The Invariance of Domain for C^1 Fredholm maps of index zero

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Summary: We give a version of the classical Invariance of Domain Theorem for nonlinear Fredholm maps of index zero between Banach spaces (and Banach manifolds). The proof is based on a finite dimensional reduction technique combined with a mod 2 degree argument for continuous maps between (finite dimensional) differentiable manifolds.

Keywords: Domain invariance, nonlinear Fredholm maps, mod 2 degree.

Classification: 58C25, 47H11.

1 Introduction

Is the one-to-one continuous image of an open set in $\mathbf{R}^n$ still open? This problem is the so called *invariance of domain* and, as H. Freudenthal pointed out in his essay on the history of topology [6] (see also [12]), it has been considered of a great importance at the beginning of this (20^{th}) century "for it was essential for the justification of the Poincaré continuation method in the theory of automorphic functions and the uniformization (of analytic functions)."

As it is well-known, a positive answer to the problem has been given by Brouwer in 1912 ([3]). This result is usually obtained as a consequence of the Odd Mapping Theorem, as it is shown, for instance, in [5]. A simpler proof can be done by using the Jordan Separation Theorem (see, e.g., [7] or [10]).

The extension of the Domain Invariance Theorem to Banach spaces is due to Schauder [9], who proved that if Ω is a bounded open subset of a Banach space E and $f : \overline{\Omega} \to E$ is a one-to-one map of the form $f = I - h$ with I the identity of E and $h : \overline{\Omega} \to E$ compact, then $f(\Omega)$ is open. Again this result can be deduced by the infinite dimensional version of the Odd Mapping Theorem. A proof displaying this connection can be found, for instance, in [2], [5] or [7].

The result of Schauder is, clearly, a nonlinear counterpart of the well-known Fredholm alternative.

In this paper, we present a version of the Invariance of Domain Theorem for nonlinear Fredholm maps of index zero between Banach spaces (or, more generally, Banach manifolds). To prove our result, we use a finite dimensional reduction method for Fredholm maps that goes back to Caccioppoli (see [4]). In his paper, Caccioppoli defines a mod 2 degree for such type of maps, in order to give an answer to this question:

When does a point y_0 belong to the interior of the image of a Fredholm map of index zero?

His answer is the following:

Let E and F be Banach spaces, $f : \Omega \to F$ a Fredholm map of index zero defined on an open subset Ω of E, and $y_0 \in f(\Omega)$. If the mod 2 degree of f in Ω at y_0 is nonzero, then $f(\Omega)$ is a neighborhood of y_0.

However, it should be observed that Caccioppoli does not provide sufficient conditions for the degree to be different from zero. It is a matter of definition that, if f is one-to-one and y is a regular value of f, belonging to $f(\Omega)$ and sufficiently close to y_0, then the mod 2 degree of f in Ω at y_0 is equal to 1. On the other hand, it seems to be a nontrivial fact to prove the existence of a regular value belonging to the image of f, unless one preliminarily shows that such an image has an interior point. Actually, this is our aim here.

2 Preliminaries

Let E and F be two real Banach spaces. We recall that a bounded linear operator is said to be *Fredholm* if both $\operatorname{Ker} L$ and $\operatorname{coKer} L$ have finite dimension. In this case its *index* is the integer

$$\operatorname{ind} L = \dim \operatorname{Ker} L - \dim \operatorname{coKer} L.$$

A map $f : M \to N$ between real Banach manifolds is Fredholm of index zero (see [11]) if it is C^1 and its Fréchet derivative $f'(x)$, from the tangent space T_xM of M at x to the tangent space $T_{f(x)}N$ of N at $f(x)$, is Fredholm of index zero for any $x \in M$.

A map $f : M \to N$ between manifolds is said to be *proper* if $f^{-1}(K)$ is compact for any compact subset K of N. In particular, let us recall that Fredholm maps are locally proper (see [11]).

The proof of the Invariance of Domain Theorem we present here is based on a notion of mod 2 degree for continuous maps between (finite dimensional) differentiable manifolds.

A classical version of such a degree can be found, for example, in the textbook of Milnor [8], where the author takes into account continuous maps between compact manifolds.

The notion of degree we use in this paper is a straightforward extension of the mod 2 degree in [8] and it is defined in the following context.

Let $f : M \to N$ be a continuous map between two differentiable boundaryless manifolds of the same dimension. Given an open subset U of M and a point $y \in N$, the triple (f, U, y) is called *admissible for the mod 2 degree*, if $f^{-1}(y) \cap U$ is compact.

The mod 2 degree, denoted by $\deg_2$, is a function defined in the class of all admissible triples, taking values in $\mathbf{Z}_2$, and verifying the following properties:

i) (*Normalization*) If f is a homeomorphism of M onto N, then

$$\deg_2 (f, M, y) = 1,$$

for all $y \in N$.

ii) (*Additivity*) If (f, M, y) is an admissible triple and U_1, U_2 are two open disjoint subsets of M such that $f^{-1}(y) \subset U_1 \cup U_2$, then

$$\deg_2 (f, M, y) = \deg_2 (f, U_1, y) + \deg_2 (f, U_2, y).$$

iii) (*Homotopy invariance*) Let $H : M \times [0, 1] \to N$ be a continuous homotopy. Then, given any (continuous) path $y : [0, 1] \to N$, such that the set $\{(x, t) \in M \times [0, 1] : H(x, t) = y(t)\}$ is compact, $\deg_2 (H(\cdot\,, t), M, y(t))$ does not depend on t.

iv) (*Topological invariance*) Let (f, M, y) be admissible. Given two differentiable manifolds W and Z with two homeomorphisms $\phi : M \to W$ and $\psi : N \to Z$, one has

$$\deg_2(f, M, y) = \deg_2(\psi \circ f \circ \phi^{-1}, W, \psi(y)).$$

The following consequences of the additivity property will be used in the proof of our main result.

v) (*Excision*) If (f, M, y) is admissible and U is an open neighborhood of $f^{-1}(y)$, then

$$\deg_2(f, M, y) = \deg_2(f, U, y).$$

vi) (*Existence*) If (f, M, y) is admissible and $\deg_2(f, M, y) = 1$, then the equation $f(x) = y$ admits a solution in M.

