A NEW THEME IN NONLINEAR ANALYSIS: CONTINUATION AND BIFURCATION OF THE UNIT EIGENVECTORS OF A PERTURBED LINEAR OPERATOR

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Abstract. TBA

1. Introduction and statement of the results

Let T be a bounded linear operator acting in a real Banach space X and let S be the unit sphere in X. Suppose that u_0 is a unit eigenvector of T, that is $u_0 \in S$ and $Tu_0 = \lambda_0 u_0$ for some $\lambda_0 \in \mathbb{R}$; we say in this case that u_0 is a *unit* λ_0*-eigenvector* of T. Also let $B : U \to X$ be a (possibly nonlinear) continuous operator defined in a neighborhood U of S and for ϵ small consider the perturbed "eigenvalue" problem

$$Tu + \epsilon B(u) = \lambda u, \qquad u \in S. \tag{1.1}$$

Definition 1.1. *Let u_0 be a unit λ_0-eigenvector of T. We say that u_0 is* continuable as a unit eigenvector of $T + \epsilon B$ $(\epsilon \neq 0)$ *if there exists a continuous function $\epsilon \mapsto (\lambda(\epsilon), u(\epsilon))$ of an interval $(-\epsilon_0, \epsilon_0)$ into $\mathbb{R} \times S$ such that $Tu(\epsilon)+\epsilon B(u(\epsilon)) = \lambda(\epsilon)u(\epsilon)$ for $|\epsilon| < \epsilon_0$ and $(\lambda(0), u(0)) = (\lambda_0, u_0)$.*

For example, u_0 is continuable if it is an "eigenvector" of B too: for if $B(u_0) = \mu u_0$ for some $\mu \in \mathbb{R}$, then putting $(\lambda(\epsilon), u(\epsilon)) = (\lambda_0 + \epsilon\mu, u_0)$ for $\epsilon \in \mathbb{R}$ yields the required continuous family. On the other hand, putting $X = \mathbb{R}^2$, T the zero operator, $B(x, y) = (-y, x)$ for $(x, y) \in \mathbb{R}^2$, we see that no 0-eigenvector of T (that is, no vector in $\mathbb{R}^2$) is continuable, for the perturbed linear operator $T + \epsilon B$ has no (real) eigenvalue for $\epsilon \neq 0$.

Assuming that λ_0 be an isolated eigenvalue of finite (geometric and algebraic) multiplicity, we have discussed in [2] and [3] conditions for the continuability of a unit λ_0-eigenvector of T. In particular, in [2] it was essentially shown that when λ_0 is a simple eigenvalue, then if B is Lipschitz continuous each of the two unit λ_0-eigenvectors is continuable (in a Lipschitz continuous fashion): see Theorem 2 and Remark 2.1 of [2]. While in [3], we have considered the case in which λ_0 has multiplicity greater than one, and have given - for B of class C^2 - necessary as well as sufficient conditions for continuability of a given unit eigenvector in the C^1 sense: see Theorem 2.2 and Remark 3.6 of [3].

To obtain further information about the solutions of (1.1) it is useful to introduce a second concept, which relaxes the requirements in Definition 1.1.

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Definition 1.2. *Let u_0 be a unit λ_0-eigenvector of T. We say that u_0 is a* bifurcation point for the unit eigenvectors of $T+\epsilon B$ $(\epsilon \neq 0)$ - *or simply a* bifurcation point for (1.1) - *if any neighborhood of* $(0, \lambda_0, u_0)$ *in* $\mathbb{R}\times\mathbb{R}\times X$ *contains a solution* (ϵ, λ, u) *of (1.1) with* $\epsilon \neq 0$.

Definition 1.2 expresses the property for a unit eigenvector of T of being *persistent* under sufficiently small perturbations of T, and can be equivalently formulated as follows: *there exists a sequence* $\{(\epsilon_n, \lambda_n, u_n)\}$ *in* $\mathbb{R}\setminus\{0\}\times\mathbb{R}\times S$ *which converges to* $(0, \lambda_0, u_0)$ *and such that* $Tu_n + \epsilon_n B(u_n) = \lambda_n u_n$, $\forall n \in \mathbb{N}$. To appreciate better this Definition, it is useful to adopt as in [3] the general point of view in bifurcation theory introduced in [8]. A solution of (1.1) is a point $p = (\epsilon, \lambda, u) \in \mathbb{R}\times\mathbb{R}\times X$ such that $F(p) = 0$, where F is the map of $\mathbb{R}\times\mathbb{R}\times X$ into $X\times\mathbb{R}$ defined via

$$F(\epsilon, \lambda, u) = (Tu + \epsilon B(u) - \lambda u, \|u\|^2 - 1) \tag{1.2}$$

($\|.\|$ is the norm in X). Put

$$S_0 \equiv S \bigcap \mathrm{Ker}(T - \lambda_0 I) \tag{1.3}$$

where I denotes the identity in X, and consider the subset

$$M \equiv \{0\} \times \{\lambda_0\} \times S_0 \tag{1.4}$$

of $\mathbb{R}\times\mathbb{R}\times X$ as the set of *trivial solutions* of (1.1), or the trivial zeroes of F. Assuming that λ_0 be an isolated eigenvalue, and considering solutions of (1.1) with λ near λ_0, we see that M is precisely the set of triples $(\epsilon, \lambda, u) \in \mathbb{R}\times\mathbb{R}\times X$ solving (1.1) for $\epsilon = 0$. Solutions (ϵ, λ, u) with $\epsilon \neq 0$ are therefore the *nontrivial* solutions of (1.1), and Definition 1.2 expresses - identifying u_0 with $p_0 \equiv (0, \lambda_0, u_0)$ and using the terminology of [8] - that $p_0 \in M$ is a *bifurcation point* (from M) for the equation $F(p) = 0$.

