of the Zero Vector Field on Manifolds

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1 Introduction

Let M be a boundaryless, smooth (not necessarily closed) differentiable manifold in $\mathbf{R}^k$, and let $f : \mathbf{R} \times M \to \mathbf{R}^k$ be a T-periodic Carathéodory tangent vector field on M. Consider the following ordinary differential equation on M:

$$\dot{x} = \lambda f(t, x), \tag{1.1}$$

where $\lambda \geq 0$ is a real parameter. We deal with the problem of the existence of T-periodic solutions of (1.1), with a special attention to the case of small values of λ. Clearly, when $\lambda = 0$, any point in M may be regarded as a constant solution of (1.1). Thus, it is natural to think about M as a subset of the set X of all the pairs (λ, x), called *T-pairs* of (1.1), with $\lambda \geq 0$ and x a T-periodic Carathéodory solution of (1.1) corresponding to the value λ of the parameter. In other words, $(\lambda, x) \in X$ means that x is an absolutely continuous, T-periodic real map into M, such that $\dot{x}(t) = \lambda f(t, x(t))$ for almost all $t \in \mathbf{R}$. As usual, let $C_T(\mathbf{R}^k) := C_T(\mathbf{R}, \mathbf{R}^k)$ denote the Banach space of all the T-periodic, continuous, $\mathbf{R}^k$-valued real functions, endowed with the standard norm of uniform convergence. Since any solution of (1.1) is (in particular) continuous, the set X will be considered embedded in the metric space $[0, \infty) \times C_T(M)$, where $C_T(M)$ is the subset of $C_T(\mathbf{R}^k)$ of those functions whose image lies in M. We will prove that X is closed in this space, no matter whether or not M is closed in $\mathbf{R}^k$ (for example, M could be an open subset of $\mathbf{R}^k$). This does not necessarily happen (even when $M = \mathbf{R}^k$) in the space one could think is the most natural one in the Carathéodory case; i.e. $[0,\infty) \times L^1_T(M)$, where $L^1_T(M)$ stands for the set of L^1_{loc}, T-periodic maps $x : \mathbf{R} \to M$, with the distance inherited by the Banach space $L^1_T(\mathbf{R}^k) \cong L^1((0,T), \mathbf{R}^k)$. Nevertheless, we will show that the topology induced on X by either one of these two spaces is the same. Moreover, with this topology, X is locally compact and M is closed when regarded as a subset of X (via the embedding $p \mapsto (0, \hat{p})$, where $\hat{p}(t) \equiv p$). With this identification in mind, M will be called the set of *trivial T-pairs*. Thus, it is natural to say that an element $p_0 \in M$ is a *bifurcation point* for (1.1) if it lies in the closure of the set $X \backslash M$ of the *nontrivial T-pairs*.

It is easy to see that a necessary condition for $p_0 \in M$ to be a bifurcation point is that the autonomous tangent vector field $w : M \to \mathbf{R}^k$, given by

$$w(p) = \frac{1}{T} \int_0^T f(t,p)\, dt,$$

vanishes at p_0. Moreover, under the assumption that w is C^1 and $w(p_0) = 0$, a sufficient condition is given by the injectivity of the derivative $w'(p_0) : T_{p_0}(M) \to \mathbf{R}^k$ (here $T_{p_0}(M) \subset \mathbf{R}^k$ stands for the tangent space of M at p_0). The above sufficient condition is a direct consequence of our main result, which is an extension of a theorem in [6] regarding the continuous case: a Rabinowitz type global bifurcation result in the space $[0,\infty) \times C_T(M)$ (Theorem 2.2 below) which involves the Hopf index (degree) of the associated autonomous tangent vector field w. Another condition ensuring the existence of a T-periodic solution to the equation

$$\dot{x} = f(t,x), \tag{1.2}$$

(see Corollary 3.2) will be deduced from the same theorem.

2 Branches of Periodic Orbits

Let M be a (not necessarily closed) boundaryless smooth manifold in the space $\mathbf{R}^k$ with standard Euclidean norm $|\cdot|$. For any $p \in M$, let $T_p(M) \subset \mathbf{R}^k$ denote the tangent space of M at p. Consider in M the first order parametrized differential equation

$$\dot{x} = \lambda f(t,x), \quad \lambda \geq 0, \tag{2.1}$$

where $f : \mathbf{R} \times M \to \mathbf{R}^k$ is a T-periodic Carathéodory tangent vector field on M, i.e. f satisfies the following conditions:

1) for each $p \in M$, the map $t \mapsto f(t,p)$ is Lebesgue measurable on $\mathbf{R}$;

2) for almost all $t \in \mathbf{R}$, the map $p \mapsto f(t,p)$ is continuous on M;

3) for any compact set $K \subset M$, there exists γ_K in the space $L^1_T(\mathbf{R})$ of the L^1_{loc}, T-periodic real functions such that $|f(t,p)| \leq \gamma_K(t)$ for a.a. $t \in \mathbf{R}$ and all $p \in K$;

4) for any $p \in M$, one has $f(t+T, p) = f(t, p) \in T_p(M)$ a.e. in $\mathbf{R}$.

