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Fredholm linear operators associated with ordinary differential equations on noncompact intervals *

Mariella Cecchi, Massimo Furi,
Mauro Marini, & Maria Patrizia Pera

Abstract

In the noncompact interval $J = [a, \infty)$ we consider a linear problem of the form $Lx = y$, $x \in S$, where L is a first order differential operator, y a locally summable function in J, and S a subspace of the Fréchet space of the locally absolutely continuous functions in J. In the general case, the restriction of L to S is not a Fredholm operator. However, we show that, under suitable assumptions, S and $L(S)$ can be regarded as subspaces of two quite natural spaces in such a way that L becomes a Fredholm operator between them. Then, the solvability of the problem will be reduced to the task of finding linear functionals defined in a convenient subspace of $L^1_{loc}(J, \mathbb{R}^n)$ whose "kernel intersection" coincides with $L(S)$. We will prove that, for a large class of "boundary sets" S, such functionals can be obtained by reducing the analysis to the case when the function y has compact support. Moreover, by adding a suitable stronger topological assumption on S, the functionals can be represented in an integral form. Some examples illustrating our results are given as well.

1 Introduction

In the noncompact interval $J = [a, \infty)$ consider the linear problem

$$\begin{gathered} Lx = y \\ y \in L^1_{\rm loc}(J, \mathbb{R}^n), \quad x \in S\,, \end{gathered} \tag{1.1}$$

where S is a subspace of $AC_{\rm loc}(J, \mathbb{R}^n)$, the Fréchet space of all the locally absolutely continuous functions $x : J \to \mathbb{R}^n$, $L : AC_{\rm loc}(J, \mathbb{R}^n) \to L^1_{\rm loc}(J, \mathbb{R}^n)$ is the first order differential operator defined by $(Lx)(t) = x'(t) + A(t)x(t)$, with $A(\cdot)$ a $n \times n$ matrix with entries in the space $L^1_{\rm loc}(J, \mathbb{R})$ of locally summable functions from J to $\mathbb{R}$. It is known that, in the case when the problem is considered in a

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compact interval I and S is a finite codimensional closed subspace of $AC(I,\mathbb{R}^n)$ (i.e. S is the intersection of the kernels of finitely many linearly independent bounded real functionals on $AC(I,\mathbb{R}^n)$), the Fredholm index of the restriction of L to S is well defined and equals $n-r$, where r is the codimension of S in $AC(I,\mathbb{R}^n)$.

However, in the problems we are interested here, the codimension of $L(S)$ in $L^1_{\mathrm{loc}}(J,\mathbb{R}^n)$ may not be finite (see e.g. Example 3.1). Thus, in general, the restriction of L to S is not a Fredholm operator. Since the theory of Fredholm operators is a useful tool in the solvability of linear boundary value problems (see, e.g., [7]), it turns out to be of some interest to investigate whether or not S and $L(S)$ can be regarded as subspaces of some "natural" and suitable smaller spaces in such a way that L becomes a Fredholm operator between these two spaces. To this end, let $AC_0(J,\mathbb{R}^n)$ denote the subspace of $AC(J,\mathbb{R}^n)$ of those functions having compact support in J and set $S_\infty = S + AC_0(J,\mathbb{R}^n)$ (here the index ∞ is suggested by the fact that, in order to decide whether or not a function belongs to the space $S + AC_0(J,\mathbb{R}^n)$, it is enough to consider its behavior in a neighborhood of ∞). Hence, S can be regarded as a subspace of S_∞ and, thus, $L(S)$ of $L(S_\infty)$. Consequently, one can restate problem (1.1) in S_∞ as follows:

$$\begin{gathered} Lx = y \\ y \in L(S_\infty), \quad x \in S \subset S_\infty\,. \end{gathered} \tag{1.2}$$

By assuming that the codimension of S in S_∞ is finite, it is easy to show (see Section 3) that $L : S \to L(S_\infty)$ is a Fredholm operator of index $m-r$, where $m = \dim \mathrm{Ker} L \cap S_\infty$ and $r = \dim S_\infty/S$. Thus S_∞ and $L(S_\infty)$ play the same role as, in a compact interval, AC and L^1 respectively, and m is the analogue of n. For this reason, it turns out convenient to restrict our attention to those $y \in L^1_{\mathrm{loc}}(J,\mathbb{R}^n)$ which actually belong to $L(S_\infty)$. In this way, after some suitable sufficient conditions for $y \in L^1_{\mathrm{loc}}(J,\mathbb{R}^n)$ to be in $L(S_\infty)$ are given, when $L(S) \neq L(S_\infty)$, the solvability of (1.1) will be reduced to the problem of finding functionals which individuate $L(S)$ in $L(S_\infty)$, that is, real linear functionals defined in a convenient subspace of $L^1_{\mathrm{loc}}(J,\mathbb{R}^n)$ whose "kernel intersection" coincides with $L(S)$. We will show that, for a large class of "boundary sets" S, such functionals can be obtained by reducing the analysis to the case when the function y in problem (1.2) has compact support.

The plan of the paper is as follows. In Section 2 we investigate the solvability of the equation $Lx = y$, with x in an asymptotic space; that is, a subspace E of $AC_{\mathrm{loc}}(J,\mathbb{R}^n)$ containing $AC_0(J,\mathbb{R}^n)$. In Section 3, the solvability of (1.2) is preliminary reduced to that of the same problem with y in the subspace $L^1_0(J,\mathbb{R}^n)$ of $L^1_{\mathrm{loc}}(J,\mathbb{R}^n)$ of the functions with essential compact support. In Section 4 we show that, by adding a suitable topological assumption on the "boundary space" S, the functionals individuating $L(S) \cap L^1_0(J,\mathbb{R}^n)$ in $L^1_0(J,\mathbb{R}^n)$ can be represented in an integral form. In our main result (Theorem (4.3) below) we prove, with a stronger topological assumption on S, that the same integral expressions individuate $L(S)$ in a convenient subspace of $L(S_\infty)$. Some examples illustrating our results are given.

There exists a vast literature concerning the solvability of linear boundary value problems on unbounded intervals. For quite recent results on this topic we suggest, e.g., [3], [4] and references therein. Let us point out that the solvability of the linear problem (1.1) arises for instance in the investigation of nonlinear boundary value problems on noncompact intervals and it is, in some sense, a useful preliminary when such a study is carried out by means of topological methods. This nonlinear point of view will be considered in a forthcoming paper. For a different approach to nonlinear problems see [1], [5], [6] and references therein.