Very recently, we have proved the existence of at least one bifurcation point for the unit eigenvectors of $T + \epsilon B$ under the assumptions that T be a self-adjoint operator in a Hilbert space, that B be of class C^1 and that one of the following conditions be satisfied:

- the multiplicity of λ_0 is *odd*;
- B is a *gradient* operator.

Our aim in the present paper is to explain these results - proved in [4] and [5] respectively - also in connection with the older ones [3], and in particular to make available the main idea followed in the (yet unpublished) paper [5] to deal with the variational case.

We first set our problem in the context of perturbations of (linear) Fredholm operators of index zero: this turns out to be a sufficiently general [functional-analytic] framework in order to state our results on a common ground, compare their strength and appreciate the different assumptions. We also indicate the main points of the proofs. This is done in Section 2, while Section 3 is addressed to exhibit some simple examples of our problem in the euclidean space $\mathbb{R}^3$. Working in this context - and even with a linear B - gives some concrete evidence of the conditions involved on T and B, and may thus help for a better understanding of the ideas before expressed in infinite-dimensional Banach spaces.

2. Finite-dimensional reduction. Necessary conditions and sufficient conditions for bifurcation

Consider equation (1.1) for a bounded linear operator $T : X \to X$, X a real Banach space. We suppose in the sequel that:

- λ_0 is an isolated eigenvalue of T.

As already said, this ensures that for $\epsilon = 0$ and λ near λ_0, the only solutions of (1.1) are those with $\lambda = \lambda_0$, that is the trivial ones. Now set

$$A = T - \lambda_0 I, \qquad \delta = \lambda - \lambda_0$$

and write the equation in (1.1) as

$$Au + \epsilon B(u) = \delta u. \tag{2.1}$$

We assume the following hypotheses upon A.

HA1) A is a Fredholm operator of index zero, that is,

- $\operatorname{Ker} A = \{u \in X : Au = 0\}$ is of finite dimension; in words, λ_0 is an eigenvalue of finite geometric multiplicity;
- $\operatorname{Im} A = \{Au : u \in X\}$ is closed and of finite codimension;
- $\dim \operatorname{Ker} A = \operatorname{codim} \operatorname{Im} A$.

HA2) $\operatorname{Ker} A \cap \operatorname{Im} A = \{0\}$.

It follows from HA1) and HA2) that

$$E = \operatorname{Ker} A \oplus \operatorname{Im} A \tag{2.2}$$

and that the projections P, $Q = I - P$ onto $\operatorname{Ker} A$, $\operatorname{Im} A$ respectively corresponding to this direct sum are *continuous*.

It is useful to recall two typical situations in which the above assumptions are satisfied:

- $T : X \to X$ is compact, $\lambda_0 \neq 0$ (ensuring HA1)) and $\operatorname{Ker} A = \operatorname{Ker} A^2$ (ensuring HA2). The last condition also implies that $\operatorname{Ker} A^n = \operatorname{Ker} A^{n+1}$ for all $n \in \mathbb{N}$, and therefore that the geometric multiplicity of λ_0 equals its *algebraic* multiplicity $\dim \bigcup_{n=1}^{\infty} \operatorname{Ker} A^n$.
- $X = H$, a Hilbert space, $T : H \to H$ is self-adjoint (that is, $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$, $\langle ., .\rangle$ denoting the scalar product in H) and $\dim \operatorname{Ker} A < \infty$. Indeed self-adjointness of T implies that $\operatorname{Ker} A = \operatorname{Im} A^{\perp} \equiv \{x \in H : \langle x, y\rangle = 0 \ \forall y \in \operatorname{Im} A\}$, and it follows that $H = \operatorname{Ker} A \oplus \overline{\operatorname{Im} A}$, where the sum is orthogonal. However as λ_0 is isolated by assumption, $\operatorname{Im} A$ is closed (see e.g. [7, pg. 1395]) and therefore $H = \operatorname{Ker} A \oplus \operatorname{Im} A$. Self-adjointness also implies that $\operatorname{Ker} A = \operatorname{Ker} A^2$, so that the geometric and algebraic multiplicity of λ_0 always coincide in this case.

Writing $u = Pu + Qu \equiv v + w$ according to (2.2) and applying in turn P, Q to both members of (2.1), we see that the latter equation is equivalent to the following two:

$$\epsilon PB(v + w) = \delta v \tag{2.3}$$

$$Aw + \epsilon QB(v + w) = \delta w. \tag{2.4}$$

This decomposition (the so-called *Lyapounov-Schmidt method*) reveals easily a **necessary condition** for bifurcation as soon as B satisfies the following "minimal" regularity asumption:

HB0) B is continuous in a neighborhood of S.

