BIFURCATION OF FIXED POINTS FROM A MANIFOLD OF TRIVIAL FIXED POINTS IN THE INFINITE DIMENSIONAL CASE

MASSIMO FURI, MARIO MARTELLI, AND MARIA PATRIZIA PERA

Abstract. We obtain a necessary as well as a sufficient condition for the existence of bifurcation points of a coincidence equation, and, in particular, of a parametrized fixed point problem. In both cases the trivial solutions are assumed to form a finite dimensional submanifold of a Banach manifold. An application is given to a delay differential equation on a manifold: we detect periodic solutions that rotate close to an equilibrium point.

1. Introduction

Let $f, h\colon X \to Y$ be continuous maps between Banach manifolds. Consider the coincidence equation

$$f(x) = h(x). \tag{1.1}$$

We regard a submanifold M of X as the set of *trivial* solutions of (1.1). An element $p \in M$ is called a bifurcation point (from M) if any neighborhood of p contains nontrivial solutions of (1.1). We are interested in finding and proving necessary as well as sufficient conditions for a point $p \in M$ to be of bifurcation.

The results obtained in this paper can be considered generalizations of those proved in [4] and [6]. In fact, X and Y are Banach spaces in [4] and finite dimensional manifolds in [6].

A particular case of (1.1) is the following fixed point equation depending on a real parameter λ:

$$z = h(\lambda, z). \tag{1.2}$$

In this case $X = \mathbb{R} \times Z$, $Y = Z$, f is the projection π_2 onto the second factor Z, and a coincidence point x is a pair (λ, z) with z a fixed point of the partial map $h(\lambda, \cdot)$. The manifold of trivial solutions is $M = \{0\} \times M_0$, where M_0 is a submanifold of Z, contained in the set of fixed points of $h(0, \cdot)$.

The obtained results are applied to determine conditions for the existence of small oscillations of the parametrized delay differential equation

$$x'(t) = \lambda F(t, x(t), x(t-1)). \tag{1.3}$$

The map F is assumed to be periodic in the first variable and tangent to a manifold N in the second one.

The paper is articulated in five different sections.

After this Introduction, we present some preliminaries (Section 2). Our readers should pay particular attention to the definition of *coincidence Hessian*, $H(f,h)(p)$,

2000 *Mathematics Subject Classification.* Primary 58E07; Secondary 34C25, 34C40.
Key words and phrases. bifurcation of fixed points, manifolds, forced oscillations.

of the pair of functions (f, h), evaluated at a coincidence point $p \in X$ of (1.1). This is an intrinsically defined symmetric bilinear operator from the kernel to the cokernel of

$$(f'(p) - h'(p))\colon T_pN \to T_{f(p)}N.$$

As usual T_pN and $T_{f(p)}N$ denote the tangent spaces of N at p and $f(p)$ $(= h(p))$, respectively. The Hessian $H(f,h)(p)$ is a well defined substitute of what, when X and Y are Banach spaces, is the bilinear operator $f''(p) - h''(p)$. In fact, as well known, the second derivative is not intrinsically defined in the framework of manifolds. Theorem 2.3 provides a practical way of computing $H(f,h)(p)$ in the case when X and Y are embedded in Banach spaces.

Section 3 contains two theoretical results, namely a necessary condition (Theorem 3.5) and a sufficient condition (Theorem 3.7) for a coincidence point $p \in M$ of (1.1) to be of bifurcation. The coincidence Hessian $H(f,h)(p)$ plays a crucial role in the statement of the sufficient condition, which, in the framework of Banach spaces, can be expressed (as in [4]) in terms of second derivatives (see Theorem 3.2).

Section 4 contains bifurcations results for the parametrized fixed point equation (1.2). Corollaries 4.1 and 4.2 are the natural reformulations of Theorems 3.5 and 3.7 for this particular case. Some additional results of this section, particularly Theorems 4.5 and 4.6, will facilitate the application contained in the next section.

Finally, in Section 5, we study the delay differential equation (1.3). We assume, for simplicity, that F is T-periodic in the first variable with $T \geq 1$. We present a possibile way of proving the existence, for small values of λ, of periodic solutions of period T rotating around equilibria where bifurcation takes place. The existence result expressed in Theorem 5.2 could be established also when $T < 1$, using a more elaborate computation. We have selected not to do so in this paper, and to focus on a straightforward application of our abstract results.

2. Notation and Preliminaries

We shall assume that all Banach spaces are real and all manifolds are smooth and modeled on Banach spaces. Although most of the statements make sense with less regularity, we are not interested, in this paper, in a more general setting.

Given two manifolds X and Y, and a C^1 map $f\colon X \to Y$, the (*first*) *derivative* of f at $x \in X$ will be denoted either by $Df(x)$ or $f'(x)$, whichever is more convenient. It is well-known that $f'(x)$ is a linear operator sending the tangent space T_xX of X at x into the tangent space $T_{f(x)}Y$ of Y at $f(x)$.

When $X = X_1 \times X_2$, the *partial derivative* with respect to the first (respectively, the second) variable at (x_1, x_2) will be indicated with $\partial_1 f(x_1, x_2)$ (respectively, $\partial_2 f(x_1, x_2)$). Thus, for any pair of tangent vectors $(u_1, u_2) \in T_{x_1}X_1 \times T_{x_2}X_2$, one has

$$Df(x_1, x_2)(u_1, u_2) = \partial_1 f(x_1, x_2)u_1 + \partial_2 f(x_1, x_2)u_2.$$

In particular, when $X_1 = \mathbb{R}$, the partial derivative $\partial_1 f(x_1, x_2)$, which is actually a linear operator from $\mathbb{R}$ to the tangent space $T_{f(x_1,x_2)}Y$, will be identified with the tangent vector $\partial_1 f(x_1, x_2)(1) \in T_{f(x_1,x_2)}Y$. With this notation, for the (total) derivative $Df(x_1, x_2)$, one has the equality

$$Df(x_1, x_2)(u_1, u_2) = u_1\partial_1 f(x_1, x_2) + \partial_2 f(x_1, x_2)u_2,$$

where $(u_1, u_2) \in \mathbb{R} \times T_{x_2}X_2$.

When X and Y are Banach spaces, the *second derivative* of a C^2 map $f\colon X \to Y$ at $x \in X$ is a symmetric bilinear operator from X to Y, i.e. an element of the space $L^2_s(X,Y)$, and will be denoted either by $D^2 f(x)$ or $f''(x)$. The following is a practical method for its computation. Given $u, v \in X$, consider the function of two real variables $\sigma(r,s) = f(x + ru + sv)$. Then, it is easy to see that

$$f''(x)(u,v) = \frac{\partial^2 \sigma}{\partial r \partial s}(0,0).$$

However, the second derivative of f at $x \in X$ is not intrinsically defined when X and Y are differentiable manifolds, since only part of this derivative is independent of the coordinates. In this case, one can define (see e.g. [1]) an intrinsic symmetric bilinear operator $Hf(x)$, called the *Hessian* of f at x, from $\operatorname{Ker} f'(x)$ to $\operatorname{coKer} f'(x) = T_{f(x)}Y / \operatorname{Im} f'(x)$, i.e. an element of $L^2_s(\operatorname{Ker} f'(x), \operatorname{coKer} f'(x))$.

Given $x \in X$, the operator $Hf(x)$ is defined as follows. Consider two charts $\varphi\colon U \subseteq X \to E$ and $\psi\colon V \subseteq Y \to F$ about x and $y = f(x)$. Then

$$Hf(x)(u,v) = \Pi\, \psi'(y)^{-1} \hat{f}''(\varphi(x))(\varphi'(x)u, \varphi'(x)v), \tag{2.1}$$

where $u, v \in \operatorname{Ker} f'(x)$, $\hat{f}$ is the composition $\psi f \varphi^{-1}$, and $\Pi\colon T_yY \to T_yY/\operatorname{Im} f'(x)$ is the canonical projection. It can be shown (see [6]) that (2.1) does not depend on the choice of the charts φ and ψ. Thus, $Hf(x) \in L^2_s(\operatorname{Ker} f'(x), \operatorname{coKer} f'(x))$ is well defined.