Observe that conditions 1), 2) and 3) are the usual Carathéodory type assumptions, while condition 4) says that f is a time-dependent vector field which is tangent to M and T-periodic with respect to the first variable.

A pair (λ, x), where λ is a nonnegative real number and $x : \mathbf{R} \to M$ is an absolutely continuous T-periodic map, will be called a *T-pair* of (2.1) if $\dot{x}(t) = \lambda f(t, x(t))$ a.e. in $\mathbf{R}$. The set of all the T-pairs of (2.1) will be denoted by X. In what follows, it is convenient to consider X as a subspace of $[0, \infty) \times C_T(M)$, and not of $[0, \infty) \times L^1_T(M)$, as one might suppose. The reason is that X is closed in the first space (as shown below) and not in the second one (see Example 2.5). To prove that X is closed in $[0, \infty) \times C_T(M)$, consider a sequence $\{(\lambda_n, x_n)\}$ in X converging to $(\lambda, x) \in [0, \infty) \times C_T(M)$. One has $\dot{x}_n(t) = \lambda_n f(t, x_n(t))$, a.e. in $\mathbf{R}$ or, equivalently,

$$x_n(t) = x_n(0) + \lambda_n \int_0^t f(s, x_n(s))\, ds, \quad \forall t \in \mathbf{R},$$

which clearly makes sense since we have assumed M embedded in $\mathbf{R}^k$. Since $x_n(t)$ converges to $x(t)$ uniformly in $\mathbf{R}$, there exists a compact subset K of M such that $x_n(t) \in K$ for any $n \in \mathbf{N}$ and all $t \in \mathbf{R}$. Hence, by the Carathéodory assumption 3), the sequence $\{f(\cdot, x_n(\cdot))\}$ is dominated by an L^1_T function; so that, by the Lebesgue Convergence Theorem, one can pass to the limit in the above equality obtaining

$$x(t) = x(0) + \lambda \int_0^t f(s, x(s))\, ds, \quad \forall t \in \mathbf{R}.$$

Thus $(\lambda, x) \in X$, and this proves that X is closed in $[0, \infty) \times C_T(M)$.

Observe that the space $[0, \infty) \times C_T(M)$ is not necessarily complete, unless M is closed in $\mathbf{R}^k$. However, due to the fact that M (as a manifold) is locally compact, one can easily prove that $[0, \infty) \times C_T(M)$ is always locally complete. Consequently X, as a closed subset of this space, is locally complete as well. Moreover, X is locally totally bounded since, as a consequence of Ascoli's theorem, a subset of $C_T(M)$ is totally bounded if (and only if) it is bounded and equicontinuous. Now, by recalling that a metric space is compact if and only if it is totally bounded (i.e. precompact) and complete, one can observe that X is actually locally compact. This fact will turn out to be useful in the sequel.

In what follows, it will be convenient to consider the commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & [0, \infty) \times M \\ \downarrow & & \downarrow \\ C_T(M) & \longrightarrow & [0, \infty) \times C_T(M), \end{array}$$

where the horizontal arrows are obtained by associating to $p \in M$ or, respectively, to $x \in C_T(M)$ the element $(0, p) \in [0, \infty) \times M$ or $(0, x) \in [0, \infty) \times C_T(M)$, and the vertical ones are defined by regarding any $p \in M$ as the constant map $\hat{p}(t) \equiv p$. With this identification, if A is any subset of $[0, \infty) \times C_T(M)$, $A \cap M$ will denote the set $\{p \in M : (0, \hat{p}) \in A\}$.

Since any element $p \in M$ may be viewed as a constant solution of (2.1) corresponding to the value $\lambda = 0$ of the parameter, the whole manifold M will be regarded as a subset of the set X of the T-pairs of (2.1). We point out that, despite the fact that $[0,\infty) \times C_T(M)$ may not be closed in $[0,\infty) \times C_T(\mathbf{R}^k)$, M is always closed in $[0,\infty) \times C_T(M)$, as well as in X. Any $p \in M$ will be called a trivial solution of (2.1) and, consequently, any $(\lambda, x) \in X \backslash M$, i.e. with $\lambda > 0$, will be a *nontrivial solution.* A trivial solution $p \in M$ will be called a *bifurcation point* of (2.1) if it lies in the closure of $X \backslash M$.

Let us associate to f the mean value (autonomous) vector field $w : M \to \mathbf{R}^k$ given by

$$w(p) = \frac{1}{T} \int_0^T f(t,p)\, dt. \tag{2.2}$$

Observe that, the Carathéodory assumptions on f and the Lebesgue Convergence Theorem yield the continuity of w.

The mean value vector field introduced above provides the following necessary condition for $p \in M$ to be of bifurcation.