Proposition 2.1. *Suppose that* HA1) , HA2) *and* HB0) *are satisfied. If $v_0 \in S_0 = S \cap \mathrm{Ker}(T - \lambda_0 I)$ is a bifurcation point for (1.1), then there exists $\mu_0 \in \mathbb{R}$ such that*

$$PB(v_0) = \mu_0 v_0. \tag{2.5}$$

Proof. If $v_0 \in S_0$ is a bifurcation point, there exists by definition a sequence $(\delta_n, \epsilon_n, u_n) \in \mathbb{R} \times \mathbb{R} \times S$, with $\epsilon_n \neq 0$ for each $n \in \mathbb{N}$, such that $(\delta_n, \epsilon_n, u_n) \to (0, 0, v_0)$ as $n \to \infty$ and

$$Au_n + \epsilon_n B(u_n) = \delta_n u_n, \qquad \forall n \in \mathbb{N}. \tag{2.6}$$

Then putting $v_n = Pu_n, w_n = Qu_n$ we have $v_n \to Pv_0 = v_0$, $w_n \to Qv_0 = 0$ and moreover from (2.3)

$$PB(v_n + w_n) = \frac{\delta_n}{\epsilon_n} v_n.$$

We claim that the sequence (δ_n/ϵ_n) is bounded. For otherwise, since $\|v_n\| \to \|v_0\| = 1$, it would follow (passing if necessary to a subsequence) that $\|\frac{\delta_n}{\epsilon_n} v_n\| \to +\infty$, contradicting the boundedness of the sequence $PB(v_n+w_n)$ which in fact converges to $PB(v_0)$. Hence we can assume (again through a subsequence) that (δ_n/ϵ_n) converges to some μ_0, so that in the limit we obtain (2.5).

1. **Comment**: For B of class C^1, the above condition was proved in [3].
2. **Necessary condition is not sufficient**: see Example 3.3.

In order to discuss **sufficient conditions** for bifurcation, we shall henceforth strengthen HB0) as follows:

HB1) B is of class C^1 in a neighborhood of S.

Indeed put

$$N = \operatorname{Ker} A, \qquad W = \operatorname{Im} A$$

and identify X with $N \times W$. Then HB1) guarantees, via the Implicit Function Theorem, that given any $v_0 \in S_0 \subset N$, equation (2.4) - the so-called *complementary equation* - can be solved uniquely w.r.t. w for each given (δ, ϵ, v) in a neighborhood $U_0 \subset \mathbb{R}\times\mathbb{R}\times N$ of $(0, 0, v_0)$. Moreover if $w(\delta, \epsilon, v)$ denotes the solution corresponding to $(\delta, \epsilon, v) \in U_0$, then $w(0, 0, v) = 0$ for any v and the mapping $(\delta, \epsilon, v) \to w(\delta, \epsilon, v)$ of U_0 into W is of class C^1 in U_0. Therefore by definition

$$Aw(\delta, \epsilon, v) + \epsilon QB(v + w(\delta, \epsilon, v)) = \delta w(\delta, \epsilon, v) \tag{2.7}$$

for any $(\delta, \epsilon, v) \in U_0$; and we see from (2.3) that in order to solve our problem (1.1), it is enough to find $(\delta, \epsilon, v) \in U_0$ satisfying the finite-dimensional equation (the *bifurcation equation*)

$$\epsilon PB(v + w(\delta, \epsilon, v)) = \delta v \tag{2.8}$$

and the additional normalization constraint

$$v + w(\delta, \epsilon, v) \in S. \tag{2.9}$$

At this stage, in order to prove that a given $v_0 \in S_0$ - satisfying (2.5) - is indeed a bifurcation point, we need find a sequence $(\delta_n, \epsilon_n, v_n)$ of solutions of the above system (2.8) - (2.9), with $\epsilon_n \neq 0$ for each $n \in \mathbb{N}$, such that $(\delta_n, \epsilon_n, v_n) \to (0, 0, v_0)$ as $n \to \infty$. While if for each sufficiently small ϵ we find $\delta(\epsilon), v(\epsilon)$ - depending continuously upon ϵ - such that $(\delta(0), v(0)) = (0, v_0)$ and $(\delta(\epsilon), \epsilon, v(\epsilon))$ solves (2.8) - (2.9), then so much the better as v_0 will be continuable by means of the equation

$$u(\epsilon) = v(\epsilon) + w(\delta(\epsilon), \epsilon, v(\epsilon)). \tag{2.10}$$

[ChiFuPe1]

When B and the space X (that is, its norm) are sufficiently smooth, the Implicit Function Theorem can be further employed to perform such construction and yield a sufficient condition for continuation.

