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On the product formula for the oriented degree for Fredholm maps of index zero between Banach manifolds

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1. Introduction and preliminaries

One of the most important and deep properties of the Leray–Schauder degree is the well-known Leray Product Formula for the computation of the degree of a composite map (see, e.g., [3,14,15,17,19]). In this paper, using the concept of boundary set of a map introduced in [1], among other results we give an extension of the Leray formula (Theorem 3.5) and we provide, as a consequence, a simple proof of the generalized Jordan–Brouwer Separation Theorem due to Leray (see [14]).

As it is well known, the integer-valued degree has been extended by several authors to the framework of Fredholm maps between real Banach manifolds. A pioneering work in this direction is due to Elworthy and Tromba (see [8,9]). In [1], still in the context of nonlinear Fredholm maps, the first two authors introduce an elementary notion of oriented map (see below) which differs from the one given in [10] in some aspects which are pointed out in [2]. By means of this notion they define an integer-valued degree which coincides, for a large variety of maps, with the degree introduced in [10] and can be considered an evolution of the oriented degree of Elworthy–Tromba.

This work contains two versions of the Product Formula for the oriented degree of [1], namely, Theorems 3.1 and 3.7. The first one is the analog of Theorem 3.5.

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The second one is a more general formula containing, as a particular case, an extended additivity property for the degree of oriented maps. At the end a Jordan's-like separation theorem in Banach manifolds is deduced from Theorem 3.1.

We need some preliminaries.

Let E and F be two real Banach spaces. We recall that a bounded linear operator is said to be *Fredholm* if both $\operatorname{Ker} L$ and $\operatorname{coKer} L$ have finite dimension. In this case, its *index* is the integer

$$\operatorname{ind} L = \dim \operatorname{Ker} L - \dim \operatorname{coKer} L.$$

A map $f : M \to N$ between real Banach manifolds is Fredholm of index zero (see [18]) if it is C^1 and its Fréchet derivative $Df(x)$, from the tangent space T_xM of M at x to the tangent space $T_{f(x)}N$ of N at $f(x)$, is Fredholm of index zero for any $x \in M$.

A map $f : M \to N$ between manifolds is said to be *proper* if $f^{-1}(K)$ is compact for any compact subset K of N. In particular, let us recall that Fredholm maps are locally proper (see [18]).

A map $f : X \to E$ defined on a subset X of a Banach space E is a *compact vector field* if it is a completely continuous perturbation of the identity; that is, if it has the form $f(x) = x - \varphi(x)$, with $\varphi : X \to E$ sending bounded subsets of X into relatively compact subsets of E. We observe that if $f : X \to E$ is a compact vector field, X is closed, and $\|f(x)\| \to \infty$ as $\|x\| \to \infty$, then f is proper. In particular, a compact vector field is proper on bounded closed sets.

2. Orientation and degree

In this section, we give a brief review of the notion of degree for oriented maps between real Banach manifolds introduced in [1]. This notion is essentially based on the concept of orientability for Fredholm maps developed in [1,2].

Let $L : E \to F$ be a bounded Fredholm linear operator of index zero between real Banach spaces. We say that a bounded linear operator $A : E \to F$ with finite-dimensional range is a *corrector* of L provided that $L + A$ is an isomorphism. Observe that the set of correctors of L is nonempty. In fact, any (possibly trivial) bounded linear operator $A : E \to F$ such that $\operatorname{Ker} A \oplus \operatorname{Ker} L = E$ and $\operatorname{Range} A \oplus \operatorname{Range} L = F$ is a corrector of L.

Let A and B be two correctors of L. Observe that the isomorphism $T = (L + B)^{-1}(L + A)$ is a finite-dimensional perturbation of the identity I. Moreover, given any finite-dimensional subspace E_0 of E containing the image of $I - T$, one has $T(E_0) \subset E_0$. Thus, the determinant of the restriction of T to E_0, $\det T|_{E_0}$, is well defined. It is not difficult to show that this determinant does not depend on the choice of the finite-dimensional space E_0 containing $\operatorname{Range}(I - T)$. This common value will be denoted $\det T$. We say that A is *equivalent* to B or, more precisely, A is L-equivalent to B, if $\det T > 0$. This is an equivalence relation on the set of correctors of L with just two equivalence classes (see [1]). An *orientation* of L is, by definition, one of the two equivalence classes.

Given an oriented operator $L : E \to F$, the elements of its orientation will be called the *positive correctors* of L.

