GENERAL BIFURCATION THEORY:

LOCAL RESULTS AND APPLICATIONS

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Abstract. We give a necessary and a sufficient condition for a point $\mathbf{p} \in M$, $M \subset E$, where E is a Banach space and M is a smooth manifold contained in the solution set S of the nonlinear equation $f(\mathbf{x})=\mathbf{0}$, to be a bifurcation point. Known results are derived as particular cases. Some applications are given.

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Introduction

Bifurcation theory has played a significant role in the study of nonlinear processes by providing critical information on the presence and stability of multiple solutions of nonlinear functional equations. Three types of bifurcation phenomena are relevant to the present paper.

The first, which may be considered classic, regards nonlinear operator equations of the form

(1) $$f(\lambda, \mathbf{x}) = \mathbf{0},$$

where λ is a scalar (or a finite dimensional vector), $\mathbf{x}$ is a vector in a Banach space G and f takes values in a Banach space F. It is assumed that $f(\lambda,\mathbf{0}) \equiv \mathbf{0}$. The space $M = \{(\lambda,\mathbf{x}): \mathbf{x} = \mathbf{0}\}$ is regarded as the set of trivial solutions of (1). The interest is focused on detecting values of λ, called bifurcation points, such that in every neighborhood of $(\lambda,\mathbf{0})$ there are non-trivial solutions of (1), i.e. pairs $(\lambda,\mathbf{x})$ with $\mathbf{x} \neq \mathbf{0}$ which satisfy the equation.

The second bifurcation phenomenon, which some authors have called atypical bifurcation (see [AP]) or cobifurcation (see [FP1]), regards semilinear operator equations of the form

(2) $$L(\mathbf{x}) + H(\lambda, \mathbf{x}) = \mathbf{0},$$

where $L: G \to F$ is a non-invertible Fredholm operator of index 0 between the Banach spaces G and F, and $H: \mathbb{R}\times G \to F$ is such that $H(0,\mathbf{x}) = \mathbf{0}$ for every $\mathbf{x} \in G$. In this case the set of trivial solutions is the space $M = \{0\}\times \mathrm{Ker}L$. The goal is to detect those points $\mathbf{x} \in \mathrm{Ker}L$, such that in every neighborhood of $(0,\mathbf{x})$ there are non-trivial solutions of (2), i.e. solutions $(\lambda,\mathbf{x})$ with $\lambda \neq 0$.

Results concerning the first type of bifurcation have been obtained by Krasnosel'skii [K], Rabinowitz [R], Crandall-Rabinowitz [CR] and others for the case when λ is a real parameter. When λ is a vector in $\mathbb{R}^m$, $m > 1$, (several-parameters bifurcation) results have been obtained among others by Alexander [A], Alexander-Yorke [AY], and Fitzpatrick-Pejsachowicz [FiPe].

Results concerning the second type of bifurcation have been obtained by Ambrosetti-Prodi [AP], Martelli [Ma], Furi-Pera [FP1] , Iannacci-Martelli [IM], Fabry-Martelli [FM], etc.

The third bifurcation phenomenon, which has some similarities with the second one, can be illustrated with the following example.

Let g be a T-periodic smooth (time dependent) tangent vector field on a differentiable submanifold $M \subseteq \mathbb{R}^k$. That is $g: \mathbb{R}\times M \to \mathbb{R}^k$ is such that $g(t,\mathbf{q}) \equiv g(t+T,\mathbf{q}) \in T_{\mathbf{q}}(M)$, the tangent space to M at $\mathbf{q}$. Consider on M the differential equation

(3) $$\dot{\mathbf{x}}(t) = \lambda g(t, \mathbf{x}(t)),$$

depending on a real parameter λ. It is evident that when $\lambda = 0$ any point $\mathbf{q}$ of M may be viewed as a constant solution of this equation (a rest point). Therefore, if one is interested in T-periodic solutions of the above equation, one may regard M as the set of trivial solutions of (3). More precisely, a solution pair of (3) is a pair $(\lambda, \mathbf{q})$, with $\lambda \in \mathbb{R}$ and $\mathbf{q} \in M$ a starting point of a T-periodic solution of (3). The set of trivial pairs $\{0\}\times M$ is identified with M, and $\mathbf{p} \in M$ is a bifurcation point (relative to M) if any neighborhood of $(0, \mathbf{p})$ in $\mathbb{R}\times M$ contains some nontrivial pairs. Conditions which ensure the existence of such bifurcation points are given in [FP2] and [FP3].

A more physically meaningful example is provided by a forced constrained system. One may regard the motion equation of a constrained system (for example, the double pendulum) as a (generalized) mass point constrained to a smooth submanifold M of some $\mathbb{R}^k$ and acted on by a (generalized) force. To study forced oscillations we assume the active force to be of the form $\lambda\mathbf{f}$, where λ is a real parameter and $\mathbf{f}: \mathbb{R}\times M \to \mathbb{R}^k$ a T-periodic tangent vector field on M (the normal component of any applied force is neutralized by an appropriate reactive force due to the constraint). We may regard M as the set of rest points of the inertial problem (i.e. the one corresponding to $\lambda = 0$). As in the previous example, we say that a rest point $\mathbf{p} \in M$ is a bifurcation point of the forced system, if every neighborhood of the associated trivial triple $(0; \mathbf{p}, \mathbf{0}) \in \mathbb{R}\times\mathbb{R}^k\times\mathbb{R}^k$ contains a nontrivial solution triple $(\lambda; \mathbf{q}, \mathbf{v})$. That is, an element $(\lambda; \mathbf{q}, \mathbf{v}) \in \mathbb{R}\times\mathbb{R}^k\times\mathbb{R}^k$ with the following properties. Either λ or $\mathbf{v}$ is nonzero. Moreover, the solution $\mathbf{x}(\cdot)$ of the motion problem associated to the force $\lambda\mathbf{f}$ and satisfying the initial conditions $\mathbf{x}(0) = \mathbf{q} \in M$, $\dot{\mathbf{x}}(0) = \mathbf{v} \in T_{\mathbf{q}}(M)$, is a forced oscillation (i.e. periodic with period T).

The study of this bifurcation phenomenon for constrained systems turned out to be crucial in the proof of the fact that the forced spherical pendulum always admits forced oscillations (see [FP5]). For further investigations about bifurcation of periodic solutions for second order equations on manifold see [FP4] and [FP6].

In order to include all the above bifurcation phenomena we propose the following general setting. Assume that in some "universe" U we have an "equation" for which there is a clear method to decide whether or not an element $\mathbf{x} \in U$ is a solution. More precisely, a set $X \subseteq U$, called the set of solutions of the given equation, is well defined. Let X be endowed with some topology, choose a distinguished subset $M \subseteq X$, and call it the set of trivial solutions. Once these minimal requirements are fulfilled one can propose the following definition of bifurcation point.