Theorem 2.1. *For $x \in X$, put $g(x) = \|x\|^2 - 1$. Suppose that B and g are* of class *C^2 in an open neighborhood of $S = g^{-1}(0)$ and that* HA1) *and* HA2) *are satisfied. Let $v_0 \in S_0$ be such that $PB(v_0) = \mu_0 v_0$, let $V = \{h \in X : g'(v_0)h = 0\}$ and let π be a continuous projection of X onto V. If v_0 satisfies the condition:*

$$h \in N \cap V, \qquad \pi PB'(v_0)h = \mu_0 h \Rightarrow h = 0, \tag{2.11}$$

then v_0 is continuable.

Remark 2.1. V is the tangent space to S at v_0, and likewise $N \cap V$ is the tangent space to $S_0 = N \cap S$ at v_0. The condition (2.11) means that the map $\pi PB'(v_0) - \mu_0 I$, restricted to $N \cap V$, is an isomorphism of $N \cap V$ onto iself.

Reference to: i)More general versions of Theorem 2.1;
ii)Applications to BVP.

Theorem 2.1 ia a special case of Theorem 3.4 in [3], where it is shown that similar results hold when the operators involved act between different Banach spaces, and when the unit sphere S is replaced by more general manifolds $M = g^{-1}(0)$ given as level sets of a C^2 functional g.

In turn, Theorem 3.4 of [3] is an application to Banach space operator equations of results formulated in [8] in the context of general bifurcation theory. This considers a C^1 map f defined in an open set U of a Banach space E and with values in a Banach space F. Given a differentiable manifold $M \subseteq f^{-1}(0)$, regard M as the set of *trivial solutions* of the equation $f(u) = 0$, so that $f^{-1}(0) \setminus M$ represents the set of nontrivial solutions. An element $p \in M$ is a *bifurcation point (from M) of* $f(u) = 0$ if any neighborhood of p contains elements of $f^{-1}(0) \setminus M$. Necessary as well as sufficient conditions for bifurcation are proved in [8] in essentially geometrical terms, starting from the observation that the condition $M \subseteq f^{-1}(0)$ implies that, for any $u \in M$, the tangent space $T_u M$ of M at u is contained in the kernel of $f'(u)$.

In particular when f is a C^2 Fredholm map of index 1, and $p \in M$ is such that $\dim \operatorname{Ker} f'(p) = \dim T_p M + 1$, then a sufficient "transversality" condition for $p \in M$ to be a bifurcation point is provided in [8], which extends that contained in the Crandall–Rabinowitz Bifurcation Theorem [6], in which $\dim M = 1$. For these general conditions see, for instance, Theorem 2.2 of [3] and the comments accompanying it.

Moreover in [3], the results about (1.1) are applied to show the existence of 2π-periodic solutions of the differential equation

$$x'' + x + \epsilon(tx + x^2) = \lambda x$$

normalized by

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^2(t)\, dt = 1,$$

and in particular to study the continuability of a given *trivial* (i.e., obtained for $\epsilon = \lambda = 0$) normalized solution: that is, of a solution of the type $x(t) = c \sin t + d \cos t$, with $c^2 + d^2 = 1$.

Proposition 2.1 and Theorem 2.1 are results of **local** nature, as they give conditions upon an individual point $v_0 \in S_0$ to be a bifurcation point for (1.1). A related question is: under which conditions (on A, B, etc.) does S_0 possess at least one bifurcation point? We are able to give some partial answer to this problem in the special case that

$X = H$**, a Hilbert space and** $T : H \to H$ **is self-adjoint.**

[ChiFuPe2]

Recall that in this case the assumptions HA1) and HA2) about the linear part $A = T - \lambda_0 I$ of our equation are satisfied - provided of course that λ_0 be isolated and of finite multiplicity, as we have always assumed. Here is our first result [4]:

Theorem 2.2. *Consider the problem (1.1) where T is a bounded self-adjoint operator acting in a real Hilbert space and B satisfies the assumption* HB1). *If λ_0 is an isolated eigenvalue of T of* odd multiplicity, *then $S_0 = S \cap \mathrm{Ker}(T - \lambda_0 I)$ possesses at least one bifurcation point.*

Sketch of the proof. Topological methods

The proof of this result relies on the fact that the Euler–Poincaré characteristic of the even dimensional sphere S_0 is nonzero, and this implies that any self-map of this sphere has a fixed point if it is homotopic to the identity: for this matter see, for instance, [1] or [9]. Therefore, the methods employed are of topological nature, and quite different from those used in [2] and [3], which rely almost entirely upon the Implicit Function Theorem.

Nevertheless, it is precisely with a strengthened version of this Theorem that we start our work in [4], to the aim of solving the complementary equation "globally" with respect to S_0. Indeed for $\eta > 0$, consider the (compact) neighborhood of S_0

$$M = \Big\{ v \in N : \big| \|v\| - 1 \big| \le \eta \Big\}.$$

Taking $\eta > 0$ small, we can assume that B be of class C^1 in an open neighborhood of $M \times \{0\} \subset N \times W$, and then it follows from Lemma 2.2 of [4] that the function $w = w(\delta, \epsilon, v)$ obtained solving (2.4) is defined and of class C^1 in an open neighborhood U_1 of $\{0\} \times \{0\} \times M \subset \mathbb{R} \times \mathbb{R} \times N$.

