TOPOLOGICAL PERSISTENCE OF THE NORMALIZED EIGENVECTORS OF A PERTURBED SELFADJOINT OPERATOR

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Abstract. Let T be a selfadjoint bounded operator acting in a real Hilbert space H, and denote by S the unit sphere of H. Assume that λ_0 is an isolated eigenvalue of T of odd multiplicity greater than 1. Given an arbitrary operator $B\colon H \to H$ of class C^1, we prove that for any $\varepsilon \neq 0$ sufficiently small there exists $x_\varepsilon \in S$ and λ_ε near λ_0, such that $Tx_\varepsilon + \varepsilon B(x_\varepsilon) = \lambda_\varepsilon x_\varepsilon$. This result was conjectured, but not proved, in a previous article by the authors.

We provide an example showing that the assumption that the multiplicity of λ_0 is odd cannot be removed.

1. Introduction

Let T be a selfadjoint bounded operator acting in a real Hilbert space H, and denote by S the unit sphere of H. Let x_0 be a *unit* λ_0*-eigenvector of* T, i.e. x_0 belongs to S and is an eigenvector of T with eigenvalue $\lambda_0 \in \mathbb{R}$. Given a (possibly nonlinear) continuous operator $B\colon H \to H$, consider the perturbed "eigenvalue" problem

$$Tx + \varepsilon B(x) = \lambda x, \quad x \in S. \tag{1.1}$$

We say that x_0 is *continuable for problem* (1.1) if there exists a continuous function $\varepsilon \mapsto (\lambda_\varepsilon, x_\varepsilon)$ of an interval $(-\delta, \delta)$ into $\mathbb{R} \times H$ such that

$$Tx_\varepsilon + \varepsilon B(x_\varepsilon) = \lambda_\varepsilon x_\varepsilon, \quad \forall \varepsilon \in (-\delta, \delta).$$

Notice that a unit λ_0-eigenvector x_0 of T is continuable if it is also an *eigenvector* of B; meaning, as in the linear case, that $B(x_0) = \mu x_0$ for some $\mu \in \mathbb{R}$. Indeed, $\varepsilon \mapsto (\lambda_0 + \varepsilon\mu, x_0)$ is the required continuous function.

For a simple example of a perturbed eigenvalue problem without any continuable eigenvector (but with nonempty sphere of unit eigenvectors), take $H = \mathbb{R}^2$, T the zero operator, and $B\colon (x, y) \mapsto (-y, x)$.

In [2] the first author proved that if λ_0 is an isolated simple eigenvalue of T, then, under the mild assumption that B is Lipschitz continuous, each of the two unit λ_0-eigenvectors is continuable (in a Lipschitz continuous fashion).

In [3] we considered the case in which λ_0 is an isolated eigenvalue of T of multiplicity greater than 1 (algebraic and geometric multiplicities coincide in the selfadjoint case) and we gave necessary, as well as sufficient, conditions for a given unit λ_0-eigenvector to be continuable. In the same article we formulated the conjecture that when the multiplicity of λ_0 is odd, then, whatever is the nonlinear operator B (provided it is continuous), the even dimensional sphere of unit λ_0-eigenvectors

Date: September 25, 2009.
2000 *Mathematics Subject Classification.* Primary 47H14; Secondary 47A55, 47J10, 47J15.

contains at least one vector x_0 which is "persistent" in the following sense: *there exists a sequence* $\{(\varepsilon_n, \lambda_n, x_n)\}$ *in* $\mathbb{R} \setminus \{0\} \times \mathbb{R} \times S$ *which converges to* $(0, \lambda_0, x_0)$ *and such that* $Tx_n + \varepsilon_n B(x_n) = \lambda_n x_n$, $\forall n \in \mathbb{N}$.

When this happens we say that such an x_0 is a *bifurcation point for problem* (1.1). In fact, identifying the sphere of the unit λ_0-eigenvectors of T with the set of *trivial solutions* $(0, \lambda_0, x)$ of the equation (1.1), any neighborhood (in $\mathbb{R} \times \mathbb{R} \times S$) of x_0 (regarded as $(0, \lambda_0, x_0)$) contains nontrivial solutions, i.e. solutions which are not of the type $(0, \lambda_0, x)$. Incidentally, we observe that any nontrivial solution $(\varepsilon, \lambda, x)$ of (1.1) must have $\varepsilon \neq 0$, due to the fact that λ_0 is assumed to be isolated.