Theorem 2.1 *Let $f : \mathbf{R} \times M \to \mathbf{R}^k$ be a T-periodic Carathéodory tangent vector field on M and $w : M \to \mathbf{R}^k$ be the mean value autonomous vector field given by (2.2). If p_0 is a bifurcation point for the equation (2.1), then $w(p_0) = 0$.*

Proof. Let p_0 be a bifurcation point for the equation (2.1). Then, there exists a sequence $\{(\lambda_n, x_n)\}$ of nontrivial T-pairs of (2.1) such that $\lambda_n \to 0$ and $x_n(t) \to p_0$ uniformly in $\mathbf{R}$. Now, by integrating from 0 to T the equality

$$\dot{x}_n(t) = \lambda_n f(t, x_n(t)), \quad t \in \mathbf{R},$$

one obtains

$$0 = x_n(T) - x_n(0) = \lambda_n \int_0^T f(t, x_n(t))\, dt,$$

which implies, λ_n being nonzero,

$$\int_0^T f(t, x_n(t))\, dt = 0.$$

Since $\{x_n(t)\}$ converges to p_0 uniformly in $\mathbf{R}$, there exists a compact set $K \subset M$ such that $x_n(t) \in K$ for all $n \in \mathbf{N}$ and $t \in \mathbf{R}$. Now, using the Lebesgue Theorem, one gets

$$w(p_0) = \frac{1}{T} \int_0^T f(t, p_0)\, dt = 0,$$

as claimed. □

In what follows, given an open subset Ω of $[0,\infty) \times C_T(M)$, by a *bifurcating branch* of (2.1) in Ω, we mean a connected component of $\Omega \cap (X \setminus M)$, whose closure in X (or, equivalently, in $[0,\infty) \times C_T(M)$) intersects $\Omega \cap M$. A *global* bifurcating branch in Ω is a bifurcating branch which is not relatively compact in $\Omega \cap X$. In particular, if M is closed in $\mathbf{R}^k$ and $\Omega = [0, \infty) \times C_T(M)$, as we shall see later, any global bifurcating branch must be unbounded.

Our aim below is to provide conditions detecting those elements $p \in M$ which are emanating points of global bifurcating branches of solutions. A sufficient condition can be obtained in terms of the index of the mean value vector field w.

Let us recall that, to any continuous tangent vector field $w : M \to \mathbf{R}^k$ which is *admissible* on M, i.e. such that the set $\{p \in M : \ w(p) = 0\}$ is compact, one can associate an integer $\chi(w)$, called the Hopf index (or Euler characteristic, or rotation, or degree) of w, which, roughly speaking, counts (algebraically) the number of zeros of w (see e.g. [7], [9], [10], [11], and [4] for an equivalent definition based on fixed point index theory). In what follows, to emphasize that the index of a tangent vector field on M reduces, in the flat case, to the classical Brouwer degree (with respect to zero), the integer $\chi(w)$ will be called the (*global*) *degree* of the vector field w and denoted by $\deg(w)$. Since any open subset U of a manifold M is still a manifold, the degree of the restriction of w to U makes sense, provided that w is admissible on U, i.e. the set $\{p \in U : w(p) = 0\}$ is compact. This condition is clearly satisfied if U is a relatively compact open subset of M and $w(p) \neq 0$ for all $p \in \partial U$. The degree of the restriction of w to U, when defined, will be denoted by $\deg(w, U)$.

We are now in a position to state our sufficient condition for the existence of a global bifurcating branch of nontrivial solutions pairs. Clearly, this result provides also a sufficient condition for the existence of a bifurcation point in a given open subset of M.

Theorem 2.2 *Let $f : \mathbf{R} \times M \to \mathbf{R}^k$ be a T-periodic Carathéodory tangent vector field on M and $w : M \to \mathbf{R}^k$ be the mean value autonomous vector field given by*

$$w(p) = \frac{1}{T} \int_0^T f(t, p)\, dt.$$

Let Ω be an open subset of $[0, \infty) \times C_T(M)$ and assume that $\deg(w, \Omega \cap M)$ is defined and nonzero. Then, the equation (2.1) admits in Ω a connected set Γ of nontrivial T-pairs whose closure in Ω is noncompact and meets $\Omega \cap M$ in the set of zeros of w. In addition, if M is a closed submanifold of $\mathbf{R}^k$, then Γ cannot be contained in a bounded and complete subset of Ω.

The proof of Theorem 2.2 requires two preliminary results (Theorem 2.3 and Lemma 2.4 below). The first one is an abstract finite dimensional global result for an equation of the form $\varphi(\lambda, p) = p$, where φ is a map defined on an open subset of $[0, \infty) \times M$ with values in M. The proof is omitted, since it is the same as the one (based on intersection theory) given in [5] for the special case of C^1 maps.

Theorem 2.3 *Let D be an open subset of $[0, \infty) \times M$ containing $\{0\} \times M$ and let $\varphi : D \to M$ be a continuous map satisfying $\varphi(0, p) = p$ for all $p \in M$. Assume that φ has* *continuous derivative with respect to* λ *at* $\lambda = 0$ *and denote by* $v : M \to \mathbf{R}^k$ *the tangent vector field*

$$v(p) = \frac{\partial \varphi}{\partial \lambda}(0, p).$$

Given an open subset W *of* D*, assume that* v *is admissible in the slice*

$$W_0 = \{p \in M : (0, p) \in W\}$$

and that $\deg(v, W_0)$ *is nonzero. Then, the equation* $\varphi(\lambda, p) = p$ *admits in* W *a connected branch of solutions* (λ, p)*, with* $\lambda > 0$*, whose closure (in* $[0, \infty) \times M$*) meets* $\{0\} \times \{p \in W_0 : v(p) = 0\}$ *and is not contained in any compact subset of* W*.*

The connectivity result stated below (see e.g. [5]) turns out to be crucial in the proof of our main result.