A trivial solution $\mathbf{p} \in M$ *is called a bifurcation point* (*relative to* M) *if it belongs to the closure of the set* $X \setminus M$ *of nontrivial solutions.*

In this paper we study the existence of bifurcation points, in the sense of the above definition, for nonlinear equations of the form

$$f(\mathbf{x}) = \mathbf{0}, \tag{4}$$

where $f: U \to F$ is a C^1 Fredholm map from an open subset U of a Banach space E into a Banach space F. We assume that the subset M of trivial solutions of (4) is a finite dimensional submanifold of E. Obviously $f^{-1}(\mathbf{0})\backslash M$ will be called the set of nontrivial solutions of (4). Since f is constant on M, the tangent space $T_{\mathbf{x}}(M)$ of M at every $\mathbf{x} \in M$ must be contained in the kernel, $\mathrm{Ker} f'(\mathbf{x})$, of the Fréchet derivative of f at $\mathbf{x}$. We shall prove that (see Theorem 1) a necessary condition for $\mathbf{p} \in M$ to be a bifurcation point is $\mathrm{Ker} f'(\mathbf{p}) \neq T_{\mathbf{p}}(M)$. A sufficient condition will be given in Theorem 2 under the assumption that f is C^2.

The bifurcation results we obtain are local, in the sense that they do not provide information on the set of non-trivial solutions away from a bifurcation point. We believe however that the reader will find them useful for the following reasons.

First, the theorems we prove are invariant under C^2 diffeomorphisms. This property, besides being interesting on its own, has also the important consequence of significantly simplifying all proofs.

Second, a well-known bifurcation theorem of Crandall-Rabinowitz [CR] for equation (1) is derived as an easy consequence of our results.

Third, the dimension of the manifold of trivial solutions does not necessarily coincide with the index of the Fredholm operator. This is in marked contrast with the several-parameter bifurcation theory mentioned before, since in that case the parameter space is identified with the space of trivial solutions and its dimension is equal to the index of the Fredholm operator.

Fourth, no compactness assumptions are needed in this paper, although we expect to include them in a forthcoming paper containing global information on the set of non-trivial solutions of (4).

Preliminaries

In this section we collect the terminology used throughout this paper. We include without proof some known facts which will be helpful in presenting and establishing the various results and applications.

Let E, F be two Banach spaces, $U \subseteq E$ be open and $f: U \to F$ be (Fréchet) differentiable on U. We denote by $f'(\mathbf{x})(\mathbf{v}) \in F$ the vector obtained by applying the derivative of f at (the point) $\mathbf{x} \in U$ to (the vector) $\mathbf{v} \in E$. A practical method for computing $f'(\mathbf{x})(\mathbf{v})$ is to use the equality

$$f'(\mathbf{x})(\mathbf{v}) = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{v}) - f(\mathbf{x})}{t}.$$

Recall that $L(E,F)$ is the Banach space of bounded linear operators from E to F. When $E = \mathbb{R}$ the space $L(\mathbb{R},F)$ can be canonically identified with F (via the map $A \longmapsto A(1)$). Consequently, in this case we shall use the notation $\lambda f'(x)$ for the vector $f'(x)(\lambda)$.

A differentiable map $f: U \to F$ is said to be C^1 if $f': U \to L(E,F)$ is continuous. More generally, f is C^k if f' is C^{k-1}.

When $E = E_1 \times E_2$ and $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in E_1 \times E_2$ we indicate with $D_1 f(\mathbf{x})$, or with $D_1 f(\mathbf{x}_1, \mathbf{x}_2)$, the Fréchet derivative at $\mathbf{x}_1$ of the function $f(\cdot, \mathbf{x}_2): E_1 \to F$. $D_2 f(\mathbf{x})$ is defined in a similar manner. The following fundamental equality holds

$$f'(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2) \; = \; D_1 f(\mathbf{x})(\mathbf{v}_1) + D_2 f(\mathbf{x})(\mathbf{v}_2).$$

Since, as pointed out before, $L(\mathbb{R},F)$ is canonically isomorphic to F, we shall simply write $\mu D_1 f(\lambda, \mathbf{x}_2)$ instead of $D_1 f(\lambda, \mathbf{x}_2)(\mu)$ in the case when $E = \mathbb{R} \times E_2$. Therefore, for C^2 maps f the symbol $\mu D_2 D_1 f(\lambda, \mathbf{x}_2)(\mathbf{v}_2)$ will stand for $D_2 D_1 f(\lambda, \mathbf{x}_2)(\mu)(\mathbf{v}_2)$.

When $F = F_1 \times F_2$, i.e. the Banach space F also is a product, then indicating with f_1 and f_2 the two component functions of f, we have

$$f'(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2) \; = \; \begin{bmatrix} D_1 f_1(\mathbf{x}) & D_2 f_1(\mathbf{x}) \\ D_1 f_2(\mathbf{x}) & D_2 f_2(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}$$

The second derivative of a C^2 function f will be denoted by f''. Recall that for every $\mathbf{x} \in U$, $f''(\mathbf{x})$ is an element of the Banach space of bounded bilinear symmetric operators from E into F. This space is usually denoted by $L^2_S(E,F)$. A practical method for computing $f''(\mathbf{p})(\mathbf{u}, \mathbf{v})$ is the following. Fix $\mathbf{u}$ and consider the map $\mathbf{x} \longmapsto f'(\mathbf{x})(\mathbf{u})$. Compute the directional derivative of this function at the point $\mathbf{p}$ and in the direction $\mathbf{v}$.

When $E = E_1 \times E_2$ we have

$$f''(\mathbf{x})((\mathbf{v}_1, 0), (0, \mathbf{v}_2)) \; = \; D_1 D_2 f(\mathbf{x})(\mathbf{v}_2)(\mathbf{v}_1) \; = \; D_2 D_1 f(\mathbf{x})(\mathbf{v}_1)(\mathbf{v}_2).$$

This formula will be useful in proving Theorem 2 and some of its consequences.

Recall that an operator $A \in L(E,F)$ is said to be Fredholm if ImA is closed and both $\dim \mathrm{Ker} A + \dim \mathrm{CoKer} A < \infty$. The index of A is

$$\mathrm{ind} A \; = \; \dim \mathrm{Ker} A - \dim \mathrm{CoKer} A.$$

Therefore, when the spaces E and F are finite dimensional, $\mathrm{ind} A = \dim E - \dim F$. The composition of Fredholm operators is again Fredholm and its index is the sum of the indices of the composite applications. Consequently, the restriction of a Fredholm operator of index k to a (closed) subspace of codimension r is Fredholm of index $k - r$.

A C^1 map $f: U \to F$ is said to be Fredholm when $f'(\mathbf{x})$ is a Fredholm operator for all $\mathbf{x} \in U$. The index of f is defined as $\mathrm{ind} f = \mathrm{ind} f'(\mathbf{x})$, $\mathbf{x} \in U$, provided that this integer is constant on U. This is clearly the case when U is connected, since the set of Fredholm operators of a given index is open in $L(E,F)$.

Given a smooth manifold M and a point $\mathbf{x} \in M$ we denote by $T_{\mathbf{x}}(M)$ the tangent space to M at the point $\mathbf{x}$. The tangent bundle of M will be denoted by TM. When M is contained in a Banach space E, $T_{\mathbf{x}}(M)$ will be regarded as a subspace of E and TM as the manifold

$$\{(\mathbf{x},\mathbf{v}) \in E\times E : \mathbf{x} \in M,\ \mathbf{v} \in T_{\mathbf{x}}(M)\}.$$

In this case, the canonical projection $\pi : TM \to M$ is given by the restriction to TM of the projection of $E\times E$ onto its first factor. Given an open interval $J \subseteq \mathbb{R}$ and a differentiable function $\gamma : J \to M$ we denote by $\dot{\gamma}(t) \in T_{\gamma(t)}(M)$ the tangent vector to the curve γ at the point $\gamma(t)$.