Once this is done, a further reduction can be made on "eliminating δ" from our equations. Indeed in the present Hilbert space context, taking scalar product in (2.8) we get

$$\langle \epsilon PB(v + w(\delta, \epsilon, v)), v \rangle = \delta \|v\|^2. \tag{2.12}$$

Dividing both members of (2.12) by $\|v\|^2$ and applying again Lemma 2.2 of [4] to the resulting equation, we see that δ can be written as a C^1 function $\delta(\epsilon, v)$ of (ϵ, v), defined in an open subset V of $\mathbb{R} \times (N \setminus \{0\})$ containing $\{0\} \times M$ and such that $\delta(0, v) = 0$ for any v, and $(\delta(\epsilon, v), \epsilon, v) \in U_1$ for $(\epsilon, v) \in V$.

Put for convenience $\phi(\epsilon, v) \equiv w(\delta(\epsilon, v), \epsilon, v)$. Then we see - from (2.8) and the normalization condition (2.9) - that in order to solve (1.1) it is enough to find $(\epsilon, v) \in V$ such that

$$\epsilon PB(v + \phi(\epsilon, v)) = \delta(\epsilon, v)v \tag{2.13}$$

and

$$[\|v + \phi(\epsilon, v)\|^2] = \|v\|^2 + \|\phi(\epsilon, v)\|^2 = 1. \tag{2.14}$$

Under the assumptions of Theorem 2.2, we show that a stronger result holds: namely, for any sufficiently small ϵ there exists $v_\epsilon \in M$ such that (ϵ, v_ϵ) satisfies (2.13) and (2.14). To this purpose, assume for simplicity that $\lambda_0 = 1$. Then adding v to both sides of (2.13) and putting $h(\epsilon, v) = 1 + \delta(\epsilon, v)$ we get

$$v + \epsilon PB(v + \phi(\epsilon, v)) = h(\epsilon, v)v. \tag{2.15}$$

Fix $\epsilon \neq 0$ and let σ be the radial projection of $N \setminus \{0\}$ onto its unit sphere S_0, defined putting $\sigma(v) = v/\|v\|$ for $v \in N, v \neq 0$. Then looking for solutions $v \in M$ of (2.15) is equivalent to finding $v \in M$ such that

$$\sigma(v + \epsilon PB(v + \phi(\epsilon, v))) = \frac{v}{\|v\|}. \tag{2.16}$$

On the other hand, using (2.14) this last equation becomes

$$f_\epsilon(v) \equiv \sqrt{1 - \|\phi(\epsilon, v)\|^2}\sigma(v + \epsilon PB(v + \phi(\epsilon, v))) = v, \tag{2.17}$$

which is a fixed point equation for the map $f_\epsilon : M \to M$. The Lefschetz number of f_ϵ equals the Euler–Poincaré characteristic of S_0 [4], and thus is not zero since S_0 is even dimensional. By the Lefschetz fixed point theorem [1], there exists $v_\epsilon \in M$ such that $f_\epsilon(v_\epsilon) = v_\epsilon$.

Now fix a sequence (ϵ_n) with $\epsilon_n \to 0$ and $\epsilon_n \neq 0$ forall $n \in \mathbb{N}$ and put $v_n \equiv v_{\epsilon_n}$; also let

$$\delta_n \equiv \delta(\epsilon_n, v_n), \qquad u_n \equiv v_n + \phi(\epsilon_n, v_n).$$

By the compactness of M we can assume - passing if necessary to a subsequence - that $v_n \to v_0$. It follows that $\phi(\epsilon_n, v_n) \to \phi(0, v_0) = 0$, which implies by (2.14) that $\|v_n\| \to 1$ and therefore that $v_0 \in S$. Moreover since $(\delta_n, \epsilon_n, u_n)$ solves (1.1) for any n and $u_n \to v_0$, it follows that $v_0 \in S_0$ and is a bifurcation point for (1.1).

[ChiFuPe3]

(HBG) B **is a gradient operator in neighborhood of** S

that is, there exists a differentiable functional b defined on a open neighborhood U of S such that

$$\langle B(x), y\rangle = b'(x)y \quad \text{for all} \quad x \in U, y \in H. \tag{2.18}$$

Here $b'(x)$ denotes the (Fréchet) derivative of b at the point $x \in U$.

Theorem 2.3. *Suppose that $T : H \to H$ is a bounded self-adjoint operator, and suppose that B satisfies* (HB1) *and* (HBG)*. If λ_0 is an isolated eigenvalue of T of finite multiplicity, then S_0 possesses at least one bifurcation point.*

Sketch of the proof. Variational methods

To indicate the main points of the proof, we keep the same notations as before and put in addition

$$F_\epsilon(u) \equiv Au + \epsilon B(u), \quad \delta_\epsilon(v) \equiv \delta(\epsilon, v), \quad \phi_\epsilon(v) \equiv \phi(\epsilon, v)$$

so that the system (2.13)-(2.14) in the unknowns ϵ and v can be written

$$PF_\epsilon(v + \phi_\epsilon(v)) = \delta_\epsilon(v)v, \qquad \|v + \phi_\epsilon(v)\|^2 = 1. \tag{2.19}$$