We close the Introduction with some notation.

We denote by $AC_{\rm loc} := AC_{\rm loc}(J,\mathbb{R}^n)$, $AC_0 := AC_0(J,\mathbb{R}^n)$, $L^1_{\rm loc} := L^1_{\rm loc}(J,\mathbb{R}^n)$, $L^1_0 := L^1_0(J,\mathbb{R}^n)$, $L^\infty_{\rm loc} := L^\infty_{\rm loc}(J,\mathbb{R}^n)$.

We will denote by $\langle\cdot,\cdot\rangle$ the standard inner product in $\mathbb{R}^n$ and by $|\cdot|$ the corresponding Euclidean norm. Moreover, $|A|$ stands for the operator norm of an $n\times n$ matrix A.

Using a well-known standard notation, given $f : J \to \mathbb{R}^n$ and $g : J \to \mathbb{R}_+$, we will write $f(t) = O(g(t))$ (as $t\to\infty$) if there exist $M > 0$ and $b > a$ such that $|f(t)| \le Mg(t)$ for all $t > b$.

Given two subspaces A and B of a vector space V, by abuse of notation, we write $\dim B/A$ instead of $\dim B/(A\cap B)$. Moreover, by codimension of A with respect to B we mean the codimension of $A\cap B$ in B.

To avoid cumbersome notation, the restriction of a linear operator $\Lambda : A \to B$ to a subspace C of A (as domain) and to a subspace D of B (as codomain) will be often denoted by $\Lambda : C \to D$.

2 Asymptotic spaces

A subspace of $AC_{\rm loc}$ containing AC_0 (respectively, of $L^1_{\rm loc}$ containing L^1_0) will be called *asymptotic*. As the examples below will illustrate, the term "asymptotic" is justified by the fact that, in order to decide whether or not a function $x \in AC_{\rm loc}$ belongs to such a subspace, it is enough to consider the behavior of x for t sufficiently large. Moreover, a subset E of $AC_{\rm loc}$ will be said *strongly asymptotic* if it contains a space of the type $AC_\varphi = \{x \in AC_{\rm loc} : x = O(\varphi)\}$ for some continuous function $\varphi : J \to (0,\infty)$. An analogous definition can be given in $L^1_{\rm loc}$ as well. Since $AC_0 \subset AC_\varphi$ for any continuous $\varphi : J \to (0,\infty)$, then any strongly asymptotic space is obviously asymptotic. Clearly, the converse is not true since, for instance, AC_0 itself is asymptotic but not strongly asymptotic.

These are some examples of asymptotic spaces:

- $\{x \in AC_{\rm loc} : x(\infty) := lim_{t\to\infty} x(t) = 0\}$;
- $AC_{\rm loc} \cap L^1(J,\mathbb{R}^n)$;
- $\{x \in AC_{\rm loc} : x \text{ is bounded }\}$;
- $\{x \in AC_{\rm loc} : x'(\infty) = 0\}$.

We observe that the first three spaces are strongly asymptotic as well.

The following space is clearly not asymptotic:

- $\{x \in AC_{\rm loc} : x(0) = x(\infty) = 0\}$.

However, this space has codimension one in the (strongly) asymptotic space $\{x \in AC_{\rm loc} : x(\infty) = 0\}$.

Others interesting examples are given by the so-called Corduneanu spaces (see [2]). Namely, given a nonnegative continuous function $g : J \to \mathbb{R}$, the space

$$\{x \in AC_{\rm loc} : \exists M > 0 \text{ such that } |x(t)| \le Mg(t), t \in J\}$$

is either asymptotic or not, provided g is strictly positive or vanishes somewhere in J.

In this section we will study the solvability of the problem

$$\begin{gathered} Lx = y \\ y \in L^1_{\rm loc}\,, \quad x \in E\,, \end{gathered} \tag{2.1}$$

where $J = [a, \infty)$, $L : AC_{\rm loc} \to L^1_{\rm loc}$ is the first order differential operator defined by

$$(Lx)(t) = x'(t) + A(t)x(t),$$

with $A(\cdot)$ a $n \times n$ matrix with entries in $L^1_{\rm loc}(J, \mathbb{R})$, and E is an asymptotic subspace of $AC_{\rm loc}$. Given $y \in L^1_{\rm loc}$, by a solution of (2.1) we mean a locally absolutely continuous function $x : J \to \mathbb{R}^n$ satisfying $Lx(t) = y(t)$ a.e. in J and belonging to E.

Clearly, the operator L maps asymptotic subspaces of $AC_{\rm loc}$ in asymptotic subspaces of $L^1_{\rm loc}$. The following result shows that a similar assertion holds true for strongly asymptotic spaces.

Theorem 2.1 *Let E be a subspace of $AC_{\rm loc}$ and assume that there exists a continuous $\varphi : J \to (0, \infty)$ such that $x = O(\varphi)$ implies $x \in E$. Then there exist a continuous function $\psi : J \to (0, \infty)$ and a linear right inverse K of $L|_E$ defined on the space*

$$L^1_\psi = \{y \in L^1_{\rm loc} : y = O(\psi)\}$$

such that $K(y) = O(\varphi)$, for all $y \in L^1_\psi$. In addition, K is continuous in the subset of L^1_ψ given by

$$D_{M\psi} = \{y \in L^1_{\rm loc} : |y(t)| \le M\psi(t), \text{ for a.a. } t \in J\}, \quad M > 0.$$

Proof. Let $X(t)$ be a fundamental matrix of $Lx = 0$. For any $y \in L^1_{\rm loc}$ such that $|X^{-1}(\cdot)||y| \in L^1(J, \mathbb{R})$, the function

$$K(y)(t) = -X(t) \int_t^\infty X^{-1}(s)y(s)ds, \quad t \ge a$$

belongs to $AC_{\rm loc}$ and solves the differential equation $Lx = y$. Thus, it is enough to find $\psi > 0$ such that, when $y \in L^1_\psi$ one gets $|X^{-1}(\cdot)||y| \in L^1(J,\mathbb{R})$ and $K(y) = O(\varphi)$. Indeed, the continuity of K in $D_{M\psi}$ is a consequence of the Lebesgue Convergence Theorem.