Notice that, when X and Y are Banach spaces, $x \in X$ and $u, v \in \operatorname{Ker} f'(x)$, then the classical second derivative $f''(x)(u,v)$ belongs to the equivalence class $Hf(x)(u,v)$.

An interesting special case, that justifies the appellative "Hessian", arises when f is a real function on X and $x \in X$ is a critical point of f. In this case $\operatorname{Ker} f'(x) = T_xX$ and $\operatorname{coKer} f'(x) = \mathbb{R}$. Thus $Hf(x)$ is the classical Hessian, which can be regarded either as a symmetric bilinear form or as a quadratic form on the tangent space T_xX.

The following theorem provides a practical way of finding an element of the equivalence class $Hf(x)(u,v)$ when X and Y are embedded in Banach spaces.

Theorem 2.1. *Let $f\colon X \to Y$ be a C^2 map between two manifolds. Assume that X and Y are embedded in Banach spaces $\check{E}$ and $\check{F}$ respectively. Given $p \in X$ and $u, v \in \operatorname{Ker} f'(p) \subseteq T_pX$, let $\alpha, \beta : (-\epsilon, \epsilon) \to \check{E}$ be two C^1 curves such that $\alpha(\tau) \in X$, $\beta(\tau) \in T_{\alpha(\tau)}X \subseteq \check{E}$, $\alpha(0) = p, \alpha'(0) = u$, and $\beta(0) = v$. Consider the curve*

$$\gamma\colon (-\epsilon, \epsilon) \to \check{F},\ \ \gamma(\tau) = f'(\alpha(\tau))\beta(\tau).$$

Then the vector $\gamma'(0)$ is tangent to Y at $f(p)$, and it is a representative of the equivalence class $Hf(p)(u,v)$.

Proof. Let $\varphi\colon U \to E$ and $\psi\colon V \to F$ be two charts about p and $q = f(p)$ respectively. We may assume, without loss of generality, that $U = X$, $V = Y$, $\varphi(X) = E$, and $\psi(Y) = F$. Moreover, we may also suppose that F is a subspace of a Banach space G and that ψ is the restriction to Y as domain and to F as codomain of a diffeomorphism $\Psi\colon \check{F} \to G$. We shall use the following maps and elements:

1. $p_0 = \varphi(p),\ u_0 = \varphi'(p)u,\ v_0 = \varphi'(p)v$;
2. $\hat{f}\colon E \to F$, $\xi \mapsto \psi(f(\varphi^{-1}(\xi)))$;
3. $\check{f}\colon E \to \check{F}$, $\xi \mapsto f(\varphi^{-1}(\xi))$.

Notice that $\hat{f}$ cannot be regarded as the composition of $\check{f}$ and ψ, since ψ is not globally defined on the codomain $\check{F}$ of $\check{f}$. However, in what follows, it is convenient to consider $\hat{f}$ as the composition of two maps between Banach spaces, namely $\check{f}$ and Ψ. Taking (2.1) into account we obtain that a vector $w \in T_qY$ is a representative of $Hf(p)(u,v)$ iff

$$\psi'(q)w - \hat{f}''(p_0)(u_0,v_0) \in \operatorname{Im}\hat{f}'(p_0).$$

Hence, we compute $\hat{f}''(p_0)(u_0,v_0)$. We obtain

$$\hat{f}''(p_0)(u_0,v_0) = \Psi''(\check{f}(p_0))(\check{f}'(p_0)u_0,\check{f}'(p_0)v_0) + \Psi'(\check{f}(p_0))\check{f}''(p_0)(u_0,v_0).$$

Since $\Psi''(\check{f}(p_0))$ is a bilinear operator and $u_0,v_0 \in \operatorname{Ker}\check{f}'(p_0)$, it follows that

$$\Psi''(\check{f}(p_0))(\check{f}'(p_0)u_0,\check{f}'(p_0)v_0) = 0$$

and, consequently,

$$\hat{f}''(p_0)(u_0,v_0) = \Psi'(\check{f}(p_0))\check{f}''(p_0)(u_0,v_0).$$

From the left-hand-side of this equality we derive that

$$\Psi'(\check{f}(p_0))\check{f}''(p_0)(u_0,v_0) \in F.$$

Therefore, since $\Psi'(\check{f}(p_0))\colon \check{F} \to G$ is an isomorphism that maps T_qY onto F, the vector $w = \check{f}''(p_0)(u_0,v_0)$ belongs to T_qY. Hence, $\hat{f}''(p_0)(u_0,v_0) = \psi'(q)w$. This implies that w is a representative of $Hf(p)(u,v)$.

It remains to prove that $\gamma'(0) - w$ belongs to $\operatorname{Im} f'(p)$, which coincides with $\operatorname{Im}\check{f}'(p_0))$. Let

$$\alpha_0(\tau) = \varphi(\alpha(\tau)),\ \ \beta_0(\tau) = \varphi'(\alpha(\tau))\beta(\tau)$$

and set $\gamma_0(\tau) = \check{f}'(\alpha_0(\tau))\beta_0(\tau)$. Hence,

$$\begin{aligned}\gamma_0'(0) &= \check{f}''(\alpha_0(0))(\alpha_0'(0),\beta_0(0)) + \check{f}'(\alpha_0(0))\beta_0'(0)\\ &= \check{f}''(p_0)(u_0,v_0) + \check{f}'(p_0)\beta_0'(0) = w + \check{f}'(p_0)\beta_0'(0).\end{aligned}$$

Therefore,

$$\gamma_0'(0) - w \in \operatorname{Im}\check{f}'(p_0) = \operatorname{Im} f'(p).$$

The final result follows from the easily verified equality $\gamma_0(\tau) = \gamma(\tau)$. □

The following property of Hf will be useful in the sequel.

Lemma 2.2. *Let $f\colon X \to Y$ be a C^2 map between two manifolds and assume that f is constant on a submanifold M of X. Let p be an element of M and $u,v \in T_pM$. Then, $u,v \in \operatorname{Ker} f'(p)$ and $Hf(p)(u,v) = 0$.*

Proof. By assumption, f is constant on the submanifold M of X. Therefore, T_pM is a subspace of $\operatorname{Ker} f'(p)$. Moreover, according to the definition introduced in (2.1), the map $\hat{f}\colon \varphi(U) \to \psi(V)$ is constant on $\varphi(U \cap M)$. We may assume that φ maps $U \cap M$ into a subspace E_0 of E. Therefore, the vectors $\varphi'(p)u$ and $\varphi'(p)v$ belong to E_0. Hence, the map $\hat{\sigma}(r,s) = \hat{f}(\varphi(p) + r\varphi'(p)u + s\varphi'(p)v)$ is constant. Consequently,

$$\hat{f}''(\varphi(p))(\varphi'(p)u,\varphi'(p)v) = \frac{\partial^2\hat{\sigma}}{\partial r\partial s}(0,0) = 0.$$

This clearly implies

$$Hf(p)(u,v) = \Pi\,\psi'(q)^{-1}\hat{f}''(\varphi(p))(\varphi'(p)u,\varphi'(p)v) = 0,$$

where $q = f(p)$. The above equality proves the lemma. □

Let, as above, X and Y be two manifolds, and let $f, h\colon X \to Y$ be of class C^2. In the particular case when X and Y are Banach spaces, given $p \in X$, the expression

$$H(f,h)(p)(u,v) := H(f-h)(p)(u,v), \tag{2.2}$$

makes sense provided that $u, v \in \operatorname{Ker}(f'(p) - h'(p))$. However, (2.2) is meaningless in the general case, since the difference $f - h$ may not be defined. Nevertheless, assume that p is a coincidence point of the pair (f, h), that is, p is a solution of the equation $f(x) = h(x)$. Then, given u and v in the subspace $\operatorname{Ker} f'(p) \cap \operatorname{Ker} h'(p)$ of $\operatorname{Ker}(f'(p) - h'(p))$, we can define

$$H(f,h)(p)(u,v) := Hf(p)(u,v) - Hh(p)(u,v),$$

since both $Hf(p)(u,v)$ and $Hh(p)(u,v)$ make sense. We would like to extend the bilinear operator $H(f,h)(p)$ to the space $\operatorname{Ker}(f'(p)-h'(p))$ in such a way that when X and Y are Banach spaces we obtain

$$H(f,h)(p) = H(f-h)(p). \tag{2.3}$$

Using the guidelines provided by (2.1) we define the *coincidence Hessian* of the pair (f, h) by

$$H(f,h)(p)(u,v) = \Pi\, \psi'(q)^{-1} \hat{g}''(\varphi(p))(\varphi'(p)u, \varphi'(p)v), \tag{2.4}$$

where, $\hat{g} = \psi f \varphi^{-1} - \psi h \varphi^{-1}$, $q = f(p) = h(p)$, and $u, v \in \operatorname{Ker}(f'(p) - h'(p))$.