Results

The following result is crucial in the proof of our necessary condition for bifurcation. Roughly speaking, it ensures that a finite dimensional submanifold M of a Banach space is a level set of a C^1 Fredholm map f provided that M is well cut by f. Notice that the index of f does not necessarily coincide with the dimension of M.

Lemma 1. *Let* U *be an open subset of a Banach space* E, $f : U \to F$ *be a* C^1 *Fredholm map into a Banach space* F, *and* $M \subseteq f^{-1}(\mathbf{0})$ *be a* C^1 *manifold of dimension* m. *Assume that* $\operatorname{Ker} f'(\mathbf{x}) = T_{\mathbf{x}}(M)$ *for all* $\mathbf{x} \in M$. *Then there exists an open subset* V *of* U *such that* $M = V \cap f^{-1}(\mathbf{0})$.

Proof. Without loss of generality we may assume U connected, which implies that ind f is well defined. Since $\operatorname{Ker} f'(\mathbf{x}) = T_{\mathbf{x}}(M)$ we have $\dim \operatorname{Ker} f'(\mathbf{x}) = m$ for every $\mathbf{x} \in M$. Therefore $\operatorname{ind} f \leq m$.

Suppose first that $\operatorname{ind} f = m$. Then $f'(\mathbf{x}) : E \to F$ is onto for every $\mathbf{x} \in M$, and the same is true in a neighborhood W of M (recall that the set of surjective bounded linear operators from E into F is open in $L(E,F)$). Consequently, $\mathbf{0}$ is a regular value for the restriction of f to W. By the Implicit Function Theorem, $f^{-1}(\mathbf{0}) \cap W$ is a manifold of dimension equal to ind f. Since $\dim M = m$ we have that the manifold M is open in $f^{-1}(\mathbf{0}) \cap W$ and there exists an open subset V of U such that $M = V \cap f^{-1}(\mathbf{0})$.

Now, assume $\operatorname{ind} f < m$. From

$$\operatorname{codim} \operatorname{Im} f'(\mathbf{x}) = \dim \operatorname{Ker} f'(\mathbf{x}) - \operatorname{ind} f'(\mathbf{x}) > 0$$

we derive that $\operatorname{Im} f'(\mathbf{x})$ is a proper subspace of F. Define $g : E \to \operatorname{Im} f'(\mathbf{x})$ as the composition $g = Pf$, where $P : F \to \operatorname{Im} f'(\mathbf{x})$ is a bounded linear projection onto the finite codimensional space $\operatorname{Im} f'(\mathbf{x})$. Since

$$g'(\mathbf{x}) = Pf'(\mathbf{x}) : E \to \operatorname{Im} f'(\mathbf{x})$$

is onto and $\mathrm{Ker}\, g'(\mathbf{x}) = \mathrm{Ker}\, f'(\mathbf{x})$, one has $\mathrm{ind}\, g'(\mathbf{x}) = m$. Consequently, g is Fredholm of index m in a neighborhood W of $\mathbf{x}$. Without loss of generality we may assume $M \subseteq W$. By the first part of the proof, there exists an open subset V of E such that $M = V \cap (Pf)^{-1}(\mathbf{0})$. Since $M \subseteq f^{-1}(\mathbf{0}) \subseteq (Pf)^{-1}(\mathbf{0})$, we obtain $M = V \cap f^{-1}(\mathbf{0})$. ∎

The following necessary condition for bifurcation at $\mathbf{p} \in M$ is a consequence of Lemma 1.

Theorem 1. *Let* $U \subseteq E$ *be open and* $f: U \to F$ *be a* C^1 *Fredholm map. Assume that* $f^{-1}(\mathbf{0})$ *contains a* C^1 *manifold of dimension* m. *Then* $\mathbf{p} \in M$ *is a bifurcation point (relative to* M*) only if* $\dim \mathrm{Ker}\, f'(\mathbf{p}) > m$.

Proof. From $M \subseteq f^{-1}(\mathbf{0})$ follows that $\dim \mathrm{Ker}\, f'(\mathbf{x}) \geq m$ for every $\mathbf{x} \in M$. The set

$$N = \{\mathbf{x} \in M : \dim \mathrm{Ker}\, f'(\mathbf{x}) < m+1\}$$

is open relative to M, since the map $\mathbf{x} \in M \longmapsto \dim \mathrm{Ker}\, f'(\mathbf{x})$ is upper semicontinuous. Let $\mathbf{p} \in M$ be such that $\mathrm{Ker}\, f'(\mathbf{p}) = T_{\mathbf{p}}(M)$. Then $\mathbf{p} \in N$. By Lemma 1 there is an open set $V \subseteq E$ such that $N = V \cap f^{-1}(\mathbf{0})$. Hence $\mathbf{p}$ cannot be a bifurcation point. ∎

Remark 1. We point out that various necessary conditions for bifurcation given by different authors are particular cases of Theorem 1. We present the following two.

• The well-known necessary condition for bifurcation of equation (1) in the case when

$$f(\lambda, \mathbf{x}) = \mathbf{x} + \lambda L(\mathbf{x}) + R(\lambda, \mathbf{x}),$$

with $L: G \to F$ linear and $D_2 R(\lambda, \mathbf{0}) \equiv \mathbf{0}$, is that λ_0 is a bifurcation point only if $1/\lambda_0$ is an eigenvalue of L.

The above condition follows easily from Theorem 1. In fact, let $E = \mathbb{R} \times G$. Since $M = \mathbb{R} \times \{0\}$, we need $\dim \mathrm{Ker}\, f'(\lambda_0, \mathbf{0}) > 1$. From $f'(\lambda_0, \mathbf{0})(\mu, \mathbf{w}) = \mathbf{w} + \lambda_0 L(\mathbf{w})$, we see that this is possible only if ($\lambda_0 \neq 0$ and) $1/\lambda_0$ is an eigenvalue of L.

• We now compare Theorem 1 with a standard necessary condition for bifurcation of equation (2) (see, for example, [Ma], [IM] and [FP1]).

A point $\mathbf{x}_0 \in \mathrm{Ker} L$ is a bifurcation point (relative to $\{0\} \times \mathrm{Ker} L$) only if the vector $D_1 H(0, \mathbf{x}_0)$ belongs to $\mathrm{Im} L$.

To illustrate how this condition follows from Theorem 1, let $E = \mathbb{R} \times G$ and set

$$f(\lambda, \mathbf{x}) = L(\mathbf{x}) + H(\lambda, \mathbf{x}).$$

We have

$$f'(0, \mathbf{x}_0)(\mu, \mathbf{w}) = L(\mathbf{w}) + \mu \mathbf{y}_0,$$

where $\mathbf{y}_0 = D_1H(0,\mathbf{x}_0)$. Let us show that $\dim\mathrm{Ker} f'(0,\mathbf{x}_0) > \dim\mathrm{Ker} L$ implies $\mathbf{y}_0 \in \mathrm{Im} L$. In fact, if $\mathbf{y}_0 \notin \mathrm{Im} L$, the equation $L(\mathbf{w}) + \mu\mathbf{y}_0 = \mathbf{0}$ implies $\mu = 0$. Consequently, $\mathrm{Ker} f'(0,\mathbf{x}_0) = \{0\}\times\mathrm{Ker} L$, against our assumption.