We point out that any isomorphism L admits a special orientation, namely the equivalence class containing the trivial operator 0. We shall refer to this equivalence class as the *natural orientation* $v(L)$ of L. However, if an isomorphism L happens to be already oriented, we define its sign as follows: $\operatorname{sign} L = 1$ if the trivial operator 0 is a positive corrector of L (i.e. if the orientation of L coincides with $v(L)$), and $\operatorname{sign} L = -1$ otherwise.

Unless otherwise stated, the composition L_2L_1 of two oriented operators will be oriented by taking as a positive corrector the operator $L_2A_1 + A_2A_1 + A_2L_1$, where A_1 and A_2 are positive correctors of L_1 and L_2, respectively.

An orientation of a bounded Fredholm operator of index zero induces, by a sort of stability, an orientation to any sufficiently close bounded operator. In fact, if A is a corrector of L, then $L' + A$ is an isomorphism whenever L' is sufficiently close to L. Thus, any such L' can be oriented by choosing A as a positive corrector.

Assume now $f : M \to N$ is a Fredholm map of index zero between real Banach manifolds. An orientation of f at a point $x \in M$ is an orientation of the Fréchet derivative $Df(x)$ of f at x. An *orientation* of f is a "continuous" assignment of an orientation at any point of M (see [1,2] for a precise notion of continuous assignment). By an oriented map we mean a Fredholm map between real Banach manifolds with a given orientation. Let us point out that, when M and N are finite-dimensional orientable connected manifolds (of the same dimension), an orientation of $f : M \to N$ can be regarded as a pair of orientations, one of M and one of N, up to an inversion of both of them. The simplest example of a nonorientable Fredholm map (of index zero) is a constant function from a finite-dimensional nonorientable manifold M into a manifold N of the same dimension as M. An example of a nonorientable map in the flat case, i.e. acting between open sets of Banach spaces, can be found in [2].

Notice that a local diffeomorphism $f : M \to N$ can be oriented by choosing the natural orientation at any $x \in M$. This makes sense since $Df(x)$ is an isomorphism for any $x \in M$. Thus, for example, the covering projection from the two-dimensional sphere S^2 onto the (nonorientable) projective space P^2 is orientable. As shown in [2], if M is simply connected, then any Fredholm map of index zero $f : M \to N$ is orientable (and, consequently, an orientation of f can be given by assigning an orientation at a chosen point of M). Thus, actually, any (C^1) map from S^2 into P^2 is orientable.

A homotopy $H : M \times [0,1] \to N$ is called an *oriented homotopy* provided that any partial map $H_\lambda := H(\cdot,\lambda)$ is Fredholm of index zero, the partial derivative $D_1H(\cdot,\lambda)$ depends continuously on (x,λ), and a "continuous" choice of an orientation of $D_1H(x,\lambda)$ is assigned for any (x,λ). Thus, an oriented homotopy induces an orientation on any partial map H_λ. In [2], it is proved that an orientation of any given partial map H_λ induces a unique compatible orientation on H. As a consequence of this we observe the following. Let $T : E \to E$ be a linear operator in a real Banach space of the form $I - K$, where I is the identity and K is a compact operator. Then T has a *canonical* orientation induced by the natural orientation of I through the homotopy $H(x,\lambda) = x - \lambda Kx$. When T happens to be an isomorphism (i.e. when 1 is not an eigenvalue of K), two associated orientations can be considered: the natural one and the canonical one. We define the sign of T to be 1 if these two orientations coincide and -1 otherwise. One can show that when E is finite dimensional (or, more generally, when $I - T$

has finite-dimensional range), sign T coincides with the sign of the determinant of T. Actually, we point out that, in general, still under the assumption that $T = I - K$ is invertible, sign T coincides with the sign of the Leray–Schauder index of T at zero (i.e. the Leray–Schauder degree of T in a ball around zero).

The orientation of the composition gf of two oriented maps, f and g, can be defined as in the linear case. With this induced orientation, gf will be called the *oriented composition* of f and g. From now on, the composition of two (or more) oriented maps will be regarded as an oriented composition.