Let us show that, given a continuous $\varphi > 0$, there exists a continuous $\psi > 0$ such that $|X^{-1}(\cdot)|\psi \in L^1(J,\mathbb{R})$ and

$$|X(t)| \int_t^\infty |X^{-1}(s)|\psi(s)ds \le \varphi(t), \quad t \ge a.$$

In fact, take any C^1 function σ satisfying $0 < \sigma(t) \le \varphi(t)/|X(t)|$ for $t \ge a$, $\sigma(\infty) = 0$, $\sigma'(t) < 0$ for $t \ge a$, $\sigma' \in L^1(J,\mathbb{R})$, and define $\psi(t) = -\sigma'(t)/|X^{-1}(t)|$.

Consequently,

$$|K(y)(t)| \le |X(t)| \int_t^\infty |X^{-1}(s)||y(s)|ds \le \varphi(t)$$

as claimed. ◇

We point out that the proof of the above theorem gives an explicit formula to get a function ψ which ensures the solvability of (2.1) whenever $y = O(\psi)$. This formula is deduced from the particular form of the operator K. This, in some sense, is an universal inverse (i.e. its expression is independent of the space E) and is a good inverse in the case when the homogeneous problem $Lx = 0$, $x \in E$, admits only the trivial solution. However, when this condition is not satisfied, an accurate choice of an inverse allows us to deduce a more convenient formula for the function ψ (clearly, the bigger is ψ, the better).

To illustrate this, consider the following example.

Example 2.2 Consider in $J = [0,\infty)$ the system

$$\begin{aligned} x' + x &= y \\ x &\in E\,, \end{aligned}$$

where $E = \{x \in AC_{\rm loc}(J,\mathbb{R}) : x = O(1)\}$ is the space of bounded (locally absolutely continuous) real functions on J. The problem is solvable for any $y \in L^1_{\rm loc}(J,\mathbb{R})$ such that $y = O(1)$. In fact, the inverse

$$K(y)(t) = \int_0^t e^{s-t}y(s)ds$$

of $Lx = x'+x$ is such that when $y = O(1)$ one has $|K(y)(t)| \le M\int_0^t e^{s-t}ds \le M$, for some $M > 0$. Thus, $K(y) \in E$. Observe that if we had used the inverse of the proof of Theorem 2.1 we would have obtained the following sufficient condition for the solvability of the system: $e^t y(t) \in L^1(J,\mathbb{R})$, which is more restrictive than $y = O(1)$.

3 Algebraic results

Let S be a subspace of $AC_{\rm loc}$ and (as in the Introduction) denote by S_∞ the subspace of $AC_{\rm loc}$ given by $S_\infty = S + AC_0$. According to the definition of Section 2, S_∞ is an asymptotic space which clearly coincides with S if and only if $AC_0 \subset S$.

In this section we are concerned with the solvability of the following linear problem:

$$\begin{gathered} Lx = y \\ y \in L(S_\infty), \quad x \in S\,. \end{gathered} \tag{3.1}$$

Assume that

$\mathbf{H_1}$) $\dim S_\infty/S$ is finite.

This assumption plays the same role as that of assigning, in the compact intervals, a finite number of independent linear conditions. For instance H_1) is not satisfied for

$$S = \big\{x \in AC_{\rm loc}(J,\mathbb{R}) : x(n) = 0 \text{ for all } n \in \mathbf{N}\big\}.$$

In this case $S_\infty = AC_{\rm loc}(J,\mathbb{R})$, so that $\dim S_\infty/S = \infty$.

As the examples below show, H_1) turns out to be satisfied in many problems where, however, the codimension of S in $AC_{\rm loc}$ is not finite.

a) Let $S = \big\{x \in AC_{\rm loc}(J,\mathbb{R}) : x(\infty) = 0\big\}$. Then, $S_\infty = S$, $\dim AC_{\rm loc}/S = \infty$, $\dim S_\infty/S = 0$;

b) Let $S = \big\{x \in AC_{\rm loc}(J,\mathbb{R}) : x(1) = x(\infty) = 0\big\}$. Then, $S_\infty = \big\{x \in AC_{\rm loc}(J,\mathbb{R}) : x(\infty) = 0\big\}$, $\dim AC_{\rm loc}/S = \infty$, $\dim S_\infty/S = 1$;

c) Let $S = \big\{x \in AC_{\rm loc}(J,\mathbb{R}) \cap L^1(J,\mathbb{R}) : \int_0^\infty x(t)dt = 0\big\}$. Then, $S_\infty = AC_{\rm loc}(J,\mathbb{R}) \cap L^1(J,\mathbb{R})$, $\dim AC_{\rm loc}/S = \infty$, $\dim S_\infty/S = 1$.

The following are the corresponding problems in a compact interval $I = [a,b]$.

a') Let $S = \big\{x \in AC(I,\mathbb{R}) : x(b) = 0\big\}$. Then, $\dim AC/S = 1$;

b') Let $S = \big\{x \in AC(I,\mathbb{R}) : x(a) = x(b) = 0\big\}$. Then, $\dim AC/S = 2$;

c') Let $S = \big\{x \in AC(I,\mathbb{R}) : \int_a^b x(t)dt = 0\big\}$. Then, $\dim AC/S = 1$.

Let us point out that, as the example below illustrates, not only the codimension of S in $AC_{\rm loc}$ but also that of $L(S)$ in $L^1_{\rm loc}$ may not be finite. This means that, even assuming H_1), the theory of Fredholm operators does not apply directly to some of the problems we are interested in. Recall that a linear operator between vector spaces is said to be a Fredholm operator if it has finite dimensional kernel and finite codimensional image. The index of a Fredholm operator is the difference between the dimension of its kernel and the codimension of its image.

Example 3.1 Consider in $J = [1, \infty)$ the system

$$\begin{gathered} Lx = y \\ y \in L^1_{\rm loc}(J, \mathbb{R}) \quad x \in S\,, \end{gathered}$$

where

$$S = \big\{x \in AC_{\rm loc}(J, \mathbb{R}) : x \text{ is bounded}\big\}.$$

Clearly, $S = S_\infty$ and the functions $y_n(t) = t^{-1/n}, t \geq 1, n \in \mathbf{N}$ are in $L^1_{\rm loc}(J, \mathbb{R})$ but, as one can check, do not belong to $L(S)$. Thus, the codimension of $L(S)$ in $L^1_{\rm loc}(J, \mathbb{R})$ is not finite.