We now turn our attention to a sufficient condition for bifurcation (see Theorem 2 below). The following known result from differential calculus will be used in the proof of the theorem.

Lemma 2. *Let* $f:E\to F$ *be a* C^k *map such that* $f(\mathbf{0}) = \mathbf{0}$. *Then* $f(\mathbf{x}) = g(\mathbf{x})(\mathbf{x})$ *for some* C^{k-1} *map* $g:E\to L(E,F)$. *Consequently* $f'(\mathbf{0}) = g(\mathbf{0})$.

<u>Proof</u>. Let $\mathbf{y}^*$ be a bounded linear functional on F. Given $\mathbf{x}\in E$ define $\varphi:[0,1]\to\mathbb{R}$ by $\varphi(t) = \mathbf{y}^*(f(t\mathbf{x}))$.
Then,

$$\varphi(1) = \int_0^1 \varphi'(t)dt = \mathbf{y}^*\Big(\int_0^1 f'(t\mathbf{x})(\mathbf{x})dt\Big) = \mathbf{y}^*\Big[\Big(\int_0^1 f'(t\mathbf{x})dt\Big)(\mathbf{x})\Big].$$

Since $\mathbf{y}^*$ is arbitrary we obtain

$$f(\mathbf{x}) = \Big(\int_0^1 f'(t\mathbf{x})dt\Big)(\mathbf{x}) = g(\mathbf{x})(\mathbf{x}). \blacksquare$$

Remark 2. By Lemma 2 we have that any C^k map $f:E_1\times E_2\to F$ such that $f(\mathbf{0},\mathbf{0}) = \mathbf{0}$ can be written in the form

$$f(\mathbf{x}_1,\mathbf{x}_2) = a_1(\mathbf{x}_1,\mathbf{x}_2)(\mathbf{x}_1) + a_2(\mathbf{x}_1,\mathbf{x}_2)(\mathbf{x}_2),$$

where $a_i: E_1\times E_2\to L(E_i,F)$ is given by

$$a_i(\mathbf{x}_1,\mathbf{x}_2) = \int_0^1 D_i f(t\mathbf{x}_1,t\mathbf{x}_2)\,dt \qquad i = 1,2.$$

Clearly $D_1f(\mathbf{0},\mathbf{0}) = a_1(\mathbf{0},\mathbf{0})$ and $D_2f(\mathbf{0},\mathbf{0}) = a_2(\mathbf{0},\mathbf{0})$.

We are now ready to give a <u>sufficient condition</u> for bifurcation.

Theorem 2. *Let* $U\subseteq E$ *be open and* $f:U\to F$ *be a* C^2 *Fredholm map of index* 1 *such that* $f^{-1}(\mathbf{0})$ *contains a* C^2 *manifold* M *of dimension* m. *Assume that* $\dim\mathrm{Ker} f'(\mathbf{p}) = m+1$ *for some* $\mathbf{p}\in M$. *Then* $\mathbf{p}$ *is a bifurcation point provided that there exists* $\mathbf{u}\in \mathrm{Ker} f'(\mathbf{p})\setminus T_{\mathbf{p}}(M)$ *satisfying*

(5) $$f''(\mathbf{p})(\mathbf{u},\mathbf{v}) \notin \mathrm{Im} f'(\mathbf{p}), \quad \forall\, \mathbf{v}\in T_{\mathbf{p}}(M)\setminus\{\mathbf{0}\}.$$

Before proving Theorem 2 we point out an important fact. Let V be an open subset of E, $h: U \to V$ be a C^2 diffeomorphism. Define $g = fh^{-1}$ and let $\mathbf{q} = h(\mathbf{p})$. Then

$$f'(\mathbf{p})(\mathbf{u}) = g'(\mathbf{q})(h'(\mathbf{p})(\mathbf{u}))$$

and

$$f''(\mathbf{p})(\mathbf{u},\mathbf{v}) = g''(\mathbf{q})(h'(\mathbf{p})(\mathbf{u}), h'(\mathbf{p})(\mathbf{v})) + g'(\mathbf{q})(h''(\mathbf{p})(\mathbf{u},\mathbf{v})),$$

for every $\mathbf{u}, \mathbf{v} \in E$. The first equation implies that $\mathrm{Im} f'(\mathbf{p})$ and $\mathrm{Im} g'(\mathbf{q})$ coincide, since $h'(\mathbf{p})$ is an isomorphism. Moreover, given $\mathbf{u}$ as in Theorem 2, we have

$$f''(\mathbf{p})(\mathbf{u},\mathbf{v}) \notin \mathrm{Im} f'(\mathbf{p})$$

if and only if

$$g''(\mathbf{q})(h'(\mathbf{p})(\mathbf{u}), h'(\mathbf{p})(\mathbf{v})) \notin \mathrm{Im} g'(\mathbf{q}).$$

This remark tells us that by replacing U with V, M with h(M), and $\mathbf{p}$ with $\mathbf{q}$ we can apply Theorem 2 to g. Hence the result is invariant under C^2 diffeomorphisms.

Proof of Theorem 2. By definition of m-dimensional C^2-manifold in E there exist an open neighborhood V of $\mathbf{p}$, a Banach space G, and a diffeomorphism $h: V \to \mathbb{R}^m \times G$ with the following properties: $h(V) = \mathbb{R}^m \times G$; $h(V \cap M) = \mathbb{R}^m \times \{\mathbf{0}\}$; $h(\mathbf{p}) = (\mathbf{0},\mathbf{0})$. From the assumption $\mathbf{u} \in \mathrm{Ker} f'(\mathbf{p}) \setminus T_{\mathbf{p}}(M)$ we derive that $h'(\mathbf{p})(\mathbf{u})$ does not belong to the subspace $\mathbb{R}^m \times \{\mathbf{0}\}$ of $\mathbb{R}^m \times G$. Thus, composing h with a suitable linear isomorphism (if necessary), we may actually assume that the subspace $\{\mathbf{0}\} \times G$ of $\mathbb{R}^m \times G$ contains the nonzero vector $h'(\mathbf{p})(\mathbf{u})$. Therefore, without loss of generality, we may suppose that $G = E_1 \times \mathbb{R}$ and $h'(\mathbf{p})(\mathbf{u}) = (\mathbf{0},\mathbf{0},1)$. Consequently, using the fact that the hypothesis of Theorem 2 are invariant under C^2 diffeomorphisms, we may regard

- f as a map from $E = \mathbb{R}^m \times E_1 \times \mathbb{R}$ into F;
- the trivial manifold M as $\mathbb{R}^m \times \{\mathbf{0}\} \times \{0\}$;
- $\mathbf{p}$ as $(\mathbf{0},\mathbf{0},0) = \mathbf{0} \in E$;
- $\mathbf{u}$ as $(\mathbf{0},\mathbf{0},1)$.