Let $f : M \to N$ be an oriented map. Given an open subset U of M and an element $y \in N$, we say that the triple (f, U, y) is *admissible* if $f^{-1}(y) \cap U$ is compact. The degree introduced in [1] is an integer-valued function defined in the class of all the admissible triples and satisfying the following main properties:

Normalization. If $f : M \to N$ is a naturally oriented diffeomorphism and $y \in N$, then

$$\deg(f, M, y) = 1.$$

Additivity. If (f, M, y) is an admissible triple and U_1, U_2 are two open disjoint subsets of M such that $f^{-1}(y) \subset U_1 \cup U_2$, then

$$\deg(f, M, y) = \deg(f, U_1, y) + \deg(f, U_2, y).$$

Homotopy invariance. Let $H : M \times [0,1] \to N$ be an oriented homotopy and let $y : [0,1] \to N$ be continuous. If the set $\{(x, \lambda) \in M \times [0,1] \colon H(x, \lambda) = y(\lambda)\}$ is compact, then $\deg(H_\lambda, M, y(\lambda))$ does not depend on λ.

The degree of an admissible triple (f, U, y) is firstly defined when y is a regular value (for f in U) as

$$\deg(f, U, y) = \sum_{x \in f^{-1}(y)} \operatorname{sign} Df(x).$$

This restrictive assumption on y is then removed by means of the following lemma of [1].

Lemma 2.1. *Let (f, U, y) be admissible and let W_1 and W_2 be two open neighborhoods of $f^{-1}(y)$ such that $\bar{W}_1 \cup \bar{W}_2 \subset U$ and f is proper in $\bar{W}_1 \cup \bar{W}_2$. Then there exists a neighborhood V of y such that for any pair of regular values $y_1, y_2 \in V$ one has*

$$\deg(f, W_1, y_1) = \deg(f, W_2, y_2).$$

Lemma 2.1 justifies the following definition of degree for general admissible triples, taking also into account that Fredholm maps are locally proper.

Definition 2.2. Let (f, U, y) be admissible and let W be any open neighborhood of $f^{-1}(y)$ such that $\bar{W} \subset U$ and f is proper on $\bar{W}$. The degree of (f, U, y) is given by

$$\deg(f, U, y) := \deg(f, W, z)$$

where z is any regular value for f in W sufficiently close to y.

As pointed out in [1], no infinite-dimensional version of the Sard Theorem is needed in the above definition, since the existence of a sequence of regular values for $f|_W$ which converges to y is a consequence of the Implicit Function Theorem and the classical Sard–Brown Lemma.

This notion of degree can be compared with the classical ones of Brouwer and Leray–Schauder as follows.

Assume that $f:M \to N$ acts between connected finite-dimensional oriented manifolds (of the same dimension) and M is compact (or, more generally, assume that f is proper). Thus, the classical Brouwer degree, $\deg_B f$, is defined. In this case, if f is C^1, the orientation associated in [1] to the pair of orientations of M and N is such that $\deg(f,M,y) = \deg_B f$, for all $y \in N$.

As regards the Leray–Schauder degree, let $f:\Omega \to E$ be a C^1 compact vector field on a bounded open subset Ω of a real Banach space E. Assume that f admits a continuous extension (still denoted by f) to the closure $\bar{\Omega}$ of Ω. If $y \notin f(\partial\Omega)$, the Leray–Schauder degree $\deg_{LS}(f,\Omega,y)$ is defined. It can be shown that if f is canonically oriented (i.e. $Df(x)$ has the canonical orientation for any $x \in \Omega$), then $\deg(f,\Omega,y)$, which is clearly defined since f is proper on $\bar{\Omega}$ and $f^{-1}(y) \cap \partial\Omega = \emptyset$, coincides with $\deg_{LS}(f,\Omega,y)$.

Given an oriented map $f:M \to N$, the degree $\deg(f,M,y)$ does not necessarily depend continuously on y. To see this, observe, for instance, that the triple $(\exp,\mathbf{R},y)$ is admissible for all $y \in \mathbf{R}$, but the map $y \mapsto \deg(\exp,\mathbf{R},y)$ is discontinuous at $y = 0$. To overcome this inconvenience, we introduce the boundary set ∂f of f, which is a subset of N with the property that the map $y \mapsto \deg(f,M,y)$ is well defined and continuous when restricted to $N \setminus \partial f$.

Given $y \in N$, we say that f is *y-proper* if there exists a neighborhood V of y such that $f^{-1}(K)$ is compact for any compact subset K of V. Clearly, the set

$$\{y \in N\colon\ f \text{ is } y\text{-proper}\}$$

is open in N. Consequently, its complement, the *boundary set of* f, denoted by ∂f, is closed. As shown in Proposition 2.3 below, a map $f:M \to N$ is proper if and only if ∂f is empty.