When X and Y are embedded in Banach spaces, the coincidence Hessian of (f, h) can be evaluated in a manner similar to the method outlined in Theorem 2.1. The following theorem provides a practical way for finding an element of the equivalence class $H(f,h)(p)(u,v)$. The result will be used in the application presented in Section 5, and the proof is omitted since it can be carried out as in Theorem 2.1.

Theorem 2.3. *Let $f, h\colon X \to Y$ be C^2 maps between manifolds. Assume that X and Y are embedded in Banach spaces $\check{E}$ and $\check{F}$ respectively. Given $p \in X$ such that $f(p) = h(p)$ and $u, v \in \operatorname{Ker}(f'(p) - h'(p)) \subseteq T_pX$, let $\alpha, \beta\colon (-\epsilon, \epsilon) \to \check{E}$ be two C^1 curves such that $\alpha(\tau) \in X$, $\beta(\tau) \in T_{\alpha(\tau)}X \subseteq \check{E}$, $\alpha(0) = p, \alpha'(0) = u$, and $\beta(0) = v$. Consider the curve*

$$\gamma\colon (-\epsilon, \epsilon) \to \check{F},\ \gamma(\tau) = \big(f'(\alpha(\tau)) - h'(\alpha(\tau))\big)\beta(\tau).$$

Then the vector $\gamma'(0)$ is tangent to Y at $f(p) = h(p)$ and it is a representative of the equivalence class $H(f,h)(p)(u,v)$.

The following result for the coincidence Hessian is the analog of Lemma 2.2. Its proof will be omitted, since it repeats, almost verbatim, the proof of Lemma 2.2.

Lemma 2.4. *Let $f, h\colon X \to Y$ be C^2 maps between manifolds and assume that $f(x) = h(x)$ for every x in a submanifold M of X. Let p be an element of M and $u, v \in T_pM$. Then, $u, v \in \operatorname{Ker}(f'(p) - h'(p))$ and $H(f,h)(p)(u,v) = 0$.*

We conclude this section by recalling some known facts about Fredholm maps. Let $L\colon E \to F$ be a bounded linear operator between Banach spaces. Recall that $\operatorname{Im} L$ is closed whenever $F/\operatorname{Im} L$ is finite dimensional, and L is said to be Fredholm if

$$\dim \operatorname{Ker} L \ + \dim F/\operatorname{Im} L < +\infty.$$

In this case, the index of L is the integer

$$\operatorname{ind} L = \dim \operatorname{Ker} L - \dim F/\operatorname{Im} L.$$

As well-known, the set of Fredholm operators of a given index is open in the space of bounded linear operators.

Let $f\colon X \to Y$ be a C^1 map between two manifolds. Recall that f is *Fredholm* at $x \in X$ if the derivative $f'(x)$ is a Fredholm operator (from T_xX to $T_{f(x)}Y$). The map f is said to be Fredholm (of index n) if it is Fredholm (of index n) at every $x \in X$.

3. General bifurcation

Let X and Y be manifolds, and $f, h\colon X \to Y$ be continuous. Denote by S the solution set of the coincidence equation

$$f(x) = h(x). \tag{3.1}$$

Suppose that one would like to regard a distinguished subset M of S as the set of *trivial* solutions of (3.1). Consequently, $S \setminus M$ is the set of *nontrivial* solutions. According to this terminology, a trivial solution $p \in M$ will be called a *bifurcation point* (from M) for (3.1) if any neighborhood of p contains elements of $S \setminus M$.

Actually, some structure is required on the set M. For instance, we may assume that

- the set M of trivial solutions of (3.1) is a manifold.

Our purpose is to prove a necessary condition (Theorem 3.5) and a sufficient condition (Theorem 3.7) for the coincidence equation (3.1) to possess bifurcation (from M). To this end, we will make use of two results for Fredholm maps between Banach spaces, namely Theorem 3.1 and Theorem 3.2 below (see [4]).

To better understand the meaning of Theorem 3.1, observe that, when g is a C^1 map between Banach spaces and M is a differentiable manifold contained in $g^{-1}(0)$, then $T_xM \subseteq \operatorname{Ker} g'(x)$ for all $x \in M$. Therefore, if g is Fredholm at a point $x \in M$, then $\operatorname{Ker} g'(x)$ is finite dimensional and, consequently, so is M.

Theorem 3.1. *Let $g\colon U \subseteq E \to F$ be a C^1 map defined on an open subset U of a Banach space E into a Banach space F, and let M be a manifold contained in $g^{-1}(0)$. Let $p \in M$ be a bifurcation point (from M) for the equation $g(x) = 0$ and assume that g is Fredholm at p. Then* $\operatorname{Ker} g'(p) \neq T_pM$.

Theorem 3.2. *Let g, U, E, F and M be as in Theorem 3.1 with the additional assumption that g is C^2 on U. Let $p \in M$ be such that g is Fredholm at p and* $\dim \operatorname{Ker} g'(p) = \dim M + 1$. *Denote by* $\Pi\colon F \to F/\operatorname{Im} g'(p)$ *the canonical projection and assume that there exists* $v \in \operatorname{Ker} g'(p) \setminus T_pM$ *such that the linear operator* $A_v\colon T_pM \to F/\operatorname{Im} g'(p)$ *given by*

$$A_v u = \Pi\, g''(p)(u, v)$$

is onto. Then p is a bifurcation point (from M) for the equation $g(x) = 0$. More precisely, there exists a C^1 curve $\delta\colon (-r, r) \to g^{-1}(0)$ such that $\delta(0) = p$ and $\delta(s) \notin M$ for $0 < |s| < r$.

Remark 3.3. The assumptions in Theorem 3.2 imply that the index of g at p is positive. In fact, from the surjectivity of A_v, we get

$$\dim(F/\operatorname{Im} g'(p)) \leq \dim T_pM = \dim M$$

Therefore, taking into account that $\dim \operatorname{Ker} g'(p) = \dim M + 1$, it follows that

$$\operatorname{ind} g'(p) = \dim \operatorname{Ker} g'(p) - \dim(F/\operatorname{Im} g'(p)) \geq 1,$$

as claimed.

Remark 3.4. In Theorem 3.2 the surjectivity of the map A_v is equivalent to the assumption

$$u \in T_pM \text{ and } g''(p)(u,v) \in \operatorname{Im} g'(p) \implies u = 0,$$

provided that $\operatorname{ind} g'(p) = 1$.

To see this, it is enough to observe that, from the assumption that $g'(p)$ is Fredholm of index 1 and that $\dim \operatorname{Ker} g'(p) = \dim M + 1$, we obtain

$$\dim(F/\operatorname{Im} g'(p)) = \dim T_pM.$$

Consequently, the map A_v acts between two finite dimensional spaces of the same dimension. Therefore, A_v is one-to-one if and only if it is onto.

Let us now go back to the coincidence equation (3.1) and assume f and h at least C^1 in a neighborhood of M. Observe first that, since any $x \in M$ is a solution of (3.1), each Fréchet derivative, $f'(x)$ and $h'(x)$, maps the tangent space T_xX into $T_{f(x)}Y$. Thus, the same occurs for the difference $f'(x) - h'(x)$. Moreover, for any $x \in M$, the following inclusion holds:

$$T_xM \subseteq \operatorname{Ker}(f'(x) - h'(x)). \tag{3.2}$$

To see this, observe that the above inclusion is invariant under diffeomorphisms. Therefore, one can regard X and Y as Banach spaces. Thus, the manifold M can be considered as a subset of $g^{-1}(0)$, where $g = f - h$. Now, the fact that g is constant on M implies, as already observed, that $T_xM \subseteq \operatorname{Ker} g'(x)$, for all $x \in M$, as claimed.