Under the assumptions of Theorem [.] we show that for any ϵ small there exist (at least) *two* distinct solutions $v = v_\epsilon$, $z = z_\epsilon$ of (2.19). To this aim, let $B = \nabla b$ - that is, suppose that (2.18) holds; then $F_\epsilon = \nabla f_\epsilon$ with

$$f_\epsilon(u) = \frac{1}{2}\langle Au, u\rangle + \epsilon b(u).$$

We follow an idea of Stuart [11] to show that for fixed ϵ, the solutions v of (2.19) are precisely the critical points of the functional α_ϵ defined by

$$\alpha_\epsilon(v) = f_\epsilon(v + \phi_\epsilon(v)) = \frac{1}{2}\langle A\phi_\epsilon(v), \phi_\epsilon(v)\rangle + \epsilon b(v + \phi_\epsilon(v)) \tag{2.20}$$

over the manifold defined by the norm constraint, that is

$$M_\epsilon = \{v \in N : \|v + \phi_\epsilon(v)\|^2 = 1\}. \tag{2.21}$$

Once this is done, the compactness of M_ϵ implies the existence of $v_\epsilon, z_\epsilon \in M_\epsilon$ such that

$$\alpha_\epsilon(v_\epsilon) = \min_{v \in M_\epsilon} \alpha_\epsilon(v), \qquad \alpha_\epsilon(z_\epsilon) = \max_{v \in M_\epsilon} \alpha_\epsilon(v). \tag{2.22}$$

and therefore implies that (for each ϵ), v_ϵ and z_ϵ solve (2.19).

Using for instance v_ϵ and reasoning as in the proof of Theorem 2.2, we can then construct a sequence $(\delta_n, \epsilon_n, u_n)$ of solutions to (1.1), with u_n converging to some $v_0 \in S_0$ which is therefore a bifurcation point.

Remark 2.2. It would be interesting to establish conditions guaranteeing that there are (at least) two different bifurcation points.

3. Examples in $\mathbb{R}^3$

In this Section we consider (2.1) in the very special case that $X = \mathbb{R}^3$ and that (besides A) also the perturbing term B is linear. Moreover we keep fixed a very simple A, namely - writing $u = (x, y, z)$ for $u \in \mathbb{R}^3$ - the projection onto the z-axis:

$$A(x, y, z) = (0, 0, z).$$

Thus, representing A with the corresponding 3×3 matrix in the canonical basis,

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.1}$$

Consider at first a generic B:

$$B = \begin{pmatrix} a & b & m \\ c & d & n \\ p & q & r \end{pmatrix} \tag{3.2}$$

Then (2.1) is

$$\begin{cases} \epsilon(ax + by + mz) = \delta x, \\ \epsilon(cx + dy + nz) = \delta y, \\ z + \epsilon(px + qy + rz) = \delta z. \end{cases} \tag{3.3}$$

The last equation can be solved in z to yield

$$z = z(\delta, \epsilon, x, y) = \frac{\epsilon}{\delta - (1 + \epsilon r)}(px + qy) \tag{3.4}$$

and we are thus reduced to solve the system (in the unknowns δ, ϵ, x, y)

$$\begin{cases} ax + by + mz(\delta, \epsilon, x, y) = (\delta/\epsilon)x, \\ cx + dy + nz(\delta, \epsilon, x, y)) = (\delta/\epsilon)y. \end{cases} \tag{3.5}$$

Example 3.1. Consider

$$B = \begin{pmatrix} a & b & m \\ c & d & n \\ 0 & 0 & r \end{pmatrix} \tag{3.6}$$

that is,

$$B(x, y, z) = (ax + by + mz, cx + dy + nz, rz).$$

We see from (3.4) that in this case $z(\delta, \epsilon, x, y) \equiv 0$, so that the bifurcation system reduces to

$$\begin{cases} ax+by=(\delta/\epsilon)x, \\ cx+dy=(\delta/\epsilon)y. \end{cases} \tag{3.7}$$

The solutions $(x, y) \neq (0, 0)$ of this system [- if any -] are the eigenvectors of the reduced 2×2 matrix

$$\hat{B} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.8}$$

corresponding to real eigenvalues. Suppose first that $\hat{B}$ has two real eigenvalues μ_1, μ_2 with $\mu_1 \neq \mu_2$. If v_1, v_2 are corresponding normalized eigenvectors, then the bifurcation branches defined putting

$$\delta_i(\epsilon) = \epsilon\mu_i, \qquad u_i(\epsilon) = v_i \qquad (i = 1, 2) \tag{3.9}$$

provide a (trivial) continuation of v_i as solution of (1.1) for $\epsilon \neq 0$; the same clearly holds for $-v_i$. Thus each eigenvector of $\hat{B}$ is continuable as a unit eigenvector of $A + \epsilon B$.

The same conclusion holds true when $\mu_1 = \mu_2 \equiv \mu_0$, save that either the geometric multiplicity of μ_0 is two - in which case all vectors of $\mathbb{R}^2$ are eigenvectors of $\hat{B}$ - or it is one, and there is (modulo reflections) just one normed eigenvector v_0 of $\hat{B}$.