To avoid the difficulty presented before, it turns out to be useful to restrict our operator L between the spaces S and $L(S_\infty)$. Actually, the surjective operator $L|_{S_\infty} : S_\infty \to L(S_\infty)$ is Fredholm of index $m = \dim \mathrm{Ker} L \cap S_\infty$. Thus, taking into account H_1), one can immediately show that $L|_S : S \to L(S_\infty)$ is Fredholm of index $m - r$, where we denote $r = \dim S_\infty / S$. In fact, since the inclusion $i : S \to S_\infty$ is Fredholm of index $-r$, the composition $L|_S = L|_{S_\infty} \circ i$ has index $\mathrm{ind} L|_S = \mathrm{ind} L|_{S_\infty} + \mathrm{ind}\, i = m - r$. In particular, the codimension of $L(S)$ in $L(S_\infty)$ is finite.

Summarizing, we have proved the following.

Proposition 3.2 *Let H_1) be satisfied. Denote $m = \dim \mathrm{Ker} L \cap S_\infty$, $p = \dim \mathrm{Ker} L \cap S$, $q = \dim L(S_\infty)/L(S)$ and $r = \dim S_\infty / S$. Then, $L : S \to L(S_\infty)$ is a Fredholm operator whose index* $\mathrm{ind} L := p - q$ *satisfies the equality*

$$p - q = m - r.$$

Consequently, $q = p - m + r$.

The following well-known elementary lemma is stated here for completeness, since will be used several times in the sequel.

Lemma 3.3 *(*Quotient's Lemma*). Let A, B and C be vector spaces and $\pi : A \to B$, $\gamma : A \to C$ be linear operators. Assume that π is surjective and* $\mathrm{Ker}\, \pi \subset \mathrm{Ker}\, \gamma$. *Then there exists a unique linear operator $\sigma : B \to C$ such that $\gamma = \sigma \circ \pi$. Moreover, σ is injective if and only if* $\mathrm{Ker}\, \pi = \mathrm{Ker}\, \gamma$, *and σ is surjective if and only if so is γ.*

Proposition 3.2 above shows that the computation of q can be reduced to the easier task of calculating the integers p, m, r.

We will prove below that, in order to determine such integers, we can reduce our attention to consider L as an operator acting between the spaces $AC_0 \oplus \mathrm{Ker} L$ and L^1_0, which are independent of S and easier to handle than S_∞ and $L(S_\infty)$ respectively, since AC_0 and L^1_0 are spaces of functions having compact support. For instance, as ii) of Lemma 3.4 will show, the integer r can be computed by determining the codimension of S in AC_0, and so on. We point out that the spaces $AC_0 \oplus \mathrm{Ker}L$ and L^1_0 are related each other since, as easily seen, $AC_0 \oplus \mathrm{Ker}L = L^{-1}(L^1_0)$.

In Lemma 3.4 and Theorem 3.5 below an algebraic analysis of problem (1.2) is carried out. As a consequence of this analysis, we prove that $L : (AC_0 \oplus \mathrm{Ker}L) \cap S \to L^1_0$ is still Fredholm and has the same index as $L : S \to L(S_\infty)$. Moreover, since these two operators have the same kernel, the codimensions of $L(S)$ in $L(S_\infty)$ and $L(S) \cap L^1_0(J, \mathbf{R}^n)$ in $L^1_0(J, \mathbb{R}^n)$ turn out to be the same.

Lemma 3.4 *Assume H_1) is satisfied. Then, the following equalities hold:*

i) $(AC_0 \oplus \mathrm{Ker}L) \cap S_\infty = (\mathrm{Ker}L \cap S_\infty) \oplus AC_0$;

ii) $\dim(AC_0/S) = \dim((AC_0 \oplus \mathrm{Ker}L) \cap S_\infty)/S) = \dim S_\infty/S$.

Proof. i) Let us show first that $(AC_0 \oplus \mathrm{Ker}L) \cap S_\infty \subset (\mathrm{Ker}L \cap S_\infty) \oplus AC_0$. Take $x \in (AC_0 \oplus \mathrm{Ker}L) \cap S_\infty$. Then, there exist $x_0 \in AC_0$ and $x_1 \in \mathrm{Ker}L$ such that $x = x_0 + x_1$. Since $S_\infty \supset AC_0$, then $x_1 = x - x_0$ belongs to S_∞. Thus, $x \in (\mathrm{Ker}L \cap S_\infty) \oplus AC_0$. The converse inclusion is obvious.

ii) Consider the commutative diagram

$$\begin{array}{ccc} AC_0 & \xrightarrow{i} & S_\infty \\ \downarrow & & \downarrow \\ AC_0/S & \xrightarrow{\sigma} & S_\infty/S \end{array}$$

where the vertical arrows are the canonical projections, i is the inclusion, and σ is well-defined and injective because of Quotient's Lemma. Since $S_\infty = S + AC_0$, one gets immediately that σ is onto; consequently, AC_0/S and S_∞/S have the same dimension. Consider now the diagram

$$\begin{array}{ccccc} AC_0 & \xrightarrow{i_1} & (AC_0 \oplus \mathrm{Ker}L) \cap S_\infty & \xrightarrow{i_2} & S_\infty \\ \downarrow & & \downarrow & & \downarrow \\ AC_0/S & \xrightarrow{\sigma_1} & ((AC_0 \oplus \mathrm{Ker}L) \cap S_\infty)/S & \xrightarrow{\sigma_2} & S_\infty/S \end{array}$$

where i_1 and i_2 are inclusions, the vertical arrows are canonical projections and, as above, σ_1 and σ_2 are well-defined and injective. The assertion follows immediately from the fact that $\dim AC_0/S = \dim S_\infty/S$. ♢

Theorem 3.5 *Assume H_1) is satisfied. Then, the operator $L : (AC_0 \oplus \mathrm{Ker}L) \cap S \to L^1_0$ is Fredholm and has the same index as $L : S \to L(S_\infty)$. In addition, $\dim L^1_0/L(S) = \dim L(S_\infty)/L(S)$.*