We now establish two theorems, that will be particularly helpful in proving the results of the next section.

Theorem 3.5 (Necessary Condition)**.** *Let $f, h\colon X \to Y$ and M be as above, with f and h of class C^1 on a neighborhood of M. Assume that $p \in M$ is a bifurcation point for the equation* (3.1)*, and that the linear operator $f'(p) - h'(p)\colon T_pX \to T_{f(p)}Y$ is Fredholm. Then,*

$$\operatorname{Ker}(f'(p) - h'(p)) \neq T_pM.$$

Proof. The notion of bifurcation and the thesis of this theorem are invariant under diffeomorphisms. Therefore, we can regard X and Y as Banach spaces. The assertion now follows from a straightforward application of Theorem 3.1 to the map $g = f - h$. □

Remark 3.6. As a consequence of (3.2), the above necessary condition implies that T_pM is strictly contained in $\operatorname{Ker}(f'(p) - h'(p))$. Thus, the assertion in Theorem 3.5 can also be stated as follows:

there exists $v \notin T_pM$ such that $f'(p)v = h'(p)v$.

Theorem 3.7 (Sufficient Condition)**.** *Let $f, h\colon X \to Y$ and M be as above. Assume that f and h are C^2 in a neighborhood of M. Let $p \in M$ be such that the linear operator*

$$f'(p) - h'(p)\colon T_pX \to T_{f(p)}Y$$

is Fredholm and $\dim \operatorname{Ker}(f'(p) - h'(p)) = \dim M + 1$*. Assume that there exists a vector*

$$v \in \operatorname{Ker}(f'(p) - h'(p)) \setminus T_pM$$

such that the linear operator

$$L_v \colon T_pM \to T_{f(p)}Y/\operatorname{Im}(f'(p) - h'(p)),\ u \mapsto H(f,h)(p)(u,v),$$

is onto. Then p is a bifurcation point (from M) for equation (3.1)*. More precisely, there exists a* C^1 *curve* $\delta\colon (-r,r) \to X$ *such that* $\delta(0) = p$, $f(\delta(s)) = h(\delta(s))$ *whenever* $|s| < r$*, and* $\delta(s) \notin M$ *for* $0 < |s| < r$*.*

Proof. As in the proof of Theorem 3.5, we may assume that X and Y are Banach spaces. Let $g = f - h$. Then $M \subseteq g^{-1}(0)$ and $v \in \operatorname{Ker} g'(p) \setminus T_pM$. Moreover, from the equality (2.3) we have $H(f,h)(p) = Hg(p)$. Consequently, the operator

$$u \in T_pM \mapsto Hg(p)(u,v) \in F/\operatorname{Im} g'(p)$$

is onto. This implies, by Theorem 3.2, the existence of a curve $\delta\colon (-r,r) \to X$ as stated. In particular, p is a bifurcation point. □

Remark 3.8. As in Remark 3.4, the surjectivity of the linear map L_v defined in Theorem 3.7 is equivalent to the assumption

$$L_v u = 0 \implies u = 0,$$

provided that $\operatorname{ind}(f'(p) - h'(p)) = 1$.

The following result shows that in Theorem 3.7 the vector v can be any element of $\operatorname{Ker}(f'(p) - h'(p)) \setminus T_pM$.

Theorem 3.9. *Let* L_v *be the linear operator defined in Theorem 3.7. Then, given* $v_1 \in \operatorname{Ker}(f'(p) - h'(p)) \setminus T_pM$*, there exists* $\alpha \neq 0$ *such that* $L_{v_1} = \alpha L_v$*.*

Proof. Let $v_1 \in \operatorname{Ker}(f'(p) - h'(p)) \setminus T_pM$. By assumption, $\dim \operatorname{Ker}(f'(p) - h'(p)) = \dim M + 1$. Thus, there exists $\alpha \neq 0$ and $w \in T_pM$ such that $v_1 = \alpha v + w$. Recall that the bilinear operator $H(f,h)(p)$ vanishes on any pair of vectors in T_pM (see Lemma 2.4). Hence, given any $u \in T_pM$, we obtain

$$H(f,h)(p)(u,v_1) = H(f,h)(p)(u,\alpha v + w)$$
$$= \alpha H(f,h)(p)(u,v) + H(f,h)(p)(u,w) = \alpha H(f,h)(p)(u,v),$$

and this proves the assertion. □

4. Bifurcation of fixed points

In this section we are concerned with bifurcation for the n parametrized fixed point equation

$$z = h(\lambda, z), \tag{4.1}$$

where z belongs to a manifold Z and h is a Z-valued map defined on $\mathbb{R} \times Z$ (or, more generally, on a neighborhood of $\{0\} \times Z$). Given $\lambda \in \mathbb{R}$, denote by $h_\lambda\colon Z \to Z$ the partial map $h_\lambda(\cdot) = h(\lambda, \cdot)$. We use the notation $\operatorname{Fix} h_\lambda$ to indicate the subset of Z of the fixed points of h_λ. In addition, we define

$$S = \{(\lambda, z) \in \mathbb{R} \times Z : z = h(\lambda, z)\}$$

and we assume that

- there exists a submanifold M_0 of Z such that $z = h(0,z)$ for all $z \in M_0$.

In other words, M_0 is a subset of $\operatorname{Fix} h_0$, and we can think of $M := \{0\} \times M_0 \subseteq S$ as the set of *trivial solutions* of (4.1). Let us point out that M_0 could be strictly contained in $\operatorname{Fix} h_0$. In fact, this is precisely the situation that occurs in the case when h_0 is the Poincaré T-translation operator generated by a T-periodic second order differential equation on a manifold (see e.g. [6]).

Clearly, equation (4.1) is a particular case of the coincidence equation (3.1) with $X = \mathbb{R} \times Z$, $Y = Z$ and $f = \pi_2$, where

$$\pi_2 \colon \mathbb{R} \times Z \to Z, \quad (\lambda, z) \mapsto z$$

denotes the projection onto the second factor Z.

We say that an element $p \in M_0$ is a *bifurcation point* (from M_0) of (4.1) if any neighborhood of $(0, p)$ contains nontrivial solutions of (4.1), i.e. pairs $(\lambda, z) \in S \setminus M$. In other words, $p \in M_0$ is a bifurcation point if, according to the terminology of Section 3, the element $(0, p) \in \{0\} \times M_0$ is a bifurcation point for the coincidence equation

$$\pi_2(\lambda, z) = h(\lambda, z).$$

We emphasize the fact that, in the present context, a pair of the form $(0, z)$, with $z \in \operatorname{Fix} h_0 \setminus M_0$, must be considered a nontrivial solution.

In this section, we are interested in establishing a necessary as well as a sufficient condition for equation (4.1) to have bifurcation from M_0. To this end, let $z \in M_0$ and assume that h is C^1 in a neighborhood of $(0, z)$. Denote by I_z the identity map on the tangent space $T_z Z$. Since z is a fixed point of h_0, the partial derivative $\partial_2 h(0, z)$ of h at $(0, z)$, which coincides with the derivative $h_0'(z)$ of the partial map h_0 at z, sends $T_z Z$ into itself. Consequently, the same is true for the linear operator $I_z - h_0'(z)$. Also observe that the partial derivative $\partial_2 \pi_2(0, z) \colon T_z Z \to T_z Z$ coincides with I_z, since the partial map $\pi_2(0, \cdot)$ is the identity on Z.

The following conditions for bifurcation of (4.1) are straightforward consequences of Theorems 3.5 and 3.7.