Remark 3.1. If $\hat{B}$ has no real eigenvalue there cannot be bifurcation points. On the grounds of Proposition 2.1, this holds for any B (and not only for B as in (3.6)).

Example 3.2. Consider

$$B = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ p & q & r \end{pmatrix} \tag{3.10}$$

that is,

$$B(x, y, z) = (ax + by, cx + dy, px + qy + rz).$$

This time $z(\delta, \epsilon, x, y)$ is given by its general expression (3.4), however since $m = n = 0$ this does not affect the bifurcation system - which maintains its reduced form (3.7) - nor the conclusion that each eigenvector of $\hat{B}$ is a bifurcation point. The difference with Ex.1.1 is that here the solutions of the full system (3.3) have a nonzero $z-$component, and consequently the bifurcation branch continuing a given eigenvector $v_0 = (x_0, y_0)$ of $\hat{B}$ corresponding to the eigenvalue μ_0 is a bit less trivial as is given by the equations

$$\delta(\epsilon) = \epsilon\mu_0, \qquad u(\epsilon) = (x_0, y_0, z(\epsilon\mu_0, \epsilon, x_0, y_0)). \tag{3.11}$$

Remark 3.2. The above examples can be clearly seen in the context of Equation (2.1). We keep the notations used in Section 2 for $N = \operatorname{Ker} A, W = \operatorname{Im} A$ as well as for the projections P, Q onto these subspaces. Pick a $v_0 \in S_0$ and consider the complementary equation (2.4) with $v = v_0$:

$$Aw + \epsilon QB(v_0 + w) = \delta w. \tag{3.12}$$

If we suppose that

$$QB(v_0) = 0, \tag{3.13}$$

then $w = 0$ solves (3.12); by uniqueness, it follows that $w(\delta, \epsilon, v_0) = 0$ for any δ and ϵ. The bifurcation equation (2.8) thus reduces (for $v = v_0$) to

$$\epsilon PB(v_0) = \delta v_0, \tag{3.14}$$

which is precisely - via the position $\delta = \epsilon\mu_0$ - the necessary condition (2.5). This remark is not new, for (3.13) and (2.5) are equivalent to say that $B(v_0) = \mu_0 v_0$ and in this case, as already noted in the Introduction, we can immediately solve (2.1) for all ϵ. Perhaps more interesting is to observe that requiring the condition (3.13) for *all* $v_0 \in S_0$ amounts to requiring that B map S_0 into N and therefore - when B is linear, of course - that B map N into itself, i.e., that N be an *invariant subspace* for B. Indeed, this is what happens in Example 1.1.

Consider instead the dual situation in which W, rather than N, is an invariant subspace for B; the B in Example 1.2 is chosen to enjoy this property. This is expressed by the condition that $PB(w) = 0$ for any $w \in W$; so that if we pick a $v_0 \in S_0$ satisfying the necessary condition (2.5), then we have in particular

$$\epsilon PB(v_0 + w(\epsilon\mu_0, \epsilon, v_0)) = \epsilon PB(v_0) = \epsilon\mu_0 v_0 \tag{3.15}$$

for any ϵ. This shows that $(\epsilon\mu_0, \epsilon, v_0)$ is a solution of (2.8) for any ϵ, implying that v_0 is continuable via the equations

$$\delta(\epsilon) = \epsilon\mu_0, \qquad u(\epsilon) = v_0 + w(\epsilon\mu_0, \epsilon, v_0). \tag{3.16}$$

Of course, in order to have unit eigenvectors we shall take $U(\epsilon) \equiv u(\epsilon)/\|u(\epsilon)\|$ rather than $u(\epsilon)$ itself and use the linearity of the equation.

Example 3.3. Here we consider the case

$$B = \begin{pmatrix} 0 & b & 0 \\ c & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{3.17}$$

For such a B, (3.4) becomes

$$z = z(\delta, \epsilon, x, y) = \frac{\epsilon}{\delta - 1} x \tag{3.18}$$

and the bifurcation system (3.5) is

$$\begin{cases} by = (\delta/\epsilon)x \\ cx + \frac{\epsilon}{\delta-1}x = (\delta/\epsilon)y. \end{cases} \tag{3.19}$$

We have to distiguish the following cases:

- $bc > 0$
- $b > 0$ (or $b < 0$) and $c = 0$
- $b = c = 0$
- $bc < 0$.

The last case is out of interest, for the reduced matrix $\hat{B}$

$$\hat{B} \equiv \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \tag{3.20}$$

then has no real eigenvalue.

- **Case** $bc > 0$**:**

[Suppose thus that $bc > 0$]. Then the first equation in (3.19) yields

$$y = \frac{\delta}{\epsilon b} x, \tag{3.21}$$

so that the solutions $u = (x, y, z)$ of the full system (3.3) have the form

$$u = x(1, \frac{\delta}{\epsilon b}, \frac{\epsilon}{\delta - 1}). \tag{3.22}$$

Moreover replacing (3.21) in the second equation of the system (3.19) gives the condition

$$c + \frac{\epsilon}{\delta - 1} = \frac{\delta^2}{\epsilon^2 b} \tag{3.23}$$

provided that $x \neq 0$; however, (3.22) implies that $u = 0$ if $x = 0$, and we look for solutions $u \neq 0$. (Note that the above equations make sense whenever $b \neq 0$, however since the l.h.s. of (3.23) has - for ϵ small and $c \neq 0$ - the sign of c, it follows that b and c must have the same sign in order that (real) solutions to (3.23) exist).