Given an open subset U of M and $y \in N$, we say that f is *y-proper in* U if it is y-proper the restriction $f|_U$ of f to U. We will denote $\partial(f,U) := \partial(f|_U)$. The symbol "$\partial$" in this notation is justified by the fact that, in many instances, $\partial(f,U)$ coincides with $f(\partial U)$, where, as usual, ∂U stands for the boundary of U.

In the following proposition, we collect some properties of the boundary set which will be useful in the next sections. Let us point out in particular that, as a consequence of (2) below, when $f:\bar{\Omega} \to E$ is a compact vector field on the closure of a bounded open subset Ω of a Banach space E, then $\partial(f,\Omega) := f(\partial\Omega)$.

Proposition 2.3. *Let* $f:M \to N$ *and* $g:N \to Z$ *be two continuous maps between Banach manifolds. The following properties hold true*:

(1) *If* K *is any compact subset of* N *such that* $K \cap \partial f = \emptyset$, *then* $f^{-1}(K)$ *is compact. In particular, if* $\partial f = \emptyset$, *then* f *is proper.*

(2) *Given any open set* $U \subset M$, *one has* $f(\partial U) \subset \partial(f, U)$. *Moreover, if* f *is proper on the closure* $\bar{U}$ *of* U, *then* $\partial(f, U) = f(\partial U)$.
(3) *If* $C \subset M$ *is a closed set, then* $f(C) \cup \partial f$ *is closed. In particular, as well known, if* $\partial f = \emptyset$ (*i.e.* f *is proper*), *then* $f(C)$ *is closed.*
(4) *Let* $y \notin \partial f$. *Let* $\mathscr{U}$ *be a family of pairwise disjoint open subsets of* M *whose union contains* $f^{-1}(y)$. *Then there exists an open neighborhood* V *of* y *such that, for any compact* $K \subset V$ *and any* $U \in \mathscr{U}$, *the set* $f^{-1}(K) \cap U$ *is compact. In particular,* $y \notin \partial(f, U)$ *for all* $U \in \mathscr{U}$.
(5) $\partial(gf) \subset g(\partial f) \cup \partial g$.

Proof. (1) By the definition of ∂f, for any $y \in K$ there exists an open neighborhood V_y of y such that f is proper as a map from $f^{-1}(V_y)$ to V_y. For any $y \in K$, let W_y be an open neighborhood of y such that $\bar{W}_y \subset V_y$. Clearly, for any $y \in K$, $f^{-1}(\bar{W}_y \cap K)$ is compact. On the other hand, the compact set K can be covered by a finite number of W_y's, say $W_{y_1}, W_{y_2}, \dots, W_{y_n}$. Therefore,

$$f^{-1}(K) = f^{-1}\left(\left(\bigcup_{i=1}^{n} \bar{W}_{y_i}\right) \cap K\right) = \bigcup_{i=1}^{n} f^{-1}(\bar{W}_{y_i} \cap K)$$

is compact, being the union of a finite number of compact sets.

(2) Take $y \in f(\partial U)$ and let $x \in \partial U$ be such that $f(x) = y$. Given a sequence $\{x_n\}$ in U converging to x, consider the compact set $K = \{f(x_n)\colon n \in \mathbf{N}\} \cup \{y\}$. Clearly, given any closed neighborhood C of y, the set $U \cap f^{-1}(C \cap K)$ is not compact, since $\{x_n\}$ converges to $x \notin U$. Thus, $f(\partial U) \subset \partial(f, U)$. Assume now that f is proper in $\bar{U}$. We need to show that $N \setminus f(\partial U) \subset N \setminus \partial(f, U)$. Take $y \notin f(\partial U)$. Then, since f is proper, $V = N \setminus f(\partial U)$ is an open neighborhood of y. Now, if K is any compact subset of V, then $f^{-1}(K) \cap \bar{U}$ is compact. Moreover, by construction, $f^{-1}(K) \cap \partial U = \emptyset$. Consequently, the set $f^{-1}(K) \cap U = f^{-1}(K) \cap \bar{U}$ is compact, i.e. f is y-proper on U.