3 The invariance of domain

The following is our version of the invariance of domain for Fredholm maps.

Theorem 1. *Let M and N be two real Banach manifolds and $f : M \to N$ be an injective Fredholm map of index zero. Then $f(M)$ is open in N.*

Proof. The above statement is clearly equivalent to the following. If E and F are real Banach spaces, U is an open subset of E and $f : U \to F$ is an injective Fredholm map of index zero, then $f(U)$ is open. Thus, we will prove our result directly in the context of Banach spaces.

Take $y_0 \in f(U)$ and denote $x_0 = f^{-1}(y_0)$. There exists an open neighborhood V of x_0 in E such that $\overline{V} \subset U$ and f is proper on $\overline{V}$ (recall that Fredholm maps are locally proper). Let F_0 be any finite dimensional subspace of F through y_0 such that $F_0 + \mathrm{Range} f'(x_0) = F$, i.e. f is transverse to F_0 at x_0. Without loss of generality, we may assume that f is transverse to F_0 at any $x \in V$ (see e.g. [1]). Since $f(\partial V)$ is closed in F and does not contain y_0, there exists an open ball B centered at y_0 such that $\overline{B} \cap f(\partial V) = \emptyset$. We will show that B is contained in $f(U)$. Let $y \in B$ and denote $Y = \mathrm{span}(F_0 \cup \{y\})$. The map f is still transverse to the subspace Y at any $x \in V$ and, consequently, the set $f^{-1}(Y) \cap V$ is a submanifold of U (containing x_0) of the same dimension as Y (again see e.g. [1]). Denote $W = f^{-1}(Y) \cap V \cap f^{-1}(B)$ and observe that the restriction $f|_W : W \to Y$ is a continuous (actually C^1) map between two manifolds of the same dimension with $(f|_W)^{-1}(y_0)$ compact, since it reduces to $\{x_0\}$. Therefore, as pointed out in the preliminaries, the mod 2 degree, $\deg_2(f|_W, W, y_0)$, is well defined. Let us show that $\deg_2(f|_W, W, y_0) = 1$. By excision, to compute this degree it suffices to restrict f to a coordinate neighborhood W_1 of x_0 in W. Hence, using local charts, by the topological invariance, one can replace the triple $(f|_{W_1}, W_1, y_0)$ with a triple (f_1, Ω, η), where Ω is a bounded open subset of $\mathbf{R}^n$, $f_1 : \Omega \to \mathbf{R}^n$ is injective, and $\eta \in f_1(\Omega)$. In this situation, as it is well-known (see e.g. [7]), the oriented Brouwer degree is ± 1. Consequently, the mod 2 degree, $\deg_2(f_1, \Omega, \eta)$, is equal to 1 and, thus, $\deg_2(f|_W, W, y_0) = 1$ as well, as claimed.

Now, since B is a ball containing y_0 and y, the line segment Γ joining y_0 and y is contained in $B \cap Y$. Moreover, $(f|_W)^{-1}(\Gamma)$ is a compact subset of W (observe, in fact, that $\Gamma \cap f(\partial W) = \emptyset$). Thus, by the homotopy invariance of the degree, $\deg_2(f|_W, W, (1-t)y_0 + ty)$ is well defined and independent of t. Consequently, $\deg_2(f|_W, W, y) = 1$, which implies, by the existence property, that $(f|_W)^{-1}(y)$ is nonempty. □

Obviously, as a consequence of Theorem 1, we get the following nonlinear Fredholm alternative:

If $f : M \to N$ is Fredholm of index zero, $x_0 \in M$, and the equation $f(x) = y$ has at most one solution (close to x_0) for any y in a convenient neighborhood of $y_0 = f(x_0)$, then $f(x) = y$ is (uniquely) solvable for y sufficiently close to y_0.

An easy improvement of our previous result is the following.

Theorem 2. *Let M be a real Banach manifold, F a real Banach space, $f : M \to F$ the sum of two maps, g and h, where g is Fredholm of index zero and h is continuous with $h(M)$ in a finite dimensional space. Assume that f is injective. Then $f(M)$ is open in F.*

Sketch of the proof. The proof is in the outline of that of Theorem 1, with only minor differences. More precisely, in the same notation as above, choose F_0 containing $\{y_0\} \cup h(M)$ and such that $F_0 + \mathrm{Range}\, g'(x_0) = F$ (observe that, F_0 being a vector space, $g(x_0) \in F_0$). In addition, consider the manifold $g^{-1}(Y) \cap V$ and take $W = g^{-1}(Y) \cap V \cap (f)^{-1}(B)$. □

We close the paper with a question whose answer we believe is affirmative, but, as far as we know, unknown.

Does the above domain invariance result hold if the perturbation h of g, instead of having finite dimensional image, is assumed to be only locally compact?

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