Lemma 2.4 *Let* Y *be a locally compact Hausdorff space and let* Y_0 *be a compact subset of* Y*. Assume that any compact subset of* Y *containing* Y_0 *has nonempty boundary. Then* $Y \backslash Y_0$ *contains a not relatively compact component whose closure intersects* Y_0*.*

Proof of Theorem 2.2. Assume first that, for each compact subset K of M, there exists $\alpha_K \in L^1_T(\mathbf{R})$ such that

$$|f(t, p_2) - f(t, p_1)| \leq \alpha_K(t)|p_2 - p_1| \tag{2.3}$$

for a.a. $t \in \mathbf{R}$ and for all $p_1, p_2 \in K$. This assumption guarantees the uniqueness of the solution of the Cauchy problem associated to equation (2.1) (see e.g. [8], [1]).

Consider the set D defined by

$$\{(\lambda, p) \in [0, \infty) \times M : \text{ the solution } x(\cdot) \text{ of (2.1) satisfying } x(0) = p \text{ is defined in } [0, T]\}$$

and let $\varphi : D \to M$ be the operator which associates to any $(\lambda, p) \in D$ the value $x(T)$ of the solution $x(\cdot)$ of (2.1) with initial condition $x(0) = p$. By known properties of differential equations it turns out that D is an open set containing $\{0\} \times M$ and φ is continuous in D. Let us show that φ has continuous derivative with respect to λ at $\lambda = 0$ and that

$$\frac{\partial \varphi}{\partial \lambda}(0, p) = Tw(p).$$

In fact, given $(\lambda, p) \in D$, let $\psi(\lambda, p, t)$ denote the value at time $t \in [0, T]$ of the solution of (2.1) with initial condition p. Clearly

$$\varphi(\lambda, p) = p + \lambda \int_0^T f(t, \psi(\lambda, p, t))\, dt,$$

and

$$\frac{\varphi(\lambda, p) - \varphi(0, p)}{\lambda} = \frac{\varphi(\lambda, p) - p}{\lambda} = \int_0^T f(t, \psi(\lambda, p, t))\, dt.$$

Take any sequence $\lambda_n \to 0$, $n \in \mathbf{N}$. The continuous dependence on data (see e.g. [1]) ensures that the sequence $\{\psi(\lambda_n, p, t)\}$ tends uniformly in $[0, T]$ to the constant solution $\psi(0, p, t) \equiv p$. Consequently, recalling the Carathéodory assumptions on f, one has

$$\frac{\partial \varphi}{\partial \lambda}(0, p) = \lim_{\lambda \to 0} \frac{\varphi(\lambda, p) - p}{\lambda} = \int_0^T f(t, p)\, dt = Tw(p),$$

as claimed.

Consider the set

$$S = \{(\lambda, p) \in D : \varphi(\lambda, p) = p\}$$

which is locally compact, since it is closed in the locally compact set D. Moreover, the fact $\varphi(0, p) = p$ for any $p \in M$, implies that any pair $(0, p)$, with $p \in M$, belongs to S. Hence, by recalling the embedding $M \hookrightarrow [0, \infty) \times M$, we will regard M as a closed subset of S.

In the set $X \subset [0, \infty) \times C_T(M)$ of all T-pairs of (2.1), let us consider the map $h : X \to S$ given by $h(\lambda, x) = (\lambda, x(0))$. Clearly h is continuous, onto and, by the assumption on f, it is also one-to-one. Moreover, the continuous dependence on data ensures the continuity of its inverse $h^{-1} : S \to X$. Let Ω be the open subset of $[0, \infty) \times C_T(M)$ considered in the statement of the theorem. Clearly, the set $S_\Omega = h(\Omega \cap X)$ is open in S. Thus, there exists an open subset W of D such that $S_\Omega = W \cap S$. Our aim is to apply Theorem 2.3 to the equation $\varphi(\lambda, p) = p$ in W. To this end observe that, according to the identifications $M \hookrightarrow X$ and $M \hookrightarrow S$, the restriction of the homeomorphism h to M turns out to be the identity. Hence, the set $\Omega \cap M$ and the slice W_0 can be identified, so that the assumption $\deg(w, \Omega \cap M) \neq 0$ is equivalent to

$$\deg\left(\frac{\partial \varphi}{\partial \lambda}(0, \cdot), W_0\right) \neq 0,$$

where, as proved above,

$$\frac{\partial \varphi}{\partial \lambda}(0, p) = Tw(p)$$

for all $p \in M$. Consequently, by Theorem 2.3, there exists a connected subset Σ of $W \cap (S \backslash M)$ whose closure in $[0, \infty) \times M$ meets $\{p \in W_0 : w(p) = 0\}$ and it is not contained in any compact subset of W. This means that the closure of Σ in the topological space S_Ω is not compact. Set $\Gamma = h^{-1}(\Sigma)$ and observe that Γ is a connected set of nontrivial T-pairs in Ω, whose closure in Ω is noncompact and meets $\Omega \cap M$ in $w^{-1}(0)$. Hence, the existence in Ω of a global branch of T-pairs possessing all the required properties is completely proved in the case of f satisfying assumption (2.3).