In this paper we give a positive answer to the above conjecture (see Theorem 3.1 below), but under the assumption that B is of class C^1. We still believe that this additional hypothesis on B is unnecessary, but, for technical reasons, up to now we are not able to remove it.

The proof of our result is based on the fact that the Euler–Poincaré characteristic of the even dimensional sphere $S \cap \operatorname{Ker}(T - \lambda_0 I)$ is nonzero (I denotes the identity on H), and this implies that any self-map of this sphere has a fixed point if it is homotopic to the identity. Therefore, the methods we use to get our result are of topological nature, and quite different from those employed in [2] and [3], which are related to the implicit function theorem.

2. Preliminaries

Let X be a compact differentiable manifold possibly with boundary or, more generally, a compact topological space homeomorphic to a polyhedron. Then, to any continuous map $f\colon X \to X$ it is possible to associate an integer $\lambda(f)$, called the *Lefschetz number of* f, satisfying the following properties (see e.g. [1, 5]):

(1) (Normalization) The Lefschetz number of the identity of X coincides with the Euler–Poincaré characteristic $\chi(X)$ of X.
(2) (Homotopy Invariance) If f is homotopic to $g\colon X \to X$, then $\lambda(f) = \lambda(g)$.
(3) (Commutativity) If $\varphi\colon X \to Y$ and $\psi\colon Y \to X$ are continuous maps, then $\lambda(f) = \lambda(g)$, where $f = \psi \circ \varphi$, $g = \varphi \circ \psi$. In particular, if $Y \subseteq X$ and $f\colon X \to X$ is such that $f(X) \subseteq Y$, then $\lambda(f) = \lambda(f|_Y)$.

The Lefschetz fixed point theorem below will play an essential role in proving the main result of this paper.

Theorem 2.1 (Lefschetz Theorem)**.** *Let X be a compact differentiable manifold possibly with boundary and $f\colon X \to X$ a continuous map. Assume that the Lefschetz number $\lambda(f)$ is different from zero. Then f has a fixed point.*

Theorem (2.1) was first proved for compact manifolds with boundary by Lefschetz in [6]. For a wide discussion of the extensions of the theorem to various classes of maps and spaces (as, for instance, to polyhedra or noncompact ANRs and to compact or locally compact maps) see e.g. [1, 5].

The following easy extension of the Implicit Function Theorem will be needed in obtaining our main result. Its proof is given here for completeness. In what follows, given two Banach spaces E and F, a subset $G \subseteq E \times F$ is called a *graph* in $E \times F$ if for any $x \in E$ there exists at most one $y \in F$ such that $(x, y) \in G$.

Lemma 2.2. *Let E, F be Banach spaces, Ω an open subset of $E \times F$, $f\colon \Omega \subseteq E \times F \to F$ a C^n map ($n \geq 1$), and G a compact graph contained in $f^{-1}(0)$.*

Assume that the derivative $D_2 f(x,y)$ *of* f *with respect to the second variable is invertible for any* $(x,y) \in G$. *Then, there exists an open neighborhood* W *of* G *such that* $W \cap f^{-1}(0)$ *is the graph of a* C^n *map* $\varphi\colon \pi_1(W) \to F$, *where* $\pi_1\colon E \times F \to E$ *denotes the projection onto* E.

Proof. Let us prove first that there exists an open neighborhood $\tilde W$ of G such that $\tilde W \cap f^{-1}(0)$ is a graph in $E \times F$.