Corollary 4.1. *Let $h \colon \mathbb{R} \times Z \to Z$ and M_0 be as above with h of class C^1 on a neighborhood of $\{0\} \times M_0$. Assume that $p \in M_0$ is a bifurcation point of (4.1), and the linear operator $(D\pi_2(0, p) - Dh(0, p)) \colon \mathbb{R} \times T_p Z \to T_p Z$ is Fredholm. Then there exists $(\mu, w) \in (\mathbb{R} \times T_p Z) \setminus (\{0\} \times T_p M_0)$ such that*

$$w - \mu \partial_1 h(0, p) - \partial_2 h(0, p) w = 0.$$

Proof. The assumption that $(0, p)$ is a bifurcation point for the coincidence equation $\pi_2(\lambda, z) = h(\lambda, z)$ implies, by Theorem 3.5,

$$\operatorname{Ker}(D\pi_2(0, p) - Dh(0, p)) \neq \{0\} \times T_p M_0.$$

As already observed in Remark 3.6, this means that the tangent space of $M = \{0\} \times M_0$ at $(0, p)$ is strictly contained in the kernel of $D\pi_2(0, p) - Dh(0, p)$. Thus, there exists $(\mu, w) \in \operatorname{Ker}(D\pi_2(0, p) - Dh(0, p))$ which does not belong to $T_{(0,p)} M = \{0\} \times T_p M_0$. This means

$$D\pi_2(0, p)(\mu, w) - Dh(0, p)(\mu, w) = w - (\mu \partial_1 h(0, p) + \partial_2 h(0, p) w) = 0,$$

which is our assertion. □

Corollary 4.2. *Assume that h is C^2 on a neighborhood of $\{0\} \times M_0$. Let $p \in M_0$ be such that $\dim \operatorname{Ker}(D\pi_2(0, p) - Dh(0, p)) = \dim M_0 + 1$, and the linear operator*

$(D\pi_2(0,p) - Dh(0,p))\colon \mathbb{R} \times T_pZ \to T_pZ$ is Fredholm. Suppose that there exists $(\mu, w) \in (\mathbb{R} \times T_pZ) \setminus (\{0\} \times T_pM_0)$ such that

$$w - \mu\partial_1 h(0,p) - \partial_2 h(0,p)w = 0.$$

If the linear operator

$$u \in T_pM_0 \mapsto H(\pi_2, h)(0,p)((0,u),(\mu,w)) \in T_pZ/\operatorname{Im}(D\pi_2(0,p) - Dh(0,p))$$

is onto, then p is a bifurcation point of (4.1) from M_0.

Proof. Consider the coincidence equation $\pi_2(\lambda, z) = h(\lambda, z)$, and observe that the equality

$$w - \mu\partial_1 h(0,p) - \partial_2 h(0,p)w = 0$$

implies that the vector $v = (\mu, w)$ belongs to $\operatorname{Ker}(D\pi_2(0,p) - Dh(0,p))$ and it is not tangent to $M = \{0\} \times M_0$. The result now follows from Theorem 3.7. □

Remark 4.3. In the case of the bifurcation equation (4.1), similar arguments to those in Remarks 3.4 and 3.8 prove that, in Corollary 4.2, the condition that the map $u \in T_pM_0 \mapsto H(\pi_2, h)(0,p)((0,u),(\mu,w))$ is onto is equivalent to assuming that

$$u \in T_pM_0 \mapsto H(\pi_2, h)(0,p)((0,u),(\mu,w))$$

is injective, provided that $\operatorname{ind}(D\pi_2(0,p) - Dh(0,p)) = 1$.

As already pointed out, the manifold M_0 may be strictly contained in the set $\operatorname{Fix} h_0$. Thus, some nontrivial solution (λ, z) of (4.1) may have $\lambda = 0$. However, an extra condition yielding that any nontrivial solution sufficiently close to $\{0\} \times M_0$ has $\lambda \neq 0$ turns out to be satisfied, for instance, in many applications to differential equations. Lemma 4.4 below shows that such a condition can be obtained by assuming that

(H) $\quad \forall z \in M_0,\ I_z - h_0'(z)$ is Fredholm and its kernel coincides with T_zM_0.

Lemma 4.4. *Assume that h is C^1 on a neighborhood of $(0,p) \in \{0\} \times M_0$ and that the following condition is satisfied:*

(H_p) $\quad$ *the operator $I_p - h_0'(p)$ is Fredholm and $T_pM_0 = \operatorname{Ker}(I_p - h_0'(p))$.*

Then, any non-trivial solution (λ, z) of (4.1), which is sufficiently close to $(0,p)$, has $\lambda \neq 0$. Consequently, if (H) is satisfied, there exists $W \subseteq Z$ which is an isolating neighborhood of M_0, i.e. $M_0 = \operatorname{Fix} h_0 \cap W$.

Proof. Assume, by contradiction, that in any neighborhood of $(0,p)$ in $\mathbb{R} \times Z$ there exists a solution $(0,z)$ of (4.1) with $z \in \operatorname{Fix} h_0 \setminus M_0$. This means that p is a bifurcation point, relatively to the manifold Z, for the coincidence equation $z = h(0,z)$. Therefore, by applying Theorem 3.5 to $h_0 = h(0,\cdot)$, to the identity of Z and to M_0, one gets $\dim \operatorname{Ker}(I_p - h_0'(p)) > \dim T_pM_0$. This contradicts condition (H_p), and the first assertion is proved. The last statement is a trivial consequence. □

The following necessary condition for bifurcation of fixed points is a consequence of Corollary 4.1 and assumption (H_p).

Theorem 4.5 (Necessary condition)**.** *Suppose that $h\colon \mathbb{R} \times Z \to Z$ is C^1 on a neighborhood of $\{0\} \times M_0$. Assume that $p \in M_0$ is a bifurcation point of (4.1) and (H_p) is satisfied. Then the vector $\partial_1 h(0,p)$ belongs to the image of the operator $(I_p - h_0'(p))\colon T_pZ \to T_pZ$.*

Proof. The operator $D\pi_2(0,p) - Dh(0,p)\colon \mathbb{R}\times T_pZ \to T_pZ$ is Fredholm, since so is its restriction $I_p - h_0'(p)$ to the finite codimensional subspace $\{0\}\times T_pZ$ of $\mathbb{R}\times T_pZ$. Therefore, by Corollary 4.1, there exists $(\mu, w) \in (\mathbb{R}\times T_pZ)\setminus(\{0\}\times T_pM_0)$ such that

$$w - \mu\partial_1 h(0,p) - \partial_2 h(0,p)w = 0.$$

This implies that $\mu\partial_1 h(0,p)$ lies in the image of $I_p - h_0'(p)$. Hence, it is enough to show that $\mu \neq 0$. If this were not the case, w would belong to the kernel of $I_p - h_0'(p)$ and, by assumption (H_p), this would imply $w \in T_pM_0$, contradicting the fact that $(\mu, w) \notin \{0\}\times T_pM_0$. □

The following sufficient condition for bifurcation of fixed points derives from Corollary 4.2. The main trust of the second part of the result is to show that condition (H_p) can be derived from its assumptions.

Theorem 4.6 (Sufficient condition)**.** *Assume that h is C^2 on a neighborhood of $\{0\}\times M_0$. Let $p \in M_0$ be such that the operator*

$$I_p - h_0'(p)\colon T_pZ \to T_pZ$$

is Fredholm of index zero. Suppose that there exists $w \in T_pZ$ such that

$$w - \partial_1 h(0,p) - \partial_2 h(0,p)w = 0.$$

If the linear operator

$$u \in T_pM_0 \mapsto H(\pi_2, h)(0,p)((0,u),(1,w)) \in T_pZ/\operatorname{Im}(I_p - h_0'(p))$$

is injective, then p is a bifurcation point of (4.1) *from M_0.*

Moreover any nontrivial solution (λ, z) of (4.1) *close to $(0,p)$ has $\lambda \neq 0$.*

Proof. The operator $(D\pi_2(0,p) - Dh(0,p))\colon \mathbb{R}\times T_pZ \to T_pZ$ is Fredholm of index one, since its restriction $I_p - h_0'(p)$ to the one codimensional subspace $\{0\}\times T_pZ$ of $\mathbb{R}\times T_pZ$ in Fredholm of index zero. Now notice that

$$\operatorname{Im}(I_p - h_0'(p)) = \operatorname{Im}(D\pi_2(0,p) - Dh(0,p)),$$

since the vector $(1,w)$, which does not belong to $\{0\}\times T_pZ$, is mapped into zero. Consequently, $\dim\operatorname{Ker}(D\pi_2(0,p) - Dh(0,p)) = \dim M_0 + 1$.