Let us now consider the case when the assumption (2.3) is not necessarily satisfied. Let $Z = \{p \in M : w(p) = 0\}$. As a consequence of the necessary condition proved in Theorem 2.1, the set $(X \backslash M) \cup Z$ is closed in X. Therefore, since as previously observed the solution set X is locally compact, it follows that $(X \backslash M) \cup Z$ is locally compact as well. Let Y_0 denote the (compact) set of zeros of w in $\Omega \cap M$, i.e. $Y_0 = \Omega \cap Z$. Put $Y = ((\Omega \cap X) \backslash M) \cup Y_0$ and observe that Y is locally compact. In fact, Y clearly coincides with the set $\Omega\cap((X\backslash M)\cup Z)$, which is locally compact as intersection of an open set with a locally compact set. Let us apply Lemma 2.4 to the pair (Y, Y_0). In order to verify all the assumptions, we need only to show that any compact subset of Y containing Y_0 has nonempty boundary. Assume the contrary. Thus, there exists a relatively open, compact subset C of Y containing Y_0. Consequently, one can find an open subset G of Ω such that $G \cap Y = C$, $\partial G \cap Y = \emptyset$. Moreover, since the set $\{(\lambda, x(t)) \in [0,\infty) \times M : (\lambda, x) \in C,\ t \in \mathbf{R}\}$ is compact, one can assume that G is such that $\{(\lambda, x(t)) \in [0,\infty) \times M : (\lambda, x) \in G,\ t \in \mathbf{R}\}$ is contained in a compact subset $\tilde{K}$ of $[0,\infty) \times M$. This implies that G is bounded with complete closure. Without loss of generality, we may also suppose the closure of G contained in Ω. Hence, in particular, $G \cap M$ is relatively compact with closure contained in $\Omega \cap M$.

Let us now approximate f by a sequence $\{f_n\}$ of T-periodic equi-Carathéodory tangent vector fields on M satisfying assumption (2.3) and such that, if $p_n \to p$, then $f_n(t, p_n) \to f(t, p)$ for a.a. $t \in \mathbf{R}$. For instance, given $n \in \mathbf{N}$, one can define f_n as follows:

$$f_n(t,p) = \pi_p \left(\int_M \varphi_n(p,q) f(t,q)\, dq \right),$$

where $\pi_p : \mathbf{R}^k \to T_p(M)$ is the orthogonal projection and $\varphi_n : M \times M \to \mathbf{R}$ is a smooth convolution kernel (i.e. a mollifier) such that $\varphi(p,q) = 0$ whenever $|p - q| > 1/n$.

For any $n \in \mathbf{N}$, let

$$w_n(p) = \frac{1}{T} \int_0^T f_n(t,p)\, dt$$

be the mean value vector field associated to f_n. The assumptions on f_n guarantee that the sequence $\{w_n(p)\}$ converges uniformly to $w(p)$ on compact subsets of M. Moreover, since the zeros of w in $\Omega \cap M$ lie in a compact subset of $G \cap M$, it is easy to see that, for n large enough, the homotopy

$$(p, \tau) \mapsto \tau w_n(p) + (1 - \tau) w(p), \quad 0 \leq \tau \leq 1,$$

is admissible for the degree in $G \cap M$. Thus, $\deg(w_n, G \cap M)$ is well-defined and, by the homotopy invariance property of the degree, it is equal to $\deg(w, G \cap M)$, which, by excision, coincides with $\deg(w, \Omega \cap M)$. This implies that $\deg(w_n, G \cap M)$ is nonzero. Therefore, by the first part of the proof, for n sufficiently large, any equation $\dot{x} = \lambda f_n(t, x)$ has in Ω a connected set of nontrivial solutions pairs Γ_n, whose closure in Ω is noncompact and meets $\Omega \cap M$ in $w_n^{-1}(0)$. Since the closure of G is a bounded and complete subset of Ω, any Γ_n must intersect the complement of G in Ω, which implies the existence of a pair $(\lambda_n, x_n) \in \partial G \cap \Gamma_n$.

Now, any function x_n satisfies

$$\dot{x}_n(t) = \lambda_n f_n(t, x_n(t)), \quad \text{for a.a. } t \in \mathbf{R}$$

or, equivalently,

$$x_n(t) = x_n(0) + \lambda_n \int_0^t f_n(s, x_n(s))\, ds, \quad \text{for all } t \in \mathbf{R}.$$

Therefore, since for any $n \in \mathbf{N}$ and $t \in \mathbf{R}$ the pair $(\lambda_n, x_n(t))$ belongs to the compact set $\tilde{K} \subset [0,\infty) \times M$ introduced above, there exists a function $\gamma \in L^1_T(\mathbf{R})$ such that $|\dot{x}_n(t)| \leq \gamma(t)$, for all $n \in \mathbf{N}$ and a.a. $t \in \mathbf{R}$. Consequently, the sequence $\{x_n\}$ is equicontinuous, so that, because of Ascoli's theorem, it is totally bounded. Hence, without loss of generality, we may assume $\{(\lambda_n, x_n)\}$ converging to $(\lambda_0, x_0) \in \partial G$. This implies that $f_n(s, x_n(s)) \to f(s, x_0(s))$ a.e. in $\mathbf{R}$. Therefore, x_0 is a T-periodic solution of the integral equation

$$x(t) = x(0) + \lambda_0 \int_0^t f(s, x(s))\, ds,$$

which is equivalent to the differential equation $\dot{x} = \lambda_0 f(t, x)$.