By contradiction, assume the existence of sequences $\{x_n\}$ in E and $\{y_n^1\}$, $\{y_n^2\}$ in F, $y_n^1 \neq y_n^2$, such that $\{(x_n, y_n^i)\} \in f^{-1}(0)$, $i = 1,2$, and $d((x_n, y_n^i), G) \to 0$ as $n \to +\infty$, where $d(\cdot, G)$ denotes the distance from G. Since G is compact, without loss of generality, we may assume that $x_n \to x_0 \in \pi_1(G)$ and $y_n^i \to y^i$, with $(x_0, y^i) \in G$, $i = 1,2$. Hence, since the set G is a graph, y^1 must coincide with y^2. Let us denote by y_0 this common value. Since $D_2 f(x_0, y_0)$ is invertible, by the Implicit Function Theorem there exist two open neighborhoods U and V of x_0 and y_0 respectively, such that $(U \times V) \cap f^{-1}(0)$ is the graph of a C^1 map $\varphi_0\colon U \to V$. On the other hand, for $n \in \mathbb{N}$ sufficiently large, we get $(x_n, y_n^i) \in U \times V$. Consequently, $y_n^1 = y_n^2 = \varphi_0(x_n)$, which is a contradiction. This proves the existence of $\tilde W$.

To complete the proof, it suffices to apply the Implicit Function Theorem to the map f restricted to $\tilde W$. More precisely, for any $(x_0, y_0) \in G$ there exist neighborhoods U_{x_0} of x_0 in E and V_{y_0} of y_0 in F such that $U_{x_0} \times V_{y_0} \subseteq \tilde W$ and $(U_{x_0} \times V_{y_0}) \cap f^{-1}(0)$ is the graph of a C^1 map $\varphi_0\colon U_{x_0} \to V_{y_0}$. Now, set

$$W = \bigcup_{(x_0,y_0)\in G} U_{x_0} \times V_{y_0}$$

and observe that, if $x \in \pi_1(W)$, then x belongs to some U_{x_0}. Thus, because of the uniqueness established in $\tilde W$, the map $\varphi\colon \pi_1(W) \to F$ given by $\varphi(x) = \varphi_0(x)$ is well-defined and $W \cap f^{-1}(0)$ coincides with its graph. □

3. Statement and proof of the result

Theorem 3.1. *Let* T *be a selfadjoint bounded operator on a real Hilbert space* H *and let* S *denote the unit sphere of* H. *Assume that* $B\colon H \to H$ *is a* C^1 *operator. If* λ_0 *is an isolated eigenvalue of* T *of odd multiplicity, then in* $\mathbb{R}\setminus\{0\} \times \mathbb{R} \times S$ *there exists a sequence* $\{(\varepsilon_n, \lambda_n, x_n)\}$ *which converges to a point* $(0, \lambda_0, x_0)$ *and such that*

$$Tx_n + \varepsilon_n B(x_n) = \lambda_n x_n\,, \quad n \in \mathbb{N}.$$

Therefore $x_0 \in S \cap \operatorname{Ker}(T - \lambda_0 I)$ *and it is a bifurcation point for problem* (1.1).

Proof. Consider the equation

$$Tx + \varepsilon B(x) = \lambda x\,, \tag{3.1}$$

with the condition

$$\|x\|^2 = 1\,. \tag{3.2}$$

Without loss of generality we can assume $\lambda_0 = 1$. Let H_1 denote the kernel of $T - I$ and H_2 its orthogonal complement. Hence, H can be identified with $H_1 \times H_2$ via the topological isomorphism $(v,w) \mapsto v + w$. Thus, the equation (3.1) can be written in block-matrix form as follows:

$$\begin{pmatrix} I & 0 \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} + \varepsilon \begin{pmatrix} B_1(v,w) \\ B_2(v,w) \end{pmatrix} = \begin{pmatrix} \lambda v \\ \lambda w \end{pmatrix}$$

or, equivalently, as

$$\begin{cases} v + \varepsilon B_1(v, w) = \lambda v, \\ T_{22} w + \varepsilon B_2(v, w) = \lambda w. \end{cases} \tag{3.3}$$

Moreover, (3.2) becomes

$$\|v\|^2 + \|w\|^2 = 1\,. \tag{3.4}$$

Consider the compact set

$$M = \left\{ v \in H_1 : \big| \|v\| - 1 \big| \le \frac{1}{2} \right\}.$$

We seek for solutions $(\varepsilon, \lambda, v, w)$ of (3.3), (3.4) with $v \in M$. In addition, since we have supposed $\lambda_0 = 1$, we may look for solutions possessing $\lambda > 0$.