We apply Corollary 4.2 with $(\mu, w) = (1, w)$. The assumption that the operator

$$u \in T_pM_0 \mapsto H(\pi_2, h)(0,p)((0,u),(1,w)) \in T_pZ/\operatorname{Im}(I_p - h_0'(p))$$

is injective is equivalent to its surjectivity (see Remark 4.3). Thus p is a bifurcation point of (4.1), since

$$T_pZ/\operatorname{Im}(I_p - h_0'(p)) = T_pZ/\operatorname{Im}(D\pi_2(0,p) - Dh(0,p)).$$

In order to prove the last assertion, according to Lemma 4.4 it is enough to show that our assumptions guarantee the validity of condition (H_p).

Recall first that $\dim\operatorname{Ker}(D\pi_2(0,p) - Dh(0,p)) = \dim M_0 + 1$. Since $(1,w)$ belongs to $\operatorname{Ker}(D\pi_2(0,p) - Dh(0,p))$ but it does not belong to $\{0\}\times T_pM_0$ we obtain that

$$\dim\Big(\operatorname{Ker}(D\pi_2(0,p) - Dh(0,p)) \cap (\{0\}\times T_pZ)\Big) < \dim M_0 + 1.$$

Hence,

$$\dim\Big(\operatorname{Ker}(D\pi_2(0,p) - Dh(0,p)) \cap (\{0\}\times T_pZ)\Big) \le \dim M_0.$$

Moreover, the inclusion

$$(\{0\} \times T_pM_0) \subseteq \operatorname{Ker}(D\pi_2(0,p) - Dh(0,p)) \cap (\{0\} \times T_pZ)$$

and the previous inequality regarding the dimension, imply that

$$\operatorname{Ker}(D\pi_2(0,p) - Dh(0,p)) \cap (\{0\} \times T_pZ) = \{0\} \times T_pM_0.$$

Consequently, (H_p) follows noting that

$$\operatorname{Ker}(D\pi_2(0,p) - Dh(0,p)) \cap (\{0\} \times T_pZ) = \{0\} \times \operatorname{Ker}(I_p - h_0'(p)).$$

In conclusion, by Lemma 4.4, any nontrivial solution (λ, z) of (4.1) close to $(0,p)$ has $\lambda \neq 0$. □

5. Applications to differential equations

In this section we give an application of our results to a parametrized first order differential equation with delay.

Let N be an m-dimensional manifold in $\mathbb{R}^k$ and let $F\colon \mathbb{R} \times N \times N \to \mathbb{R}^k$ be a continuous map. We say that F is *tangent* to N in the second variable or, for short, that F is a *vector field* on N, if $F(t,p,q) \in T_pN$ for all $(t,p,q) \in \mathbb{R} \times N \times N$.

Given a vector field F on N, consider the following parametrized delay differential equation:

$$x'(t) = \lambda F(t, x(t), x(t-1)), \quad \lambda \in \mathbb{R}. \tag{5.1}$$

By a *solution* of (5.1), corresponding to a given value of λ, we shall mean a continuous function $x\colon J \to N$, defined on a (possibly unbounded) real interval with length greater than 1, which is of class C^1 on the subinterval $(\inf J + 1, \sup J)$ of J and verifies $x'(t) = \lambda F(t, x(t), x(t-1))$ for all $t \in J$ with $t > \inf J + 1$.

Assume that F is smooth and T-periodic in the first variable. A pair (λ, x) will be called a *T-periodic pair* if $x(\cdot)$ is a T-periodic solution of (5.1) defined on the whole real line and corresponding to λ. A T-periodic pair of the type $(0, x)$ is said to be *trivial*. In this case the function x is constant.

The set of T-periodic pairs is regarded as a subset of $\mathbb{R} \times C_T(N)$, where $C_T(N)$ is the manifold of continuous T-periodic maps from $\mathbb{R}$ to N. In $C_T(N)$ we consider the topology induced by the Banach space $C_T(\mathbb{R}^k)$ of continuous T-periodic $\mathbb{R}^k$-valued maps with the standard supremum norm. The manifold N is regarded as the set of trivial T-periodic pairs, meaning that any $p \in N$ is identified with $(0, \bar{p})$, where $\bar{p}(t) = p$ for all $t \in \mathbb{R}$.

An element $p \in N$ is called a *bifurcation point* of equation (5.1) if every neighborhood of $(0, \bar{p})$ in $\mathbb{R} \times C_T(N)$ contains nontrivial T-periodic pairs (i.e. T-periodic pairs (λ, x) with $\lambda \neq 0$). Roughly speaking, p is a bifurcation point if, for $\lambda \neq 0$ sufficiently small, there are T-periodic orbits of (5.1) rotating arbitrarily close to p.

The following two results provide a necessary condition (Theorem 5.1) and a sufficient condition (Theorem 5.2) for an element $p \in N$ to be a bifurcation point of (5.1). The tangent vector field $\omega\colon N \to \mathbb{R}^k$ defined by

$$\omega(p) = \frac{1}{T} \int_0^T F(t,p,p)\, dt$$

plays a crucial role in the statements of these conditions.

Theorem 5.1. *Assume that $p \in N$ is a bifurcation point of* (5.1). *Then,* $\omega(p) = 0$.

Theorem 5.2. *Assume that $p \in N$ is a nondegenerate zero of ω. Then, p is a bifurcation point of* (5.1).

Although these two results can be proved without any restriction on the period T, we shall assume, for the sake of simplicity, that $T \geq 1$.

Before proving Theorems 5.1 and 5.2 we present some preliminary notions. Let $\tilde{N}$ denote the set of continuous functions from $[-1,0]$ to $N \subseteq \mathbb{R}^k$, endowed with the topology induced by the Banach space $C([-1,0],\mathbb{R}^k)$ with the supremum norm. It is known that $\tilde{N}$ is a (smooth) infinite dimensional manifold (see e.g. [3]). A pair $(\lambda,\xi) \in \mathbb{R} \times \tilde{N}$ will be called a *T-starting pair* of (5.1) if there exists $x \in C_T(N)$ such that $x(t) = \xi(t)$ for all $t \in [-1,0]$ and (λ, x) is a T-periodic pair. A T-starting pair of the type $(0,\xi)$ will be called *trivial*. Notice that in this case the map ξ must be constant, being the restriction of a constant map defined on $\mathbb{R}$.

Clearly, the map $\rho\colon (\lambda,x) \mapsto (\lambda,\xi)$ which associates to a T-periodic pair (λ,x) the corresponding T-starting pair (λ,ξ) is continuous, ξ being the restriction of x to the interval $[-1,0]$. Moreover, since F is smooth, ρ is actually a homeomorphism between the set $\Gamma \subseteq \mathbb{R} \times C_T(N)$ of T-periodic pairs and the set $\Sigma \subseteq \mathbb{R} \times \tilde{N}$ of T-starting pairs.

Given $p \in N$, by $\hat{p}$ we denote the map $t \mapsto p$, $t \in [-1,0]$, and by $\hat{N}$ the finite dimensional submanifold of $\tilde{N}$ of constant maps. Notice that $\hat{p} \neq \bar{p}$, since, we recall, the constant function $\bar{p}$ is defined on the whole real line. With this notation, a trivial T-periodic pair is of the form $(0,\bar{p})$, and the corresponding trivial T-starting pair is $(0,\hat{p})$; that is, $\rho(0,\bar{p}) = (0,\hat{p})$. Clearly, N can be identified in a natural way both with the set $\{0\} \times \{\bar{p} : p \in N\} \subseteq \Gamma$ of trivial T-periodic pairs and with the set $\{0\} \times \hat{N} \subseteq \Sigma$ of trivial T-starting pairs. In other words, the restriction of the map ρ to $\{0\} \times \{\bar{p} : p \in N\}$ as domain and to $\{0\} \times \hat{N}$ as codomain can be regarded as the identity on N. Therefore, an element $p \in N$ is a bifurcation point for (5.1) if and only if $(0,\hat{p})$ lies in the closure of the set $\Sigma \setminus (\{0\} \times \hat{N})$ of nontrivial T-starting pairs of (5.1).