Thus, (λ_0, x_0) is a T-pair of (2.1) that, if $\lambda_0 > 0$, clearly belongs to Y. Otherwise, if $\lambda_0 = 0$, then x_0 is a constant function, say $x_0(t) \equiv p_0$. An argument similar to the one used in proving the necessary condition for bifurcation given in Theorem 2.1, shows that $w(p_0) = 0$, i.e. $p_0 \in Y_0$. Therefore, in any case, $(\lambda_0, x_0) \in \partial G \cap Y$, which is a contradiction. Consequently, a straightforward application of Lemma 2.4 to the pair (Y, Y_0) implies the first part of our assertion.

Assume now that M is a closed submanifold of $\mathbf{R}^k$ and let $\Gamma \subset \Omega$ be the global branch obtained above. Suppose Γ bounded. We need to show that its closure in Ω is not complete. In fact, since Γ is bounded and M is closed, the set $\{(\lambda, x(t)) \in [0,\infty) \times M : (\lambda, x) \in \Gamma,\ t \in [0,T]\}$ is contained in a compact subset of $[0,\infty) \times M$. Hence, as above, by Ascoli's theorem, Γ is totally bounded. Consequently, the closure of Γ in Ω is not complete since, otherwise, it would be compact. □

We point out that, throughout the paper, the set X of T-pairs of (2.1) is considered as a subspace of the metric space $[0,\infty) \times C_T(M)$. We have a good reason to do this: as pointed out above, X is closed in $[0,\infty) \times C_T(M)$, as it happens in the less general case of a continuous tangent vector field. However, one could expect that, in the Carathéodory context, the natural setting for X would be $[0,\infty) \times L^1_T(M)$. Unfortunately, X is not necessarily closed in this space. To convince oneself about this peculiarity, one can consider in $\mathbf{R}$ the simple case of a (non-parametrized) differential equation

$$\dot{x} = f(t, x),$$

with $f : [-1,1] \times \mathbf{R} \to \mathbf{R}$ continuous, where the set of those solutions which happen to be globally defined in $[-1,1]$ is not closed in the Banach space $L^1((-1,1), \mathbf{R})$ (but, certainly, closed in $C([-1,1], \mathbf{R})$).

The following example enlightens this phenomenon.

Example 2.5 Consider the family of (bell shaped) real functions, $\xi_c : [-1,1] \to \mathbf{R}$, given by

$$\xi_c(t) = \frac{1}{\sqrt[3]{t^2 + c^2 + 2c}}, \quad c > 0.$$

These may be regarded as global solutions of a time-dependent scalar differential equation $\dot{x} = f(t,x)$. In fact, it is sufficient to define $f : [-1,1] \times \mathbf{R} \to \mathbf{R}$ by

$$f(t,x) = \begin{cases} \xi'_{c(t,x)}(t), & \text{if } 0 \le x \le 1/\sqrt[3]{t^2} \\ \psi(t,x), & \text{otherwise,} \end{cases}$$

where $c = c(t,x)$ is the (unique) solution of the equation $x = 1/(\sqrt[3]{t^2+c^2}+2c)$, $t \in [-1,1]$, $0 \le x \le 1/\sqrt[3]{t^2}$, and ψ is any continuous extension of the map $(t,x) \mapsto \xi'_{c(t,x)}(t)$, $t \in [-1,1]$, $0 \le x \le 1/\sqrt[3]{t^2}$, whose existence is guaranteed by Tietze's theorem. Now, observe that the sequence of solutions $\{\xi_{1/n}\}$ converges in L^1 to the function $t \mapsto 1/\sqrt[3]{t^2}$, which is not a solution of the above equation (in the Carathéodory sense), since it does not admit a continuous extension to the whole interval $[-1,1]$.

In spite of the fact that the set X is closed in $[0,\infty) \times C_T(M)$ and not necessarily in $[0,\infty) \times L^1_T(M)$, we want to show that the topologies induced on X by these two spaces coincide. In fact, since the topology of $L^1_T(M)$ is weaker than that of $C_T(M)$, it is enough to prove that if $\{(\lambda_n, x_n)\}$ is a sequence in X converging in $[0,\infty) \times L^1_T(M)$ to a pair $(\lambda, x) \in X$, then $\{x_n\}$ converges to x in $C_T(M)$. Without loss of generality, we may assume that $x_n(t) \to x(t)$ a.e. in $\mathbf{R}$. Since M is locally compact and x is a T-periodic continuous function, there exists $\varepsilon > 0$ such that the set

$$K_\varepsilon = \{p \in M : |x(t) - p| \le \varepsilon, \text{ for some } t \in [0,T]\}$$