Proof. The operator $L : (AC_0 \oplus \mathrm{Ker}L) \cap S_\infty \to L^1_0$ is clearly onto and, by i) of Lemma 3.4, its kernel coincides with $\mathrm{Ker}L \cap S_\infty$. Thus, it is Fredholm of index $m = \dim \mathrm{Ker}L \cap S_\infty$. Now, as previously, the index of the restriction $L : (AC_0 \oplus \mathrm{Ker}L) \cap S \to L^1_0$ can be computed by means of the composition

$$(AC_0 \oplus \mathrm{Ker}L) \cap S \xrightarrow{i} (AC_0 \oplus \mathrm{Ker}L) \cap S_\infty \xrightarrow{L} L^1_0$$

where, by H_1) and *ii*) of Lemma 3.4, the inclusion i has index $-r$ (recall that $r = \dim S_\infty/S$). Consequently, the operator L has index $m-r$ either considered between S and $L(S_\infty)$ (as previously shown) or between $(AC_0 \oplus \mathrm{Ker}L) \cap S$ and L^1_0 (recall that $L^1_0 \subset L(S_\infty)$). This proves the first part of the assertion. Now, by Proposition 3.2 we have $m - r = p - q$, where $p = \dim \mathrm{Ker}L \cap S$ and $q = \dim L(S_\infty)/L(S)$. Hence, we also obtain $\mathrm{ind}L|_{(AC_0 \oplus \mathrm{Ker}L)\cap S} = p - q$. On the other hand, by the definition of Fredholm index,

$$\mathrm{ind}L|_{(AC_0\oplus\mathrm{Ker}L)\cap S} = \dim \mathrm{Ker}L|_{(AC_0\oplus\mathrm{Ker}L)\cap S} - \dim L^1_0/L((AC_0 \oplus \mathrm{Ker}L) \cap S).$$

Observe now that $\mathrm{Ker}L|_{(AC_0\oplus\mathrm{Ker}L)\cap S}$ still coincides with $\mathrm{Ker}L\cap S$ and, thus, has dimension p. Moreover, one has $L((AC_0 \oplus \mathrm{Ker}L) \cap S) = L(S) \cap L^1_0$. Therefore, $\dim L^1_0/L(S) = q = \dim L(S_\infty)/L(S)$, which achieves the proof. ♢

4 The main result

In Theorem 3.5 we have shown that $q := \dim(L(S_\infty)/L(S))$ equals $\dim(L^1_0/L(S))$ and, by Proposition 3.2 we have obtained that q can be computed by the formula $q = p-m+r$, with $p = \dim(\mathrm{Ker}L\cap S)$, $m = \dim(\mathrm{Ker}L\cap S_\infty)$, $r = \dim(S_\infty/S)$. Thus, if $q = 0$ the solvability of (1.1) is equivalent to the solvability of the associated problem

$$\begin{gathered} Lx = y \\ y \in L^1_{\mathrm{loc}}\,, \quad x \in S_\infty\,. \end{gathered} \tag{4.1}$$

That is, given $y \in L^1_{\mathrm{loc}}$, (1.1) is solvable if and only if the same is true for (4.1). Assume now we know how to solve this associated problem, for example by means of Theorem 2.1, and consider the case when $q > 0$. We want to find conditions on $y \in L(S_\infty)$ ensuring the solvability of the original problem (1.1). We will show that, when S satisfies some reasonable conditions, this can be done by imposing y to belong to the kernel of q linearly independent functionals expressed in a convenient integral form.

Theorem 4.1 below shows that such "integral" functionals exist in the special case when y has compact support. Moreover, with an additional topological assumption on S, the same functionals can be adapted to ensure the solvability of problem (1.1) (see Theorem 4.3).

To this end, given any $b > a$, consider the following closed subspace of AC_0:

$$AC_{[a,b]} = \big\{x \in AC_0 : x(t) = 0 \text{ for } t \geq b\big\}.$$

We will say that S is *locally closed* in S_∞ if $S \cap AC_{[a,b]}$ is closed in S_∞ for any $b > a$.

Theorem 4.1 *Let S be a subspace of AC_{loc} such that*

H$_1$) $\dim S_\infty/S$ *is finite;*

H$_2$) *S is locally closed in S_∞.*

Suppose the codimension q of $L(S)$ in $L(S_\infty)$ is nonzero. Then there exist $\alpha_1,\dots,\alpha_q \in L^\infty_{\rm loc}$ such that, given $y \in L^1_0$, $y \in L(S)$ if and only if

$$\int_a^\infty \langle \alpha_i(t), y(t)\rangle dt = 0, \quad \forall\, i = 1,\dots,q.$$

Proof. By Theorem 3.5, $q = \dim L^1_0/L(S)$. Hence, there exist q linearly independent functionals on L^1_0, say $\lambda_1,\dots,\lambda_q$, such that $y \in L(S)\cap L^1_0$ if and only if $\lambda_i(y) = 0$ for all $i = 1,..,q$. Let us show that any λ_i is locally continuous, i.e. $\forall b > a$ any λ_i is continuous in the Banach subspace

$$L^1_{[a,b]} = \big\{y \in L^1_0 : y(t) = 0 \text{ for a.a. } t \ge b\big\}$$

of L^1_0. Since $L(S)\cap L^1_{[a,b]}$ has finite codimension in $L^1_{[a,b]}$, it is enough to prove that it is closed in $L^1_{[a,b]}$. Observe first that the subspace $AC_{[a,b]}\oplus{\rm Ker}L$ of $AC_{\rm loc}$ is a Banach space, being the sum of a Banach space and a finite dimensional space, and that the restriction $L : AC_{[a,b]} \oplus {\rm Ker}L \to L^1_{[a,b]}$ is continuous, onto and has finite dimensional kernel. Moreover, since $AC_{[a,b]}$ is contained in S_∞, by assumption H_2) it follows that $S \cap AC_{[a,b]}$ is closed in $AC_{[a,b]}$. Hence, it is not difficult to prove that, Ker$L$ being finite dimensional, the subspace $S\cap (AC_{[a,b]} \oplus {\rm Ker}L)$ is closed in $AC_{[a,b]} \oplus {\rm Ker}L$. Thus, recalling that a continuous Fredholm operator between Banach spaces sends closed subspaces into closed subspaces, it follows immediately that $L(S\cap(AC_{[a,b]} \oplus {\rm Ker}L)) = L(S)\cap L^1_{[a,b]}$ is closed in $L^1_{[a,b]}$, as claimed.