In this new formulation the map f satisfies the following conditions:

(α) $f(\mathbf{x},\mathbf{0},0) = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^m$;

(β) $D_2 f(\mathbf{0},\mathbf{0},0) \in L(E_1, F)$ is 1-1;

(γ) $D_3 f(\mathbf{0},\mathbf{0},0) = \mathbf{0} \in L(\mathbb{R}, F) \cong F$.

From (α) and (γ) we derive $\mathrm{Im} f'(\mathbf{0},\mathbf{0},0) = \mathrm{Im} D_2 f(\mathbf{0},\mathbf{0},0)$. By assumption, $f'(\mathbf{0},\mathbf{0},0)$ is a Fredholm operator of index one. Thus, F may be regarded as a product $\mathbb{R}^m \times F_1$ with $\{\mathbf{0}\} \times F_1 = \mathrm{Im} f'(\mathbf{0},\mathbf{0},0)$. From these observations and using Remark 2 we obtain that the map f can be written in the form

$$f(\mathbf{x},\mathbf{y},\mu) = \begin{bmatrix} f_1(\mathbf{x},\mathbf{y},\mu) \\ \\ f_2(\mathbf{x},\mathbf{y},\mu) \end{bmatrix} = \begin{bmatrix} 0 & \alpha(\mathbf{x},\mathbf{y},\mu) & \beta(\mathbf{x},\mathbf{y},\mu) \\ \\ 0 & \gamma(\mathbf{x},\mathbf{y},\mu) & \delta(\mathbf{x},\mathbf{y},\mu) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mu \end{bmatrix}$$

where

$$\alpha : E \to L(E_1, \mathbb{R}^m), \qquad \beta : E \to L(\mathbb{R}, \mathbb{R}^m) \cong \mathbb{R}^m$$
$$\gamma : E \to L(E_1, F_1), \qquad \delta : E \to L(\mathbb{R}, F_1) \cong F_1$$

are C^1 maps. Thus, the equation $f(\mathbf{x},\mathbf{y},\mu) = \mathbf{0} \in F$ is equivalent to the following system

$$(6) \qquad \begin{cases} \alpha(\mathbf{x},\mathbf{y},\mu)(\mathbf{y}) + \mu\beta(\mathbf{x},\mathbf{y},\mu) = \mathbf{0} \\ \gamma(\mathbf{x},\mathbf{y},\mu)(\mathbf{y}) + \mu\delta(\mathbf{x},\mathbf{y},\mu) = \mathbf{0}. \end{cases}$$

Since $\mathrm{Im} f'(\mathbf{0},\mathbf{0},0) = \{\mathbf{0}\}\times F_1$, one has $f_1'(\mathbf{0},\mathbf{0},0) = \mathbf{0} \in L(E,\mathbb{R}^m)$ and $\mathrm{Im} f_2'(\mathbf{0},\mathbf{0},0) = F_1$. Thus $D_2f_1(\mathbf{0},\mathbf{0},0) = \alpha(\mathbf{0},\mathbf{0},0) = \mathbf{0}$ and $D_3f_1(\mathbf{0},\mathbf{0},0) = \beta(\mathbf{0},\mathbf{0},0) = \mathbf{0}$. Moreover, from condition (γ) one gets $D_3f_2(\mathbf{0},\mathbf{0},0) = \delta(\mathbf{0},\mathbf{0},0) = \mathbf{0} \in F_1$. This implies that the operator $\gamma(\mathbf{0},\mathbf{0},0)$ is onto, since $\mathrm{Im}\gamma(\mathbf{0},\mathbf{0},0) = \mathrm{Im}D_2f_2(\mathbf{0},\mathbf{0},0) = \mathrm{Im}f_2'(\mathbf{0},\mathbf{0},0)$.

Condition (β) implies that the operator $\gamma(\mathbf{0},\mathbf{0},0) = D_2f_2(\mathbf{0},\mathbf{0},0)$ is also 1-1. Therefore the Implicit Function Theorem applies to the equation $f_2(\mathbf{x},\mathbf{y},\mu) = \mathbf{0}$, and one can express $\mathbf{y}$ as a C^2 function of $(\mathbf{x},\mu)$ in a suitable neighborhood of $(\mathbf{0},0) \in \mathbb{R}^m\times\mathbb{R}$; say $\mathbf{y} = \varphi(\mathbf{x},\mu)$. Since $\gamma(\mathbf{0},\mathbf{0},0)$ is invertible, and the set of isomorphisms is open in $L(E_1,F_1)$, from the second equation of (6) we derive

$$\varphi(\mathbf{x},\mu) = \gamma(\mathbf{x},\varphi(\mathbf{x},\mu),\mu)^{-1}(-\mu\delta(\mathbf{x},\varphi(\mathbf{x},\mu),\mu)),$$

for $(\mathbf{x},\mu)$ in a convenient neighborhood of $(\mathbf{0},0)$. Hence $\varphi(\mathbf{x},\mu) = \mu\psi(\mathbf{x},\mu)$, where

$$\psi(\mathbf{x},\mu) = -\gamma(\mathbf{x},\varphi(\mathbf{x},\mu),\mu)^{-1}(\delta(\mathbf{x},\varphi(\mathbf{x},\mu),\mu))$$

is C^1 (by Lemma 2). Also observe that $\psi(\mathbf{0},0) = \mathbf{0}$, since $\varphi(\mathbf{0},0) = \mathbf{0}$ and $\delta(\mathbf{0},\mathbf{0},0) = \mathbf{0}$.

Substitution into the first equation of system (6) gives

$$\mu\alpha(\mathbf{x},\varphi(\mathbf{x},\mu),\mu)(\psi(\mathbf{x},\mu)) + \mu\beta(\mathbf{x},\varphi(\mathbf{x},\mu),\mu) = \mathbf{0},$$

which for $\mu \neq 0$ is equivalent to $S(\mathbf{x},\mu) = \mathbf{0}$, where

$$S(\mathbf{x},\mu) = \alpha(\mathbf{x},\mu\psi(\mathbf{x},\mu),\mu)(\psi(\mathbf{x},\mu)) + \beta(\mathbf{x},\mu\psi(\mathbf{x},\mu),\mu).$$

Clearly S is C^1 in a neighborhood of $(\mathbf{0},0) \in \mathbb{R}^m\times\mathbb{R}$ and such that $S(\mathbf{0},0) = \mathbf{0}$. We want to apply the Implicit Function Theorem to the equation $S(\mathbf{x},\mu) = \mathbf{0}$, in order to express $\mathbf{x}$ as a C^1 function of μ in a neighborhood of $\mu = 0$. Thus, we need to show that $D_1S(\mathbf{0},0)$ is an isomorphism of $\mathbb{R}^m$. The transversality assumption (5) of Theorem 2 will be crucial to accomplish this. Observe first that (5) says that the derivative at $\mathbf{0}$ of the map

$$\omega : \mathbf{x} \in \mathbb{R}^m \longmapsto D_3f_1(\mathbf{x},\mathbf{0},0) \in \mathbb{R}^m$$

is 1-1, and hence an isomorphism (since the source space and the target space have the same dimension).