In order to prove the two results stated above we will apply Theorems 4.5 and 4.6 to a Poincaré-type T-translation operator $P\colon \mathcal{D} \to \tilde{N}$, where $\mathcal{D}$ is an open neighborhood of $\{0\} \times \tilde{N}$ in $\mathbb{R} \times \tilde{N}$. From the definition of P it will turn out (see Lemma 5.3) that (λ,ξ) is a T-starting pair if and only if it is a solution of the equation

$$\xi = P(\lambda,\xi). \tag{5.2}$$

Moreover, it will be clear that the manifold $\hat{N}$ coincides with the set of fixed points of the partial map $P_0 = P(0,\cdot)$. Therefore, taking into account the properties of the map ρ, we can say that $p \in N$ is a bifurcation point of the delay equation (5.1) if and only if $\hat{p}$ is a bifurcation from $\hat{N}$ of (5.2).

We now describe how P is defined. Given $\lambda \in \mathbb{R}$ and $\xi \in \tilde{N}$, consider the following delay differential (initial value) problem in N:

$$\begin{cases} x'(t) = \lambda F(t, x(t), x(t-1)), & t > 0, \\ x(t) = \xi(t), & t \in [-1,0]. \end{cases} \tag{5.3}$$

Let

$$\mathcal{D} = \big\{(\lambda,\xi) \in \mathbb{R} \times \tilde{N} : \text{the maximal solution of (5.3) is defined on } [-1,T]\big\}.$$

An argument analogous to that given in [8] for the ODE case shows that $\mathcal{D}$ is open in $\mathbb{R} \times \tilde{N}$, and clearly contains $\{0\} \times \tilde{N}$.

Denote by $x_{(\lambda,\xi)}$ the maximal solution of problem (5.3) corresponding to $(\lambda, \xi) \in \mathcal{D}$ and let

$$P \colon \mathcal{D} \to \tilde{N}$$

be the Poincaré-type operator defined by $P(\lambda, \xi)(t) = x_{(\lambda,\xi)}(t + T)$, $t \in [-1, 0]$.

It can be shown that the smoothness of F implies that P is smooth.

The following lemma regards a property of P mentioned above. The proof is standard and will be omitted.

Lemma 5.3. *The fixed points of $P(\lambda, \cdot)$ correspond to the T-periodic solutions of the equation* (5.1) *in the following sense: ξ is a fixed point of $P(\lambda, \cdot)$ if and only if it is the restriction to* $[-1, 0]$ *of a T-periodic solution corresponding to λ.*

Observe that, as a consequence of Lemma 5.3, the submanifold $\hat{N}$ of $\tilde{N}$ coincides with the set of fixed points of the partial map P_0.

We now apply Theorems 4.5 and 4.6 with $Z = \tilde{N}$, $M_0 = \hat{N}$, and $h = P$. Since we are dealing with local bifurcation, we shall assume, without loss of generality, that the open set $\mathcal{D}$ coincides with $\mathbb{R} \times \tilde{N}$.

Given any $p \in N$, let us compute the action of the linear map

$$(I_{\hat{p}} - P_0'(\hat{p})) \colon T_{\hat{p}}\tilde{N} \to T_{\hat{p}}\tilde{N},$$

where the tangent space of $\tilde{N}$ at $\hat{p}$ is given by

$$T_{\hat{p}}\tilde{N} = \big\{\eta \in C([-1, 0], \mathbb{R}^k) : \eta(t) \in T_pN,\ t \in [-1, 0]\big\}.$$

We need to evaluate the derivative of P_0 at $\hat{p}$. Since $P(0, \xi)(t) = x_{(0,\xi)}(t + T)$, where

$$\begin{cases} x'_{(0,\xi)}(t) = 0, & t > 0, \\ x_{(0,\xi)}(t) = \xi(t), & t \in [-1, 0], \end{cases}$$

we obtain

$$x_{(0,\xi)}(t) = \begin{cases} \xi(t) & \text{for } -1 \le t \le 0, \\ \xi(0) & \text{for } t \ge 0. \end{cases}$$

Taking into account that $T \ge 1$, we get $P(0, \xi)(t) = \xi(0)$ for $t \in [-1, 0]$. Hence, given $\eta \in T_{\hat{p}}\tilde{N}$, we arrive at

$$P_0'(\hat{p})\eta = \widehat{\eta(0)},$$

where $\widehat{\eta(0)}$ is the constant map $t \mapsto \eta(0)$. Consequently,

$$(I_{\hat{p}} - P_0'(\hat{p}))\eta = \eta - \widehat{\eta(0)},$$

and the kernel of $I_{\hat{p}} - P_0'(\hat{p})$ is the subspace $T_{\hat{p}}\hat{N}$ of $T_{\hat{p}}\tilde{N}$ of constant T_pN-valued functions. Moreover, one can easily verify that its image is the space of functions that vanish for $t = 0$, and the operator is the identity on its image. In other words, $I_{\hat{p}} - P_0'(\hat{p})$ is a projector with finite dimensional kernel. Therefore, it is Fredholm of index zero.

Proof of Theorem 5.1. As observed above $I_{\hat{p}} - P_0'(\hat{p})$ is Fredholm and

$$\operatorname{Ker}(I_{\hat{p}} - P_0'(\hat{p})) = T_{\hat{p}}\hat{N}.$$

Thus, condition $(H_{\hat{p}})$ is verified. Consequently, the vector $\partial_1 P(0, \hat{p})$, which is a function from $[-1, 0]$ to $T_pN \subseteq \mathbb{R}^k$, belongs, by Theorem 4.5, to the image of the operator $I_{\hat{p}} - P_0'(\hat{p})$. To compute the partial derivative $\partial_1 P(0,\hat{p})$ we need to derive at $\lambda = 0$ the partial map $\lambda \mapsto P(\lambda,\hat{p})$. Recall that $P(\lambda,\hat{p})(t) = x_{(\lambda,\hat{p})}(t+T)$, where $x_{(\lambda,\hat{p})}(t)$ is such that

$$\begin{cases} x'_{(\lambda,\hat{p})}(t) = \lambda F(t, x_{(\lambda,\hat{p})}(t), x_{(\lambda,\hat{p})}(t-1)), & t > 0, \\ x_{(\lambda,\hat{p})}(t) = p, & t \in [-1,0]. \end{cases} \tag{5.4}$$

Thus,

$$x_{(\lambda,\hat{p})}(t) = p + \lambda \int_0^t F(s, x_{(\lambda,\hat{p})}(s), x_{(\lambda,\hat{p})}(s-1))\, ds,$$

and

$$P(\lambda,\hat{p})(t) = p + \lambda \int_0^{t+T} F(s, x_{(\lambda,\hat{p})}(s), x_{(\lambda,\hat{p})}(s-1))\, ds,$$

which makes sense since N is contained in $\mathbb{R}^k$. It follows that

$$P(\lambda,\hat{p}) = \hat{p} + \lambda G(\lambda), \tag{5.5}$$

where, given λ, $G(\lambda) \in C([-1,0],\mathbb{R}^k)$ is the function

$$G(\lambda)(t) = \int_0^{t+T} F(s, x_{(\lambda,\hat{p})}(s), x_{(\lambda,\hat{p})}(s-1))\, ds.$$

Incidentally, we observe that, even though $\hat{p} + \lambda G(\lambda)$ is N-valued, this is not the case for $G(\lambda)$.

From (5.5), we derive

$$\partial_1 P(0,\hat{p})(t) = G(0)(t) = \int_0^{t+T} F(s,p,p)\, ds.$$

Recalling that the image of $I_{\hat{p}} - P_0'(\hat{p})$ is the space of functions that vanish for $t = 0$, we get

$$0 = \int_0^T F(s,p,p)\, ds = T\omega(p).$$

This completes the proof. □

Proof of Theorem 5.2. Let $p \in N$ be a nondegenerate zero of ω. Our goal is to show that the assumptions of Theorem 4.6 are satisfied.

As already seen (just before the proof of Theorem 5.1), the operator $I_{\hat{p}} - P_0'(\hat{p})$ is Fredholm of index zero. Let us show that the condition $\omega(p) = 0$ implies the existence of $w \in T_{\hat{p}}\tilde{N}$ such that

$$w - \partial_1 P(0,\hat{p}) - \partial_2 P(0,\hat{p}) w = 0. \tag{5.6}$$

Recalling that

$$\partial_1 P(0,\hat{p})(t) = \int_0^{t+T} F(s,p,p)\, ds$$

and

$$(\partial_2 P(0,\hat{p})\eta)(t) = (P_0'(\hat{p})\eta)(t) = \eta(0),$$

we need to find w such that

$$w(t) - \int_0^{t+T} F(s,p,p)\, ds - w(0) = 0.$$

Define the vector $w \in T_{\hat{p}}\tilde{N}$ by

$$w(t) = \int_0^{t+T} F(s,p,p)\,ds$$

and observe that

$$w(0) = \int_0^{T} F(s,p,p)\,ds = T\omega(p) = 0.$$

Hence w satisfies equation (5.6).