Let $b_j \to \infty$, $b_j > a$, be an increasing sequence of numbers and set $I_j = [a, b_j]$. Since $L^\infty_{[a,b_j]}$ is the dual of $L^1_{[a,b_j]}$, then, for any $j \in \mathbf{N}$ and $i = 1,\dots,q$, there exists $\alpha^i_j \in L^\infty_{[a,b_j]}$ such that $\lambda_i(y) = \int_{I_j}\langle \alpha^i_j(t), y(t)\rangle dt$ for any $y \in L^1_{[a,b_j]}$. Clearly, if $j < k$, then $\alpha^i_j(t) = \alpha^i_k(t)$ a.e. in I_j. Hence, for any $i = 1,\dots,q$, one can define $\alpha_i \in L^\infty_{\rm loc}$ by setting $\alpha_i(t) = \alpha^i_j(t)$ for $t \in I_j$. Now, if $y \in L^1_0$, there exists $j \in \mathbf{N}$ such that $y(t) = 0$ for a.a. $t \ge b_j$. Thus,

$$\lambda_i(y) = \int_{I_j}\langle \alpha^i_j(t), y(t)\rangle dt = \int_a^\infty \langle \alpha_i(t), y(t)\rangle dt,$$

so that $y \in L(S)\cap L^1_0$ if and only if $\int_a^\infty\langle \alpha_i(t), y(t)\rangle dt = 0$ for any $i = 1,\dots,q$. $\diamondsuit$

We give now an example illustrating how one can find the functions α_i of Theorem 4.1.

Example 4.2 Consider in $J = [0,\infty)$ the system

$$x' + x = y$$
$$x = O(1) \quad x(0) = x(1)e$$

Here,

$$S = \big\{x \in AC_{\rm loc}(J,\mathbb{R}) : x = O(1) \text{ and } x(0) - x(1)e = 0\big\}$$

and

$$S_\infty = \big\{x \in AC_{\mathrm{loc}}(J,\mathbb{R}) : x = O(1)\big\}.$$

Moreover, it is easy to verify that $r = 1$, $m = 1$ and $p = 1$, so that $q = p - m + r = 1$. In order to find the function α of Theorem 4.1, consider in L^1_0 the inverse

$$x(t) = K_0(y)(t) = -e^{-t}\int_t^\infty e^s y(s)ds.$$

One has $x(0) = -\int_0^\infty e^s y(s)ds$ and $x(1)e = -\int_1^\infty e^s y(s)ds$. By imposing the boundary condition $x(0) = x(1)e$, one obtains $\int_0^1 e^s y(s)ds = 0$. Thus,

$$\alpha(t) = \begin{cases} e^t & \text{if} \quad t \in [0,1] \\ 0 & \text{if} \quad t > 1\,. \end{cases}$$

Notice that, given two subspaces F_1 and F_2 of a vector space F, if F_1 and F_2 have the same finite codimension and one is contained into the other, then they necessarily coincide. Referring to Theorem 4.1, let $F_1 = L(S) \cap L^1_0$ and $F_2 = \big\{y \in L^1_0 : \int_a^\infty \langle \alpha_i(t), y(t)\rangle dt = 0 \text{ for all } i = 1,\dots,q\big\}$. Assume that the functions α_i are linearly independent. Then, F_2 has codimension q in L^1_0. Consequently, if in Theorem 4.1 one of the two conditions "if" or "only if" is satisfied, then the other one is satisfied as well.

Now, we are ready to state our main result which depends on the preliminary investigation in the compact support context contained in Theorem 4.1 above.

Given a positive continuous function $\varphi : J \to \mathbb{R}$, let us denote by A_φ the closed subset of AC_{loc} given by

$$A_\varphi = \big\{x \in AC_{\mathrm{loc}} : |x(t)| \le \varphi(t),\ t \in J\big\}.$$

We have the following

Theorem 4.3 *Let S be a subspace of AC_{loc} such that*

H$_1$) $\dim S_\infty/S$ *is finite;*

H$_2'$) *there exists a continuous real function $\varphi : J \to (0,\infty)$ such that A_φ is contained in S_∞ and $S \cap A_\varphi$ is a closed subset of AC_{loc}.*

Suppose the codimension q of $L(S)$ in $L(S_\infty)$ is positive and let $\alpha_1,\dots,\alpha_q \in L^\infty_{\mathrm{loc}}$ be as in Theorem 4.1. Let $\psi : J \to (0,\infty)$ be continuous and satisfying the following properties:

a) $|\alpha_i|\psi \in L^1(J,\mathbb{R})$*, for all* $i = 1,\dots,q$*;*

b) there exists a linear right inverse K of $L|_{S_\infty}$, defined on the space

$$L^1_\psi = \big\{y \in L^1_{\mathrm{loc}} : y = O(\psi)\big\}$$

and continuous in the subset

$$D_\psi = \big\{y \in L^1_{\mathrm{loc}} : |y(t)| \le \psi(t), \text{ for a.a. } t \in J\big\}$$

of L^1_ψ, such that $K(y) = O(\varphi)$, for all $y \in L^1_\psi$.

Then, given $y = O(\psi)$, the problem

$$\begin{gathered} Lx = y \\ y \in L^1_{\rm loc}\,, \quad x \in S \end{gathered}$$

has a solution if and only if

$$\int_a^\infty \langle \alpha_i(t), y(t) \rangle dt = 0, \quad \textit{for any } i = 1, \dots, q.$$

Before giving the proof of Theorem 4.3, let us make some comments about its statement.

Remark 4.4 Assumption $H_2')$ above is stronger than $H_2)$ of Theorem 4.1. To see this, observe first that the condition that $S \cap A_\varphi$ is closed in $AC_{\rm loc}$ for some $\varphi > 0$ clearly implies that $S \cap A_{M\varphi}$ is closed in $AC_{\rm loc}$ for all $M > 0$. To get $H_2)$ we need to show that if $\{x_n\}$ is a sequence in $S \cap AC_{[a,b]}$ converging to $x \in S_\infty$, then $x \in S \cap AC_{[a,b]}$. Obviously, $x \in AC_{[a,b]}$. Let us show that x belongs to S as well. First, notice that there exists $c > 0$ such that $|x_n(t)| \le c$ for all $n \in \mathbf{N}$ and $t \in [a, b]$. Let $M > 0$ be such that $c \le M\varphi(t)$ for all $t \in [a, b]$. This means that $\{x_n\}$ is a sequence in $S \cap A_{M\varphi}$ which, as observed above, is closed in $AC_{\rm loc}$. Thus, x belongs to S, so that S is locally closed in S_∞, i.e. $H_2)$ is satisfied.