As the eigenvalue λ_0 is assumed to be isolated, the image of $T - I$ coincides with the subspace H_2 of H (see e.g. [4, pg. 1395]). Thus, in system (3.3), the linear operator $w \in H_2 \mapsto T_{22} w - w \in H_2$ is invertible. Moreover, for any $v \in M$, the element $(0, 1, v, 0)$ satisfies the equation $T_{22} w + \varepsilon B_2(v, w) = \lambda w$. Therefore, Lemma 2.2 applies to this equation with $G = \{0\} \times \{1\} \times M \times \{0\}$ and allows us to express w as a C^1 function of $(\varepsilon, \lambda, v)$, say $w = g(\varepsilon, \lambda, v)$, on an open subset U of $\mathbb{R} \times (0, +\infty) \times (H_1 \setminus \{0\})$ containing the compact set $\{0\} \times \{1\} \times M$. Hence,

$$T_{22} g(\varepsilon, \lambda, v) + \varepsilon B_2(v, g(\varepsilon, \lambda, v)) = \lambda g(\varepsilon, \lambda, v)\,, \quad (\varepsilon, \lambda, v) \in U,$$

and, clearly, $g(0, 1, v) = 0$ for all v. By replacing w with $g(\varepsilon, \lambda, v)$ in the first equation of system (3.3) and in (3.4), we obtain

$$v + \varepsilon B_1(v, g(\varepsilon, \lambda, v)) = \lambda\, v \tag{3.5}$$

and

$$\|v\|^2 + \|g(\varepsilon, \lambda, v)\|^2 = 1\,. \tag{3.6}$$

Let $S_1 \subseteq M$ be the unit sphere of H_1 and let $\pi\colon H_1 \setminus \{0\} \to S_1$ denote the radial projection onto S_1. Due to the continuity of B and the compactness of $\{0\} \times \{1\} \times M$, we can assume, in case taking U smaller, that the map $(\varepsilon, \lambda, v) \in U \mapsto B_1(v, g(\varepsilon, \lambda, v))$ is bounded and, since v is nonzero, that $v + \varepsilon B_1(v, g(\varepsilon, \lambda, v))$ belongs to $H_1 \setminus \{0\}$. Therefore, we can apply π to the equation (3.5) and, recalling that $\lambda > 0$, we get

$$\pi(v + \varepsilon B_1(v, g(\varepsilon, \lambda, v))) = \pi(\lambda\, v) = \frac{\lambda\, v}{\|\lambda\, v\|} = \frac{v}{\|v\|}\,. \tag{3.7}$$

Moreover, by taking the norms in both sides of (3.5), we obtain

$$\lambda = \frac{\|v + \varepsilon B_1(v, g(\varepsilon, \lambda, v))\|}{\|v\|}. \tag{3.8}$$

Now, the partial derivative with respect to λ of the real valued map

$$(\varepsilon, \lambda, v) \in U \mapsto \lambda - \frac{\|v + \varepsilon B_1(v, g(\varepsilon, \lambda, v))\|}{\|v\|}$$

at $(0, 1, v)$ is clearly equal to 1 and, in equation (3.8), for $\varepsilon = 0$ and any v we get $\lambda = 1$. Therefore, by applying Lemma 2.2 to equation (3.8), we can write λ as a C^1 function of (ε, v), say $\lambda = h(\varepsilon, v)$, on an open subset V of $\mathbb{R} \times (H_1 \setminus \{0\})$ containing $\{0\} \times M$ and such that $(\varepsilon, h(\varepsilon, v), v) \in U$, for $(\varepsilon, v) \in V$. Hence, h satisfies

$$h(\varepsilon, v) = \frac{\|v + \varepsilon B_1(v, g(\varepsilon, h(\varepsilon, v), v))\|}{\|v\|}\,, \quad (\varepsilon, v) \in W,$$

and, clearly, $h(0,v) = 1$ for all v. By replacing λ with $h(\varepsilon, v)$ in (3.7), we get the equation