We have $f_1(\mathbf{x},\mathbf{y},\mu) = \alpha(\mathbf{x},\mathbf{y},\mu)(\mathbf{y}) + \mu\beta(\mathbf{x},\mathbf{y},\mu)$. Therefore,

$$D_3 f_1(\mathbf{x},\mathbf{y},\mu) = D_3\alpha(\mathbf{x},\mathbf{y},\mu)(\mathbf{y}) + \beta(\mathbf{x},\mathbf{y},\mu) + \mu D_3\beta(\mathbf{x},\mathbf{y},\mu).$$

This implies $\omega(\mathbf{x}) = \beta(\mathbf{x},\mathbf{0},0)$, and consequently $\omega'(\mathbf{0}) = D_1\beta(\mathbf{0},\mathbf{0},0)$.

Let us prove that $D_1 S(\mathbf{0},0) = D_1\beta(\mathbf{0},\mathbf{0},0)$. To do this notice that

$$S(\mathbf{x},0) = \alpha(\mathbf{x},\mathbf{0},0)(\psi(\mathbf{x},0)) + \beta(\mathbf{x},\mathbf{0},0).$$

Therefore, $\forall\, \mathbf{v} \in \mathbb{R}^m$,

$$D_1 S(\mathbf{0},0)(\mathbf{v}) = \big(D_1\alpha(\mathbf{0},\mathbf{0},0)(\mathbf{v})\big)(\psi(\mathbf{0},0)) + \alpha(\mathbf{0},\mathbf{0},0)\big(D_1\psi(\mathbf{0},0)(\mathbf{v})\big) + D_1\beta(\mathbf{0},\mathbf{0},0)(\mathbf{v}).$$

Now, as observed above, we have $\alpha(\mathbf{0},\mathbf{0},0) = \mathbf{0}$ and $\psi(\mathbf{0},0) = \mathbf{0}$. Thus

$$D_1 S(\mathbf{0},0)(\mathbf{v}) = D_1\beta(\mathbf{0},\mathbf{0},0)(\mathbf{v}), \text{ for all } \mathbf{v} \in \mathbb{R}^m,$$

as claimed.

Since, as pointed out before, $D_1\beta(\mathbf{0},\mathbf{0},0) = \omega'(\mathbf{0})$ is an isomorphism, the Implicit Function Theorem implies that there exists a C^1 function $\mathbf{x} = \mathbf{x}(\mu)$, defined in a neighborhood of $\mu = 0$, such that $\mathbf{x}(0) = \mathbf{0}$ and $S(\mathbf{x}(\mu),\mu) = \mathbf{0}$. Consequently

$$f(\mathbf{x}(\mu), \varphi(\mathbf{x}(\mu),\mu),\mu) = 0$$

in a neighborhood of $\mu = 0$, and this finally implies that $\mathbf{0} \in E$ is a bifurcation point relative to the trivial set of solutions $M = \mathbb{R}^m \times \{\mathbf{0}\} \times \{0\}$. ∎

Remark 3. The proof of Theorem 2 actually shows that the closure of the set of non-trivial solutions of (4) in a neighborhood of $\mathbf{p} \in M$ is a C^1 curve parametrized by $\mu \longmapsto (\mathbf{x}(\mu), \varphi(\mathbf{x}(\mu),\mu),\mu)$.

Remark 4. The assumption

there exists $\mathbf{u} \in \operatorname{Ker} f'(\mathbf{p}) \setminus T_{\mathbf{p}}(M)$ *such that*

$$f''(\mathbf{p})(\mathbf{u},\mathbf{v}) \notin \operatorname{Im} f'(\mathbf{p}), \ \forall\, \mathbf{v} \in T_{\mathbf{p}}(M)\setminus\{\mathbf{0}\},$$

can be replaced by the equivalent condition

there exists $\mathbf{u} \in \operatorname{Ker} f'(\mathbf{p}) \setminus T_{\mathbf{p}}(M)$ *such that the map*

$$\mathbf{v} \in T_{\mathbf{p}}(M) \longmapsto \pi f''(\mathbf{p})(\mathbf{u},\mathbf{v}) \in F/\operatorname{Im} f'(\mathbf{p}),$$

where $\pi: F \to F/\operatorname{Im} f'(\mathbf{p})$ *is the canonical projection, is an isomorphism.*

Observe that any surjective bounded linear operator $Q: F \to \mathbb{R}^m$ such that $\operatorname{Ker} Q = \operatorname{Im} f'(\mathbf{p})$ would play in Remark 4 the same role as π. In particular, since $\operatorname{Im} f'(\mathbf{p})$ is m-codimensional, Q could be a projection, parallel to $\operatorname{Im} f'(\mathbf{p})$, onto a direct summand of $\operatorname{Im} f'(\mathbf{p})$.

Remark 5. Theorem 2 is still true under the more general condition

$$\operatorname{ind} f \geq 1 \textit{ and the map } \mathbf{v} \in T_{\mathbf{p}}(M) \longmapsto \pi f''(\mathbf{p})(\mathbf{u},\mathbf{v}) \in F/\operatorname{Im} f'(\mathbf{p}) \textit{ is onto},$$

where π is as in Remark 4.

The result can be established as follows. The case $\operatorname{ind} f = 1$ is proved in Theorem 2. Let $\operatorname{ind} f > 1$. As in the proof of Theorem 2, we can assume without loss of generality

$$U = E = \mathbb{R}^m \times E_1 \times \mathbb{R},\ F = \mathbb{R}^m \times F_1,\ M = \mathbb{R}^m \times \{\mathbf{0}\} \times \{0\},\ \mathbf{p} = (\mathbf{0},\mathbf{0},0),\ \mathbf{u} = (\mathbf{0},\mathbf{0},1).$$

Let X be a subspace of $\mathbb{R}^m$ such that the restriction of $\pi f''(\mathbf{p})(\mathbf{u},\cdot)$ to X is an isomorphism between X and $F/\operatorname{Im} f'(\mathbf{p})$. Theorem 2 implies (see Remark 4) that $\mathbf{0} \in E$ is a bifurcation point relative to $X \times \{\mathbf{0}\} \times \{0\}$ for the restriction of f to $X \times E_1 \times \mathbb{R}$. Hence $\mathbf{0}$ is a bifurcation point for f relative to M, since any element of $X \times E_1 \times \mathbb{R}$ which is not in $X \times \{\mathbf{0}\} \times \{0\}$ does not belong to M.

The following bifurcation result of Crandall-Rabinowitz [CR] can be easily derived from Theorem 2.