is a compact subset of M. Choose $r > 0$ in such a way that $0 \le \lambda_n \le r$ for any $n \in \mathbf{N}$. By recalling the Carathéodory assumptions on f, one can find a function $\gamma_\varepsilon \in L^1_T(\mathbf{R})$ such that $|f(t,p)| \le \gamma_\varepsilon(t)$ for all $p \in K_\varepsilon$ and a.a. $t \in \mathbf{R}$. Thus, there exists $\delta = \delta(\varepsilon)$ such that for $\vartheta, \tau \in [0,T]$ with $0 \le \tau - \vartheta < \delta$ one has

$$\int_\vartheta^\tau \gamma_\varepsilon(t)\,dt < \frac{\varepsilon}{3r}.$$

Hence, if ξ is any solution of $\dot{x} = \lambda f(t,x)$ corresponding to $\lambda \in [0,r]$ and satisfying $|\xi(t) - x(t)| < \varepsilon$ for all $t \in [\vartheta, \tau]$, then

$$|\xi(\tau) - \xi(\vartheta)| \le \lambda \int_\vartheta^\tau |f(t,\xi(t))|\,dt \le r \int_\vartheta^\tau \gamma_\varepsilon(t)\,dt < \frac{\varepsilon}{3}.$$

Now, take a finite number of points in $[0,T]$, say $t_0, t_1, \ldots, t_N$, such that $x_n(t_i) \to x(t_i)$ as $n \to \infty$ and $|t_i - t_j| < \delta$ for $i,j = 0,1,\ldots,N$. Assume also $|t_0| < \delta/2$ and $|t_N - T| < \delta/2$. Let us show that, if $n \in \mathbf{N}$ is such that $|x_n(t_i) - x(t_i)| < \varepsilon/3$ for $i = 0,1,\ldots,N$, then $|x_n(t) - x(t)| \le \varepsilon$ for all $t \in [0,T]$. In fact, given $t \in [0,T]$ and $i \in \{0,\ldots,N\}$ with $|t - t_i| \le \delta$, one clearly has

$$|x_n(t) - x(t)| \le |x_n(t) - x_n(t_i)| + |x_n(t_i) - x(t_i)| + |x(t_i) - x(t)| < |x_n(t) - x_n(t_i)| + \varepsilon/3 + \varepsilon/3.$$

Moreover, it is easy to show that $|x_n(s) - x(s)| \le \varepsilon$ for all s in the interval with end points t, t_i. Hence, one obtains $|x_n(t) - x_n(t_i)| < \varepsilon/3$, so that $|x_n(t) - x(t)| < \varepsilon$. Thus, $\{x_n(t)\}$ converges to $x(t)$ uniformly in $[0,T]$, as claimed.

Remark 2.6 Results analogous to the ones obtained throughout the paper are still valid for an equation of the form $\dot{x} = \lambda f(\lambda, t, x)$, with f continuous with respect to λ, provided the vector field w is replaced by $w_1(p) = \frac{1}{T}\int_0^T f(0,t,p)\,dt$. Observe that this includes the case of a vector field $(\lambda, t, p) \mapsto g(\lambda, t, p)$ satisfying $g(0,t,p) = 0$ and continuously differentiable with respect to λ, with $(\lambda, t, p) \mapsto \frac{\partial g}{\partial \lambda}(\lambda, t, p)$ a Carathéodory map. In fact $g(\lambda, t, p)$ can be written in the form $\lambda f(\lambda, t, p)$ by defining

$$f(\lambda, t, p) = \int_0^1 \frac{\partial g}{\partial \lambda}(s\lambda, t, p)\,ds.$$

3 Some consequences

We give now some corollaries illustrating the global bifurcation result expressed in Theorem 2.2. A first straightforward application is the following existence result for T-periodic solutions on compact manifolds.

Corollary 3.1 *Let f be as in Theorem 2.2. If M is compact with nonzero Euler-Poincaré characteristic, then there exists a connected branch of T-pairs whose projection on the λ-axis is* $[0, \infty)$.

Proof. By the Poincaré-Hopf theorem the degree, $\deg(w)$, of the mean value autonomous vector field w associated to f coincides with the Euler-Poincaré characteristic $\chi(M)$ of M. Thus, applying the last assertion of Theorem 2.2 to the open set $\Omega = [0, \infty) \times C_T(M)$, one gets the existence of an unbounded connected set Γ of nontrivial T-pairs whose closure $\overline{\Gamma}$ meets the slice $\lambda = 0$. The assertion now follows from the fact that the metric space $C_T(M)$ is bounded and, consequently, the projection on the λ-axis of $\overline{\Gamma}$ must be a connected unbounded subset of $[0, \infty)$ containing 0. □

The following continuation principle for periodic solutions extends Corollary 2.5 in [6], in which f is continuous, and Theorem 2.4 in [4], in which f is continuous and the open set Ω_0 has the special form $\Omega_0 = \{x \in C_T(M) : x(t) \in V \text{ for all } t \in [0, T]\}$, with V a relatively compact open subset of M. We point out that an interesting extension of the last mentioned result has been obtained in [2], Corollary 2, in the case when M is a complete Euclidean Neighborhood Retract.