$$\|v\|\,\pi(v + \varepsilon B_1(v, g(\varepsilon, h(\varepsilon, v), v))) = v\,,$$

where, from (3.6), v also verifies

$$\|v\|^2 + \|g(\varepsilon, h(\varepsilon, v), v)\|^2 = 1\,.$$

The last two conditions give raise to the equation

$$\sqrt{1 - \|g(\varepsilon, h(\varepsilon, v), v)\|^2}\,\pi(v + \varepsilon B_1(v, g(\varepsilon, h(\varepsilon, v), v))) = v\,, \tag{3.9}$$

that can be interpreted as a fixed point equation in v depending on the real parameter ε. More precisely, let $f\colon V \subseteq \mathbb{R} \times (H_1 \setminus \{0\}) \to H_1 \setminus \{0\}$ be the continuous map

$$f(\varepsilon, v) = \sqrt{1 - \|g(\varepsilon, h(\varepsilon, v), v)\|^2}\,\pi(v + \varepsilon B_1(v, g(\varepsilon, h(\varepsilon, v), v)))$$

and let $f_\varepsilon\colon V_\varepsilon \to H_1 \setminus \{0\}$ be the partial map $f_\varepsilon = f(\varepsilon, \cdot)$, where

$$V_\varepsilon = \{v \in H_1 : (\varepsilon, v) \in V\}$$

denotes the slice of V at ε. Since V contains the compact set $\{0\} \times M$, there exists $\epsilon_0 > 0$ such that $V_\varepsilon \supseteq M$ for $|\varepsilon| \le \varepsilon_0$. Moreover, for $\varepsilon = 0$, we have $g(0, h(0, v), v) = 0$ since $h(0, v) = 1$. Consequently, the map f_0 is a retraction of M onto S_1, i.e. $f_0(M) \subseteq S_1$ and $f_0|_{S_1}(v) = v$ for all $v \in S_1$. Therefore, in case reducing ε_0, we can assume $f_\varepsilon(M) \subseteq M$ for all $|\varepsilon| \le \varepsilon_0$. Our aim is to apply Theorem 2.1 to the map f_ε in M that is clearly a compact smooth manifold with boundary. To this end, we need to compute the Lefschetz number $\lambda(f_\varepsilon)$. By the homotopy invariance of the Lefschetz number, we get

$$\lambda(f_\varepsilon) = \lambda(f_0)\,,$$

and, by its commutativity,

$$\lambda(f_0) = \lambda(f_0|_{S_1})\,.$$

On the other hand, recalling that $f_0|_{S_1}$ is the identity of S_1, the normalization property implies

$$\lambda(f_0|_{S_1}) = \chi(S_1)\,.$$

By assumption, the eigenvalue $\lambda_0 = 1$ has odd multiplicity. Hence, the sphere S_1 is even dimensional and, thus, its Euler–Poincaré characteristic $\chi(S_1)$ is 2. Thus,

$$\lambda(f_\varepsilon) = \chi(S_1) \neq 0\,,$$

and the Lefschetz fixed point theorem applies yielding the existence, for $|\varepsilon| \le \varepsilon_0$, of $v_\varepsilon \in M$ such that $f_\varepsilon(v_\varepsilon) = v_\varepsilon$. Consequently, taking $\lambda_\varepsilon = h(\varepsilon, v_\varepsilon)$ and $w_\varepsilon = g(\varepsilon, \lambda_\varepsilon, v_\varepsilon)$, the element $(\varepsilon, \lambda_\varepsilon, v_\varepsilon, w_\varepsilon) \in [-\varepsilon_0, \varepsilon_0] \times (0, +\infty) \times M \times H_2$ is a solution of (3.3), (3.4).

Now, let us take a real sequence $\{\varepsilon_n\}$, $0 < |\varepsilon_n| \le \varepsilon_0$, $\varepsilon_n \to 0$. As proved above, for any $n \in \mathbb{N}$ there exists a fixed point $v_n \in M$ of f_{ε_n} and, thus, $\lambda_n := h(\varepsilon_n, v_n)$ and $w_n := g(\varepsilon_n, \lambda_n, v_n)$ such that $(\varepsilon_n, \lambda_n, v_n, w_n)$ solves (3.3), (3.4). Without loss of generality we may also assume $v_n \to v_0$. Therefore, $\lambda_n = h(\varepsilon_n, v_n) \to h(0, v_0) = 1$ and, thus, $w_n = g(\varepsilon_n, \lambda_n, v_n) \to g(0, 1, v_0) = 0$.