Now the latter equation - expressing the eigenvalues in function of the parameter ϵ - can be written equivalently as

$$(\delta - 1)(\delta^2 - \epsilon^2 bc) = \epsilon^3 b, \tag{3.24}$$

and by direct inspection we then find that it has for each ϵ three real solutions $\delta_i(\epsilon), 1 \leq i \leq 3$, with the property

$$\delta_i(\epsilon) \to 0,\ i = 1, 2; \qquad \delta_3(\epsilon) \to 1 \qquad (\epsilon \to 0). \tag{3.25}$$

Therefore, using (3.23) in (3.22), we see that the eigenvectors of interest are given by the formula

$$u(\epsilon) = x(1, \frac{\delta}{\epsilon b}, \frac{\delta^2}{\epsilon^2 b} - c) \qquad (x \neq 0) \tag{3.26}$$

where $\delta = \delta_i(\epsilon),\ i = 1, 2$. Equation (3.26) shows that the ratio δ/ϵ is the significant parameter here. Now since $\epsilon/(\delta - 1)$ approaches zero as $\epsilon \to 0$, it follows from (3.23) that $\delta^2/(\epsilon^2 b) \to c$ as $\epsilon \to 0$, and therefore

$$\frac{\delta}{\epsilon} \to \pm\sqrt{bc} \equiv \pm\mu_0 \qquad (\epsilon \to 0). \tag{3.27}$$

Thus if we let in (3.26) $x = 1$ and $\delta = \delta_i(\epsilon)\,(i = 1, 2)$, and denote with $u_i(\epsilon)$ the corresponding vector, then as $\epsilon \to 0$

$$u_i(\epsilon) = (1, \frac{\delta_i(\epsilon)}{\epsilon b}, \frac{\delta_i^2(\epsilon)}{\epsilon^2 b} - c) \to (1, \pm\frac{\mu_0}{b}, 0) \equiv u_\pm \tag{3.28}$$

where the signs $+$ and $-$ refer to $i = 1$ and $i = 2$ respectively. It follows that if $U_i(\epsilon)$, $U_\pm$ denote the normalized vectors corresponding respectively to $u_i(\epsilon)$ and $u_\pm$, then

$$U_1(\epsilon) \to U_+, \qquad U_2(\epsilon) \to U_- \qquad (\epsilon \to 0), \tag{3.29}$$

and this finally shows that $U_\pm$ (together of course with their opposites $-U_\pm$) are the bifurcation points in this case.

Conclusions ($bc > 0$):

Since $\mu_0 > 0$ in this case, we thus have precisely four bifurcation points. Note that $\pm\mu_0$ are precisely the eigenvalues of the matrix in (3.20) and $U_\pm$ (together with their opposites $-U_\pm$) the corresponding unit eigenvectors. Thus also in this case (as in the Examples 1.1 and 1.2), every $v_0 \in S_0$ satisfying the necessary condition (2.5) is in fact a bifurcation point.

- **Case $b > 0$ (or $b < 0$) and $c = 0$:**

The previous analysis remains true save that in this case $\mu_0 = 0$. Therefore,

$$U_{\pm} = (1, 0, 0) \equiv e_1$$

is the only bifurcation point (modulo reflections). However, it is important to note that there exist two distinct *bifurcation branches* bifurcating from e_1: indeed it is easily seen from (3.24) that $\delta_1(\epsilon) \neq \delta_2(\epsilon)$ for each $\epsilon \neq 0$, and this shows - via the formula (3.28) - that $u_1(\epsilon) \neq u_2(\epsilon)$ for $\epsilon \neq 0$.

- **Case $b = c = 0$:**

The situation is quite different when $b = c = 0$. Indeed in this case the bifurcation system (3.19) reduces to

$$\begin{cases} 0 = \delta x \\ \frac{\epsilon^2}{\delta - 1} x = \delta y. \end{cases} \tag{3.30}$$

Solutions $(x, y) \neq (0, 0)$ of (3.30) exist only for $\delta = 0$, in which case they are (for $\epsilon \neq 0$)

$$(0, y), \qquad y \neq 0.$$

Thus, the only nontrivial normalized solution of the full system (3.3) are (for any $\epsilon \neq 0$)

$$(0, \pm 1, 0) \equiv \pm e_2.$$

This shows that $\pm e_2$ are also the only bifurcation points of the system in this case (and we have the trivial bifurcation branch $\delta(\epsilon) = 0, U_\epsilon = \pm e_2$). On the other hand, as $\hat{B} = 0$, any $v = (x, y) \in N = \mathbb{R}^2$ satisfies the necessary condition with $\mu_0 = 0$.

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