Finally, let us point out that condition $H_2')$ is clearly not equivalent to $H_2)$ as one can immediately see by taking $S = AC_0$.

Remark 4.5 The existence of a function ψ satisfying $b)$ of Theorem 4.3 is ensured by our Theorem 2.1 above. Moreover, by reducing ψ, if necessary, we can also assume that $a)$ is satisfied.

Proof of Theorem 4.3. Define $Q_0 : L^1_0 \to \mathbf{R}^q$ by

$$Q_0(y) = \Big(\int_a^\infty \langle \alpha_1(t), y(t) \rangle dt, \dots, \int_a^\infty \langle \alpha_q(t), y(t) \rangle dt \Big).$$

By Theorem 4.1, given $y \in L^1_0$, one has $Q_0(y) = 0$ if and only if $y \in L(S)$. Consequently Q_0 is onto, its image being isomorphic to the q-dimensional quotient $L^1_0/L(S)$. Let

$$L^1_\psi = \big\{ y \in L^1_{\rm loc} : y = O(\psi) \big\}$$

and $Q_\psi : L^1_\psi \to \mathbb{R}^q$ be the linear extension of Q_0 given by

$$Q_\psi(y) = \Big(\int_a^\infty \langle \alpha_1(t), y(t) \rangle dt, \dots, \int_a^\infty \langle \alpha_q(t), y(t) \rangle dt \Big).$$

We need to show that $Q_\psi(y) = 0$ if and only if $y \in L(S) \cap L^1_\psi$. This will be obtained by proving the existence of a convenient linear operator onto $\mathbb{R}^q$ coinciding with our integral operator Q_ψ in L^1_ψ and whose kernel is $L(S)$. In order to construct such an operator, as previously let r denote the codimension of S in S_∞. Hence, there exists a surjective linear operator $R : S_\infty \to \mathbb{R}^r$ such that $x \in S$ if and only if $R(x) = 0$. Observe that the restriction $R|_{AC_0}$ of R to AC_0 is still surjective. Indeed, since $S \cap AC_0 = \mathrm{Ker} R|_{AC_0}$, the image of $R|_{AC_0}$ is isomorphic to the quotient AC_0/S, which, by Lemma 3.4, has dimension r.

Let $K_0 : L^1_0 \to AC_0$ be the restriction of K to the pair of spaces L^1_0 and AC_0. Since $S_\infty \supset AC_0$, the composition $T : L^1_0 \to \mathbb{R}^r$ given by $T = R \circ K_0$ is well-defined. Observe that T is onto, since K_0 is an isomorphism and $R|_{AC_0}$ is onto. Moreover, $\mathrm{Ker}T$ is contained in $\mathrm{Ker}Q_0 = L(S) \cap L^1_0$ (indeed if $T(y) = 0$, then $K_0(y) \in S \cap AC_0$ so that $y = (L \circ K_0)(y) \in L(S) \cap L^1_0$). Consequently, because of the Quotient Lemma, there exists a unique $\rho : \mathbb{R}^r \to \mathbb{R}^q$ such that $Q_0 = \rho \circ T$. Since $K(L^1_\psi)$ is contained in S_∞, the composition $Q := \rho \circ R \circ K$ is an extension of Q_0 to L^1_ψ.

Let us show that, similarly to the case of Q_0, one has $\mathrm{Ker}Q = L(S) \cap L^1_\psi$. Observe first that, since Q is onto, then $\mathrm{Ker}Q$ has codimension q in L^1_ψ. The same is true for $L(S)$ since it is easy to see that $L^1_\psi/L(S)$ is isomorphic to $L^1_0/L(S)$ (or, equivalently, to $L(S_\infty)/L(S)$). Therefore, it is enough to prove that, for instance, $\mathrm{Ker}Q$ is contained in $L(S) \cap L^1_\psi$. To this end, take $y \in \mathrm{Ker}Q$, so that $(R \circ K)(y) \in \mathrm{Ker}\rho$. Clearly, $(R \circ K)(L(S) \cap L^1_\psi) \supset (R \circ K_0)(L(S) \cap L^1_0)$ and $(R \circ K_0)(L(S) \cap L^1_0) = \mathrm{Ker}\rho$. Hence, $(R \circ K)(y) \in (R \circ K)(L(S) \cap L^1_\psi)$. Thus, there exists $\tilde{y} \in L(S) \cap L^1_\psi$ such that $K(y - \tilde{y}) \in \mathrm{Ker}R = S$. Consequently, $y - \tilde{y} = (L \circ K)(y - \tilde{y}) \in L(S) \cap L^1_\psi$, which implies that y belongs to $L(S) \cap L^1_\psi$ as claimed.

Let us now go back to the map $Q_\psi : L^1_\psi \to \mathbb{R}^q$ introduced previously. By using the Lebesgue Convergence Theorem, it is immediately seen that Q_ψ is continuous in any subset of L^1_ψ of the form

$$D_{M\psi} = \big\{ y \in L^1_{\mathrm{loc}} : |y(t)| \le M\psi(t), \text{ for a.a. } t \in J \big\}, \quad M > 0.$$

Let us show that Q is continuous in $D_{M\psi}$ as well. To this end, consider the (closed) subset

$$A_{M\varphi} = \big\{ x \in AC_{\mathrm{loc}} : |x(t)| \le M\varphi(t), t \in J \big\}, \quad M > 0$$

of AC_{loc} and let AC_φ be the vector space

$$AC_\varphi = \big\{ x \in AC_{\mathrm{loc}} : x = O(\varphi) \big\} = \cup_{M>0} A_{M\varphi}.$$

Suppose AC_φ equipped with the finest topology which makes any inclusion $A_{M\varphi} \to AC_\varphi$ continuous. That is, $U \subset AC_\varphi$ is open in AC_φ if and only if $U \cap A_{M\varphi}$ is open in $A_{M\varphi}$ for all $M > 0$. It is not hard to show (apply, e.g., Theorems 3.3F and 3.3G of [8]) that, with this topology AC_φ becomes a Hausdorff topological vector space.