Summarizing, we get the existence of a sequence $\{(\varepsilon_n, \lambda_n, x_n)\}$ (where $x_n = (v_n, w_n)$) of solutions of (3.1),(3.2) converging to a point $(0, 1, x_0)$, $x_0 = (v_0, 0)$. This proves our assertion. □

We conclude with an example in ℓ^2 which shows that in Theorem 3.1 one cannot drop the assumption that the multiplicity of λ_0 is odd.

Example 3.2. Given $k \in \mathbb{N}$, let $T_k : \ell^2 \to \ell^2$ be the bounded linear operator that associates to any $x = (\xi_1, \xi_2, \dots) \in \ell^2$ the element

$$T_k x = (0, 0, \dots, 0, \xi_{k+1}, \xi_{k+2}, \dots)$$

and define $B : \ell^2 \to \ell^2$ by

$$Bx = (-\xi_2, \xi_1, -\xi_4, \xi_3, \dots, -\xi_{2i}, \xi_{2i+1}, \dots).$$

Clearly T_k is selfadjoint and its kernel is the k-dimensional space

$$\operatorname{Ker} T_k = \{x \in \ell^2 : x = (\xi_1, \xi_2, \dots \xi_k, 0, 0, \dots)\}.$$

Hence, $\lambda_0 = 0$ is an eigenvalue of T_k of multiplicity k. Let us consider the perturbed eigenvalue problem

$$T_k x + \varepsilon B(x) = \lambda x.$$

It is easy to verify that if k is even then, for any ε and for any λ sufficiently small, the above equation has no solutions $x \neq 0$. On the other hand, if k is odd, then according to Theorem 3.1 there exists a sequence $\{(\varepsilon_n, \lambda_n, x_n)\}$ which converges to a point $(0, 0, x_0)$ such that $\varepsilon_n \neq 0$, $\|x_n\| = 1$,

$$T_k x_n + \varepsilon_n B(x_n) = \lambda_n x_n, \quad \forall n \in \mathbb{N}.$$

For example, if $k = 3$, then for any $\varepsilon \neq 0$ we get the eigenvalue

$$\lambda_\varepsilon = \frac{1 - \sqrt{1 - 4\varepsilon^2}}{2}$$

of $T_3 + \varepsilon B$ to which corresponds the eigenspace spanned by the eigenvector

$$v_\varepsilon = (0, 0, 1, \xi_4(\varepsilon), 0, \dots),$$

where

$$\xi_4(\varepsilon) = \frac{\sqrt{1 - 4\varepsilon^2} - 1}{2\varepsilon} = -\varepsilon + \mathrm{o}(\varepsilon).$$

Thus, one gets exactly two unit 0-eigenvectors of T_3 which are bifurcation points for the perturbed eigenvalue problem $T_3 x + \varepsilon B(x) = \lambda x$, namely $(0, 0, \pm 1, 0, \dots)$.

References

[1] Brown R. F., *The Lefschetz Fixed Point Theorem*, Scott and Foresman, 1971.

[2] Chiappinelli R., *Isolated Connected Eigenvalues in Nonlinear Spectral Theory*, Nonlinear Funct. Anal. Appl. **8** (2003), 557–579.

[3] Chiappinelli R. - Furi M. - Pera M.P., *Normalized Eigenvectors of a Perturbed Linear Operator via General Bifurcation*, Glasgow Mathematical Journal **50** (2008), 303–318.

[4] Dunford N. - Schwartz J., *Linear Operators*, part II, Interscience Publishers John Wiley & Sons, New York-London, 1963.

[5] Granas A. - Dugundji J., *Fixed Point Theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[6] Lefschetz S., *Manifolds with a boundary and their transformations*, Trans. AMS **29** (1927), 429–462.

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