Corollary 3.2 *Let f and w be as in Theorem 2.2. Let Ω_0 be a bounded open subset of $C_T(M)$ with complete closure and such that the family of maps $\{f(\cdot, x(\cdot)) \in L^1_T(M) : x \in \Omega_0\}$ is dominated by a function in $L^1_T(\mathbf{R})$. Assume that $\deg(w, M \cap \Omega_0)$ is defined and nonzero. Then, the equation (2.1) has in $[0, \infty) \times \Omega_0$ a connected branch of nontrivial T-pairs whose closure in $[0, \infty) \times C_T(M)$ meets $M \cap \Omega_0$ in $w^{-1}(0)$ and is either unbounded (with respect to λ) or intersects $[0, \infty) \times \partial\Omega_0$. In particular, the equation $\dot{x} = f(t, x)$ has a T-periodic solution in Ω_0, provided that in addition $w(p) \neq 0$ for all $p \in M \cap \partial\Omega_0$ and the following a priori bound is satisfied:*

- *if (λ, x) is a T-pair of (2.1) in $(0, 1] \times \overline{\Omega}_0$, then $x \notin \partial\Omega_0$.*

Proof. Apply Theorem 2.2 to the open set $\Omega = [0, \infty) \times \Omega_0$. Then, there exists in Ω a connected bifurcating branch Γ of nontrivial T-pairs whose closure in Ω is noncompact. Suppose Γ bounded with respect to λ. Hence, as in the last part of the proof of Theorem 2.2, Γ turns out to be totally bounded. Consequently, since Ω_0 has complete closure, the closure $\overline{\Gamma}$ of Γ in $[0, \infty) \times C_T(M)$ must be compact. On the other hand, since by Theorem 2.2 $\overline{\Gamma} \cap \Omega$ is not compact, if Γ is bounded, $\overline{\Gamma}$ must contain a pair $(\lambda, x) \in [0, \infty) \times \partial\Omega_0$, as claimed.

Assume now, in particular, $w(p) \neq 0$ for all $p \in M \cap \partial\Omega_0$ and $\overline{\Gamma} \cap ((0, 1] \times \partial\Omega_0) = \emptyset$. This means that $\overline{\Gamma}$ must intersect either $\{0\} \times \partial\Omega_0$ or $\{1\} \times \Omega_0$. Because of Theorem 2.1 (and the assumption $w(p) \neq 0$ on $M \cap \partial\Omega_0$) the first situation does not occur. □

Corollary 3.3 below extends a result in [6] in which f is continuous. It contains also a theorem in [3] regarding the flat case (i.e. when M is an open subset of $\mathbf{R}^k$) and obtained as an application of some abstracts results involving nonlinear compact perturbations of linear Fredholm operators of index zero.

Corollary 3.3 *Let f and w be as in Theorem 2.2, and let U be an open subset of M. If* $\deg(w, U)$ *is defined and nonzero, then the equation (2.1) admits in* $[0, \infty) \times C_T(M)$ *a connected branch of nontrivial T-pairs whose closure meets U in a zero of w and is not contained in any compact subset of* $([0, \infty) \times C_T(M)) \setminus (M \setminus U)$. *In addition, if M is closed in $\mathbf{R}^k$, the closure of this branch satisfies at least one of the following properties:*

a) it is unbounded;

b) it contains a bifurcation point in $M \setminus U$.

Proof. Since M is closed in $[0, \infty) \times C_T(M)$, the set

$$\Omega = ([0, \infty) \times C_T(M)) \setminus (M \setminus U)$$

is open in $[0, \infty) \times C_T(M)$. Observe that $\Omega \cap M = U$. The assertion now follows immediately from Theorem 2.2. In particular, if M is closed, $[0, \infty) \times C_T(M)$ is a complete metric space. Thus the closure of the branch in $[0, \infty) \times C_T(M)$ is complete. Therefore, if bounded, as a consequence of the last assertion of Theorem 2.2, this closure must contain a bifurcation point in $M \setminus U$. □

In Theorem 2.1, we have proved that a necessary condition for $p \in M$ to be a bifurcation point is that the mean value vector field w vanishes at p. The following direct consequence of the above Corollary provides a sufficient condition for an open subset on M to contain bifurcation points.

Corollary 3.4 *Let f and w be as in Theorem 2.2, and let U be an open subset of M. If* $\deg(w, U)$ *is defined and nonzero, then the equation (2.1) admits at least a bifurcation point in U.*

As an easy consequence of Corollary 3.4 we get the following sufficient condition for a point $p \in M$ to be of bifurcation.

Corollary 3.5 *Let p be a zero of the mean value vector field w. Assume that w is differentiable at p and that $w'(p) : T_p(M) \to \mathbf{R}^k$ is one-to-one. Then p is a bifurcation point of the equation (2.1).*

Proof. The assumption $w(p) = 0$ implies that $w'(p)$ maps $T_p(M)$ into itself (see e.g. [10]). Consequently, $w'(p)$ is an automorphism of $T_p(M)$ and $\det w'(p)$ is well defined and nonzero. This implies that p is an isolated zero. Thus, there exists an open neighborhood U of p in M such that $\deg(w, U) = \operatorname{sign} \det w'(p) \neq 0$. □

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