In order to apply Theorem 4.6 we need to show that the linear operator

$$\hat{y} \in T_{\hat{p}}\hat{N} \mapsto H(\pi_2, P)(0,\hat{p})((0,\hat{y}),(1,w)) \in T_{\hat{p}}\tilde{N}/\operatorname{Im}(I_{\hat{p}} - P_0'(\hat{p}))$$

is injective. That is, we need to show that, given any $y \in T_pN$, if a representative of the class $H(\pi_2,P)(0,\hat{p})((0,\hat{y}),(1,w))$ belongs to $\operatorname{Im}(I_{\hat{p}} - P_0'(\hat{p}))$, then $y = 0$ (or, equivalently, $\hat{y} = 0$).

To compute the coincidence Hessian $H(\pi_2, P)$ we apply Theorem 2.3. For this purpose, notice that the source and target manifolds $\mathbb{R} \times \tilde{N}$ and $\tilde{N}$ are embedded in the Banach spaces $\check{E} = \mathbb{R} \times C([-1,0],\mathbb{R}^k)$ and $\check{F} = C([-1,0],\mathbb{R}^k)$, respectively.

Observe that (5.6) is the difference of two derivatives at the point $(0,\hat{p})$, applied to the vector $(1,w)$, i.e.

$$w - \partial_1 P(0,\hat{p}) - \partial_2 P(0,\hat{p})w = D\pi_2(0,\hat{p})(1,w) - DP(0,\hat{p})(1,w).$$

Analogously, given any $q \in N$, $\mu \in \mathbb{R}$, and $\eta \in T_{\hat{q}}\tilde{N}$ we obtain

$$D\pi_2(0,\hat{q})(\mu,\eta) - DP(0,\hat{q})(\mu,\eta) = \eta - \mu\partial_1 P(0,\hat{q}) - \partial_2 P(0,\hat{q})\eta,$$

where, as can be easily checked,

$$\partial_1 P(0,\hat{q})(t) = \int_0^{t+T} F(s,q,q)\,ds,$$

and

$$(\partial_2 P(0,\hat{q})\eta)(t) = (P_0'(\hat{q})\eta)(t) = \eta(0).$$

Given any $y \in T_pN$, let $\nu(\cdot)\colon (-\varepsilon,\varepsilon) \to N$ be a C^1 curve such that $\nu(0) = p$ and $\nu'(0) = y$. Define $\sigma(\cdot)\colon (-\varepsilon,\varepsilon) \to C([-1,0],\mathbb{R}^k)$ by

$$\sigma(\tau)(t) = \int_0^{t+T} F(s,\nu(\tau),\nu(\tau))\,ds,$$

and observe that $\sigma(0) = w$ and $\sigma(\tau)(t)$ belongs to $T_{\nu(\tau)}N$ for all for all $\tau \in (-\varepsilon,\varepsilon)$ and all $t \in [-1,0]$. Hence, $\sigma(\tau) \in T_{\widehat{\nu(\tau)}}\tilde{N}$, $\forall \tau \in (-\varepsilon,\varepsilon)$.

We now apply Theorem 2.3 with

$$\alpha(\tau) = (0,\widehat{\nu(\tau)}), \quad \beta(\tau) = (1,\sigma(\tau)).$$

Consequently, the curve $\gamma\colon (-\varepsilon,\varepsilon) \to C([-1,0],\mathbb{R}^k)$ in Theorem 2.3 is given by

$$\gamma(\tau) = D\pi_2(\alpha(\tau))\beta(\tau) - DP(\alpha(\tau))\beta(\tau) = \sigma(\tau) - \partial_1 P(0,\widehat{\nu(\tau)}) - \partial_2 P(0,\widehat{\nu(\tau)})\sigma(\tau).$$

Hence

$$\gamma(\tau)(t) = \sigma(\tau)(t) - \int_0^{t+T} F(s,\nu(\tau),\nu(\tau))\,ds - \sigma(\tau)(0) = -\sigma(\tau)(0).$$

Incidentally, we observe that, given $\tau \in (-\varepsilon,\varepsilon)$, $\gamma(\tau)$ is a constant function from $[-1,0]$ to $\mathbb{R}^k$. Thus, γ can be regarded as a curve in $\mathbb{R}^k$.

According to Theorem 2.3, we obtain that

$$\gamma'(0)(t) = -\sigma'(0)(0) = \int_0^T \big(\partial_2 F(s,p,p) + \partial_3 F(s,p,p)\big) y\, ds = T\omega'(p)y$$

lies in the equivalence class $H(\pi_2, P)(0,\hat{p})((0,\hat{y}),(1,w))$. It remains to prove that the condition $\gamma'(0) \in \mathrm{Im}(I_{\hat{p}} - P_0'(\hat{p}))$ implies $y = 0$. This is true since, by assumption, the operator $\omega'(p)\colon T_pN \to \mathbb{R}^k$ is injective (it is actually an automorphism of T_pN), the function $\gamma'(0)\colon [-1,0] \to \mathbb{R}^k$ is constant, and any function in the image of the operator $I_{\hat{p}} - P_0'(\hat{p})$ vanishes for $t = 0$. □

Our final comment regards a result which is more precise than the one stated in Theorem 5.2. That is, taking into account Theorem 3.7, one could prove the existence of a C^1 curve $\delta\colon (-r,r) \to \mathbb{R} \times C_T(\mathbb{R}^k)$, $s \mapsto (\lambda_s, x_s)$, with the following properties:

- (λ_s, x_s) is a T-periodic pair for $s \in (-r,r)$;
- $\delta(0) = (\lambda_0, x_0) = (0, \hat{p})$;
- $\lambda_s \neq 0$ for $0 < |s| < r$.

References

[1] Agrachev A., Sachkov Y., *Control Theory from the Geometric Viewpoint*, Encyclopedia of Math. Sciences **87**, Control Theory and Optimization, 2004.

[2] H. Amann, *Ordinary Differential Equations. An Introduction to Nonlinear Analysis*, de Gruyter Studies in Math. **13**, Walter de Gruyter and Co., Berlin, 1990.

[3] J. Eells, *A setting for global analysis*, Bull. Amer. Math. Soc. **72** (1966), 751–807.

[4] Furi M., Martelli M., Pera M.P., *General Bifurcation Theory: Some Local Results and Applications*, Differential Equations and Applications to Biology and to Industry, Cooke K., Cumberbatch E., Martelli M., Tang B. and Thieme H. Editors, World Scientific, 1996, 101–115.

[5] Furi M., Pera M.P., *Co-Bifurcating Branches of Solutions for Nonlinear Eigenvalue Problems in Banach Spaces*, Ann. Mat. Pura Appl. **135** (1983), 119–131.

[6] Furi M., Pera M.P., *Bifurcation of Fixed Points from a Manifold of Trivial Fixed Points*, Nonlinear Funct. Anal. Appl., **11** (2006), 265–292.

[7] Hirsch M.W., *Differential Topology*, Graduate Texts in Math. **33**, Springer-Verlag, New York, 1976.

[8] S. Lang, *Introduction to differentiable manifolds*, John Wiley & Sons Inc., New York, 1966.

Massimo Furi - Dipartimento di Matematica Applicata 'G. Sansone', Via S. Marta 3, I-50139 Florence, Italy - *E-mail address:* massimo.furi@unifi.it

Mario Martelli - Department of Mathematics, Claremont McKenna College, Claremont, CA, 91711, USA - *E-mail address:* mmartelli@cmc.edu

Maria Patrizia Pera - Dipartimento di Matematica Applicata 'G. Sansone', Via S. Marta 3, I-50139 Florence, Italy - *E-mail address:* mpatrizia.pera@unifi.it