By assumption H_2'), $S \cap A_\varphi$ is closed, which clearly means that S is closed in AC_φ. Consequently, the quotient space AC_φ/S is a finite dimensional Hausdorff topological vector space. Hence, there exists an injective map $\sigma : AC_\varphi/S \to \mathbb{R}^r$ such that $R = \sigma \circ \pi$, where $\pi : AC_\varphi \to AC_\varphi/S$ is the canonical projection. Such a map σ is clearly continuous, since AC_φ/S is Hausdorff and finite dimensional. Therefore, $R : AC_\varphi \to \mathbb{R}^r$ is continuous and, so, any restriction of R to $A_{M\varphi}$ is still continuous. Observe now that in $A_{M\varphi}$ the topology induced by the one of AC_φ clearly coincides with the usual topology of $A_{M\varphi}$ (as a subspace of $AC_{\rm loc}$). Since $K : D_{M\psi} \to A_{M\varphi}$ is continuous (see Theorem 2.1), Q is continuous in $D_{M\psi}$ as claimed. Consequently, Q_ψ and Q, which are both defined in L^1_ψ, continuous in $D_{M\psi}$, $\forall M > 0$, and coinciding in $L^1_0 \cap D_{M\psi}$ (which is dense in $D_{M\psi}$), must coincide in $D_{M\psi}$ and, thus, in $L^1_\psi = \cup_{M>0} D_{M\psi}$. That is, for any $y \in L^1_\psi$, one has

$$Q(y) = \Big(\int_a^\infty \langle \alpha_1(t), y(t)\rangle dt, \dots, \int_a^\infty \langle \alpha_q(t), y(t)\rangle dt \Big).$$

This implies $\int_a^\infty \langle \alpha_i(t), y(t)\rangle dt = 0$ for any $i = 1, \dots, q$ if and only if $y \in L(S) \cap L^1_\psi$, which is the assertion. ♢

The following example illustrates our main result.

Example 4.6 Consider in $J = [0, \infty)$ the problem

$$\begin{gathered} x' - x = y \\ x \in L^1(J, \mathbb{R}) \\ \textstyle\int_0^\infty x(t)dt = 0 \end{gathered}$$

Here, $S_\infty = L^1(J, \mathbb{R})$, $r = 1$, $m = 0$, $p = 0$, so that $q = p - m + r = 1$. In order to find the function α of Theorem 4.1, consider in L^1_0 the inverse

$$x(t) = K_0(y)(t) = -e^t \int_t^\infty e^{-s} y(s)ds.$$

By imposing the condition

$$\int_0^\infty K_0(y)(t)dt = 0$$

one obtains

$$\begin{aligned} \int_0^\infty \Big(\int_t^\infty e^t e^{-s} y(s)ds \Big) dt &= \int_0^\infty \Big(\int_0^s e^t e^{-s} y(s)dt \Big) ds \\ &= \int_0^\infty (1 - e^{-s}) y(s)ds = 0. \end{aligned}$$

Thus, $\alpha(t) = 1 - e^{-t}$.

Let $\varphi : J \to (0, \infty)$ be any continuous L^1 function. Observe that, as a consequence of the Lebesgue Convergence Theorem, the assumption $H_2')$ is satisfied for such a φ. Let us find a function ψ as in Theorem 4.3. First of all we must have $|\alpha|\psi \in L^1(J, \mathbb{R})$, which implies $\psi \in L^1(J, \mathbf{R})$. In addition, take φ be C^1, positive and such that the function $\sigma(t) = \varphi(t)e^{-t}$ satisfies the following conditions:

- $\sigma(\infty) = 0$;
- $\sigma'(t) < 0$ for $t \in J$;
- $\sigma' \in L^1(J, \mathbb{R})$.

For example take $\varphi(t) = 1/(1 + t^2)$.

As in the proof of Theorem 2.1, the function $\psi(t) = -\sigma'(t)e^t$ is such that $K(D_\psi)$ is contained in A_φ, where $K : L^1_\psi \to S_\infty$ is the natural extension of K_0 defined above, and, by the Lebesgue Convergence Theorem, is clearly continuous in D_ψ.

Therefore, all the assumptions of Theorem 4.3 are satisfied, so that we may conclude that, if $y = O(\psi)$ and $\int_0^\infty (1 - e^{-t})y(t)dt = 0$, then our problem has a solution.

Remark 4.7 In many situations, the assumption

H$_3$) S_∞ *strongly asymptotic and* S *closed in* S_∞,

which is clearly stronger than $H_2')$, is satisfied. However, as the following simple example shows, assumption $H_3)$ might not be satisfied in some cases which turn out to be of some interest in the applications. Take, for instance,

$$S = \Big\{x \in L^1(J, \mathbb{R}) \cap AC_{\text{loc}}(J, \mathbb{R}) : \int_a^\infty x(t)dt = 0\Big\}.$$

Thus, S is not closed in $S_\infty = L^1(J, \mathbb{R}) \cap AC_{\text{loc}}(J, \mathbb{R})$, but, by taking a continuous $\varphi > 0$ such that $\int_a^\infty \varphi(t)dt < \infty$, one obtains immediately that A_φ is contained in S_∞ and $S \cap A_\varphi$ is closed, i.e. $H_2')$ is satisfied.

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Mariella Cecchi
Dipartimento di Elettronica,
Università di Firenze,
Via S. Marta, 3 - 50139 Firenze
e-mail address: cecchi@diefi.die.unifi.it

Massimo Furi
Dipartimento di Matematica Applicata,
Università di Firenze,
Via S. Marta, 3 - 50139 Firenze
e-mail address: furi@dma.unifi.it

Mauro Marini
Dipartimento di Elettronica,
Università di Firenze,
Via S. Marta, 3 - 50139 Firenze
e-mail address: marini@ingfi1.ing.unifi.it

Maria Patrizia Pera
Dipartimento di Matematica Applicata,
Università di Firenze,
Via S. Marta, 3 - 50139 Firenze
e-mail address: pera@dma.unifi.it