Corollary 1. *Let* G, F *be real Banach spaces,* Ω *an open neighborhood of* $\mathbf{0}$ *in* G *and* $I \subseteq \mathbb{R}$ *an open interval. Let* $f: I \times \Omega \to F$ *be a* C^2*map satisfying* $f(\lambda,\mathbf{0}) = \mathbf{0}$ *for every* $\lambda \in I$. *Assume that there exists* $\lambda_0 \in I$ *such that*

(i) $\dim \operatorname{Ker} D_2 f(\lambda_0,\mathbf{0}) = \operatorname{codim} \operatorname{Im} D_2 f(\lambda_0,\mathbf{0}) = 1$

(ii) *there exists* $\mathbf{x}_0 \in G$ *spanning* $\operatorname{Ker} D_2 f(\lambda_0,\mathbf{0})$ *such that*

$$D_1 D_2 f(\lambda_0,\mathbf{0})(\mathbf{x}_0) \notin \operatorname{Im} D_2 f(\lambda_0,\mathbf{0}).$$

Let G_0 *be any complement of* $\operatorname{span}\{\mathbf{x}_0\}$ *in* E. *Then there exists an open interval* J *containing* 0 *and continuously differentiable functions* $\lambda: J \to \mathbb{R}$ *and* $\psi: J \to G_0$ *such that* $\lambda(0) = \lambda_0$, $\psi(0) = \mathbf{0}$, *and, if* $\mathbf{x}(s) = s\mathbf{x}_0 + s\psi(s)$, *then* $f(\lambda(s),\mathbf{x}(s)) = \mathbf{0}$. *Moreover,* $f^{-1}(\mathbf{0})$ *near* $(\lambda_0,\mathbf{0})$ *consists precisely of the curves* $\mathbf{x} = 0$ *and* $(\lambda(s),\mathbf{x}(s))$, $s \in J$.

Proof. Let $E = \mathbb{R} \times G$, $U = I \times \Omega$, $M = I \times \{\mathbf{0}\}$, $\mathbf{p} = (\lambda_0,\mathbf{0})$. It is evident that f is a C^2 Fredholm map of index 1 in a sufficiently small ball around $\mathbf{p}$ and, by (i), $\dim \operatorname{Ker} f'(\mathbf{p}) = 2$ (see Theorem 1). Moreover, with

$$\mathbf{u} = (0,\mathbf{x}_0) \in \operatorname{Ker} f'(\mathbf{p}) \setminus T_{\mathbf{p}}(M)$$

we obtain, by (ii), that

$$f''(\mathbf{p})(\mathbf{u},\mathbf{v}) \notin \operatorname{Im} f'(\mathbf{p}),\ \ \forall \mathbf{v} \in T_{\mathbf{p}}(M) \setminus \{\mathbf{0}\}.$$

According to Theorem 2, $\mathbf{p}$ is a bifurcation point and, by Remark 3, we obtain a C^1 parametrization of the bifurcating branch of non-trivial solutions. ∎

Observe that the above Corollary contains as a particular case the classical bifurcation theorem of Krasnosel'skii [K] when the eigenvalue $1/\lambda_0$ is simple.

We want to show now that the following bifurcation result (see [Ma], [FP1], [IM]) can be easily derived from Theorem 2.

Corollary 2. *Let* G, F *be Banach spaces,* $L: G \to F$ *be a linear Fredholm operator of index* 0, $H: \mathbb{R}\times G \to F$ *be a* C^2 *compact operator such that* $H(0, \mathbf{x}) = \mathbf{0}$ *for every* $\mathbf{x} \in G$. *Assume that there exists* $\mathbf{x}_0 \in \mathrm{Ker}L$ *such that* $D_1H(0, \mathbf{x}_0) \in \mathrm{Im}L$. *Let* $m = \dim\mathrm{Ker}L$, $Q: F \to \mathbb{R}^m$ *be a bounded linear map such that* $\mathrm{Ker}Q = \mathrm{Im}L$, *and* $K: G \to F$ *be the operator* $D_2D_1H(0, \mathbf{x}_0)$. *Then* $\mathbf{x}_0$ *is a bifurcation point of*

$$L(\mathbf{x}) + H(\lambda, \mathbf{x}) = \mathbf{0},$$

with respect to $\{0\}\times\mathrm{Ker}L$, *provided that the restriction* $QK|_{\mathrm{Ker}L}: \mathrm{Ker}L \to \mathbb{R}^m$ *of the linear operator* QK *to* KerL *is an isomorphism.*

Proof. Let $E = \mathbb{R}\times G$, $M = \{0\}\times\mathrm{Ker}L$, $f(\lambda, \mathbf{x}) = L(\mathbf{x}) + H(\lambda, \mathbf{x})$. Then $\dim M = m$. Moreover, f is a Fredholm operator of index 1, since its restriction to the one-codimensional subspace $\{0\}\times G$ of E is Fredholm of index 0.

Set $\mathbf{p} = (0, \mathbf{x}_0)$. Since

$$f'(\mathbf{p})(\mu, \mathbf{w}) = L(\mathbf{w}) + \mu D_1H(\mathbf{p}),$$

we see that the condition $\dim\mathrm{Ker}f'(\mathbf{p}) = m+1$ is satisfied. In fact every vector of the form $(0, \mathbf{w})$, $\mathbf{w} \in \mathrm{Ker}L$, belongs to $\mathrm{Ker}f'(\mathbf{p})$. Moreover, there is a vector $\mathbf{h} \in G$ such that

$$L(\mathbf{h}) = -D_1H(\mathbf{p}).$$

Consequently $\mathrm{Ker}f'(\mathbf{p}) = \mathrm{Span}\{\{0\}\times\mathrm{Ker}L, \mathbf{u}\}$, where $\mathbf{u} = (1, \mathbf{h})$. Hence

$$\dim\mathrm{Ker}f'(\mathbf{p}) = m+1 \quad \text{and} \quad \mathrm{Im}f'(\mathbf{p}) = \mathrm{Im}L.$$

It remains to show that for every $\mathbf{v} = (0, \mathbf{k})$, $\mathbf{k} \in \mathrm{Ker}L\backslash\{\mathbf{0}\}$, we have

$$f''(\mathbf{p})(\mathbf{u}, \mathbf{v}) \notin \mathrm{Im}f'(\mathbf{p}) = \mathrm{Im}L.$$

We know that

$$D_2D_1H(0, \mathbf{x}_0)(\mathbf{k}) \notin \mathrm{Im}L,$$

since $QK|_{\mathrm{Ker}L}$ is an isomorphism and $\mathrm{Ker}Q = \mathrm{Im}L$. Therefore the result will be established if we prove that

$$D_2D_1H(0, \mathbf{x}_0)(\mathbf{k}) = f''(\mathbf{p})(\mathbf{u}, \mathbf{v}).$$

Using the strategy outlined in Preliminaries and noticing that

$$f'(0, \mathbf{x})(\mathbf{u}) = L(\mathbf{h}) + D_1H(0, \mathbf{x}),$$

we have $f''(\mathbf{p})(\mathbf{u}, \mathbf{v}) = \varphi'(0)$, where $\varphi(t) = f'(0, \mathbf{x}_0 + t\mathbf{k})(\mathbf{u})$. Since we also have

$$\varphi'(0) = D_2D_1H(0, \mathbf{x}_0)(\mathbf{k}),$$

the result follows. ∎

We now give a consequence of Theorem 2 concerning the existence of non-trivial fixed point of an operator equation depending on a parameter. This result will be later used to establish the existence of non-trivial periodic solutions of a first order system depending on a parameter, in the case when the set of stationary states (resting points) is a differentiable manifold.

Let M be a C^2 m-dimensional manifold and $\psi: U \to M$ be a C^2 map defined on an open set U of $\mathbb{R}\times M$. Assume that $\{0\}\times M \subseteq U$ and $\psi(0,\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \in M$. Consider the problem

$$\mathbf{x} = \psi(\lambda,\mathbf{x}). \tag{7}$$

The set of trivial solutions of (7) is $\{(0,\mathbf{x}): \mathbf{x} \in M\}$, which we identify with M. An element $\mathbf{p} \in M$ is a bifurcation point if in any neighborhood of $(0,\mathbf{p})$ in $\mathbb{R}\times M$ we can find solutions of (7) with $\lambda \neq 0$.

Corollary 3 below gives a necessary condition and a sufficient condition for this to happen. The corollary is a consequence of Theorems 1 and 2. To clarify its statement we need the following facts. First, consider the curve $\psi(\cdot,\mathbf{x})$, $\mathbf{x} \in M$. From $\psi(0,\mathbf{x}) = \mathbf{x}$, we derive $D_1\psi(0,\mathbf{x}) \in T_{\mathbf{x}}(M)$. Consequently, $D_1\psi(0,\cdot)$ can be regarded as a tangent vector field on M. Second, recall that given a C^1 tangent vector field $v: M \to TM$ and a point $\mathbf{p} \in M$ such that $v(\mathbf{p}) = \mathbf{0} \in T_{\mathbf{p}}(M)$, we can regard $v'(\mathbf{p})$ as an endomorphism of $T_{\mathbf{p}}(M)$ (see, for example [Mi], §6, Lemma 5).

Corollary 3. *An element* $\mathbf{p} \in M$ *can be a bifurcation point for* (7) *only when*

(i) *the vector field* $v(\mathbf{x}) = D_1\psi(0,\mathbf{x})$ *vanishes at* $\mathbf{p}$.

Moreover, if i) *is satisfied and the linear operator*

(ii) $v'(\mathbf{p})$ *is an isomorphism of* $T_{\mathbf{p}}(M)$,

then $\mathbf{p}$ *is a bifurcation point for* (7).

Proof. Since assumptions (i) and (ii) are local and invariant under C^2 diffeomorphisms we may assume, without loss of generality, $M = \mathbb{R}^m$. Define $f(\lambda,\mathbf{x}) = \mathbf{x} - \psi(\lambda,\mathbf{x})$. By Theorem 1, $(0,\mathbf{p})$ can be a bifurcation point only if $f'(0,\mathbf{p})$ is the null operator, which implies that the vector field

$$v(\mathbf{x}) = D_1\psi(0,\mathbf{x})$$

vanishes at $\mathbf{p}$ (condition (i)).

Assume that (i) and (ii) are satisfied. Then $\dim \operatorname{Ker} f'(0,\mathbf{p}) = m+1$. It is easily seen that given the vector $(1,\mathbf{0})$, which is clearly not tangent at $(0,\mathbf{p})$ to the manifold of trivial solutions, the map

$$\mathbf{h} \longmapsto f''(0,\mathbf{p})((0,\mathbf{h}),(1,\mathbf{0}))$$

is an isomorphism of $\mathbb{R}^m$. The result now follows from Remark 4. ∎

We use Corollary 3 to give a different proof of a result (see Corollary 4) due to Furi-Pera [FP3]. Let M be a m-dimensional smooth manifold and

$g:\mathbb{R}\times M \to TM$ be a smooth time-dependent tangent vector field. Assume that g is T-periodic in t and consider the first order system depending on $\lambda \in \mathbb{R}$,

$$\dot{\mathbf{x}}(t) = \lambda g(t,\mathbf{x}). \tag{8}$$

We say that $(\lambda,\mathbf{q})$ is a solution pair if $\mathbf{q}$ is the starting point of a T-periodic solution of (8). Obviously $(0,\mathbf{q})$, which will be identified with $\mathbf{q}$, is the starting point of the trivial T-periodic (constant) solution $\mathbf{x}(t) \equiv \mathbf{q}$. For this reason we say that M is the manifold of trivial solutions (resting points), and therefore the concept of bifurcation point for the equation (8) is well defined. Let $w:M\to TM$ be the average wind velocity associated to g, i.e.

$$w(\mathbf{q}) = \frac{1}{T}\int_0^T g(t,\mathbf{q})dt.$$

We want to show that the following result is an easy consequence of Corollary 3.

Corollary 4. *An element* $\mathbf{p}\in M$ *is a bifurcation point of* (8) *provided that* $w(\mathbf{p}) = \mathbf{0}$ *and* $w'(\mathbf{p}):T_{\mathbf{p}}(M)\to T_{\mathbf{p}}(M)$ *is an isomorphism.*

Proof. Denote by $t\longmapsto \varphi(\lambda,\mathbf{q},t)$ the maximal solution of (8) with initial condition $\varphi(\lambda,\mathbf{q},0) = \mathbf{q}$. Since $\varphi(0,\mathbf{q},t) = \mathbf{q}$ for all $t\in\mathbb{R}$, $\mathbf{q}\in M$, known results on differential equations imply that the map

$$\psi(\lambda,\mathbf{q}) = \varphi(\lambda,\mathbf{q},T)$$

is defined in a neighborhood of $\{0\}\times M$. As in Corollary 3 we set

$$v(\mathbf{q}) = D_1\psi(0,\mathbf{q}).$$

The result follows if we show that $v(\mathbf{q}) = Tw(\mathbf{q})$. To establish this equality we first embed M in $\mathbb{R}^k$ for some k. It now makes sense to write the equality

$$\psi(\lambda,\mathbf{q}) = \mathbf{q} + \lambda\int_0^T g(t,\varphi(\lambda,\mathbf{q},t))dt.$$

Hence

$$v(\mathbf{q}) = D_1\psi(0,\mathbf{q}) = \int_0^T g(t,\varphi(0,\mathbf{q},t))dt = \int_0^T g(t,\mathbf{q})dt = Tw(\mathbf{q}). \quad \blacksquare$$

We conclude with an additional application of Theorems 1 and 2 to a bifurcation problem for the discrete (logistic) dynamical system

$$x_{n+1} = ax_n(1-x_n),$$

where $x_n\in(0,1)$ and $a\in(0,4)$. The result we obtain is well-known and our aim is just to show how our theory works in this case.

Suppose we are interested in finding the pairs (a,x) such that x is a periodic point of period 2 of the map $x\longmapsto g(a,x)$, where $g(a,x) = ax(1-x)$. Among those pairs we define as trivial the ones such that x is a fixed point of this map. Notice that the set of trivial solutions is the manifold

$$M = \{(a,x) \in (0,4)\times(0,1) : 1 = a(1-x)\}.$$

Define $h(a,x) = g(a,g(a,x))$. Since M is one-dimensional, according to Theorem 1, $(a,x) \in M$ can be a bifurcation point only if $h'(a,x)$ is the null operator. This means

$$D_1 h(a,x) = 0,$$

since M does not admit horizontal tangent vectors. Consequently, we need to solve the system

$$\begin{cases} D_1 h(a,x)(a,x) = 0 \\ a(1-x) = 1 \end{cases}$$

which gives (3,2/3) as the only possible bifurcation point of M.

To prove that (3,2/3) is actually a bifurcation point we define, using the notations of Theorem 2, $f(a,x) = x - h(a,x)$ and $v = (1,0)$. Since $f'(3,2/3)(1,0) = D_1 f(3,2/3)$ and x is a suitable coordinate on M, we obtain that condition (5) is equivalent to

$$D_2 D_1 h(3,2/3) \neq 0.$$

One can easily check that this is true, and the result follows.

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