(3) Let $\{y_n\}$ be a sequence in $f(C) \cup \partial f$ converging to $\bar{y} \in N$. If $y_n \in \partial f$ for infinitely many n, then there exists in ∂f a subsequence of $\{y_n\}$ converging to $\bar{y}$ so that, ∂f being closed, $\bar{y} \in \partial f$. Otherwise, there exists $\bar{n} \in \mathbf{N}$ such that $y_n \in f(C)$ for $n > \bar{n}$. Thus, for any $n > \bar{n}$, there exists $x_n \in C$ such that $f(x_n) = y_n$. Suppose $\bar{y} \notin \partial f$. Since $N \setminus \partial f$ is open, without loss of generality we may assume $y_n \in N \setminus \partial f$ for all $n > \bar{n}$. Therefore, $K = \{y_n : n > \bar{n}\} \cup \{\bar{y}\}$ is a compact subset of $N \setminus \partial f$. Hence, as proved above, $f^{-1}(K)$ is compact and, consequently, $f^{-1}(K) \cap C$ is a compact subset of M containing $\{x_n : n > \bar{n}\}$. Thus, passing to a subsequence if necessary, we can assume $x_n \to \bar{x} \in C$, so that $f(\bar{x}) = \bar{y} \in f(C)$.

(4) Consider the closed set $M \setminus (\bigcup_{U \in \mathscr{U}} U)$. By (3), $f(M \setminus (\bigcup_{U \in \mathscr{U}} U)) \cup \partial f$ is a closed subset of N not containing y. Therefore, $V = N \setminus (f(M \setminus (\bigcup_{U \in \mathscr{U}} U)) \cup \partial f)$ is an open neighborhood of y. Now, if K is any compact subset of V, then, by (1), the set $f^{-1}(K)$ is compact and, taking into account that any U is also closed in $\bigcup_{U \in \mathscr{U}} U$, we have that $f^{-1}(K) \cap U$ is compact too.

(5) We can prove, equivalently, that if g is z-proper and $g^{-1}(z) \cap \partial f = \emptyset$, then gf is z-proper. To this end, take $z \notin \partial g$. By (3), the set $g(\partial f) \cup \partial g$ is closed and, since $z \notin g(\partial f)$, $V = Z \setminus (g(\partial f) \cup \partial g)$ is an open neighborhood of z. Therefore, for any compact

$K \subset V$, $g^{-1}(K)$ is compact and, since $g^{-1}(K) \cap \partial f = \emptyset$, $(gf)^{-1}(K) = f^{-1}(g^{-1}(K))$ is compact as well. □

Let us now go back to the degree and conclude this section by introducing a notation which will be used in some of our statements below.

If $f : M \to N$ is an oriented map (between real Banach manifolds), then, given $y \in N \setminus \partial f$, $\deg(f, M, y)$ is well defined and, because of the Homotopy Property of the degree, depends only on the component V of $N \setminus \partial f$ containing y. This common value will be denoted $\deg(f, M, V)$. More generally, given a not necessarily connected open subset V of N, with the symbol $\deg(f, M, V)$ we shall understand that $V \cap \partial f = \emptyset$ and that $\deg(f, M, y)$ is independent of $y \in V$.

3. The multiplicativity property

In this section, we are interested in obtaining some extensions of the classical Leray Product Theorem (see, e.g., [19]) both for oriented maps between Banach manifolds and, in the not necessarily C^1 case, for compact vector fields in Banach spaces.

The first result is the following multiplicativity formula for the degree of oriented maps between Banach manifolds.

Theorem 3.1 (Multiplicativity). *Let M, N and Z be real Banach manifolds, $f : M \to N$ and $g : N \to Z$ oriented maps, C a closed subset of N containing ∂f. Then, for any $z \notin g(C) \cup \partial g$ one has*

$$\deg(gf, M, z) = \sum_{V \in \mathscr{V}} \deg(g, V, z) \deg(f, M, V),$$

where gf is the oriented composition of f and g, and $\mathscr{V}$ denotes the family of the components of $N \setminus C$. Therefore, if W is any connected open subset of $Z \setminus (g(C) \cup \partial g)$ one has

$$\deg(gf, M, W) = \sum_{V \in \mathscr{V}} \deg(g, V, W) \deg(f, M, V).$$

Before proving Theorem 3.1, it is convenient to make the following preliminary comments to the statement.

(a) By assumption, g is z-proper and $g^{-1}(z) \cap \partial f = \emptyset$. Therefore, by (5) of Proposition 2.3, it follows that $z \notin \partial(gf)$. Thus, $\deg(gf, M, z)$ is defined.

(b) Since $g^{-1}(z) \cap C = \emptyset$, by (4) of Proposition 2.3 it follows that $z \notin \partial(g, V)$ for any component V of $N \setminus C$. Thus, $\deg(g, V, z)$ is defined. Moreover, all but a finite number of the terms $\deg(g, V, z)$ are equal to zero, since $\mathscr{V}$ is an open covering of pairwise disjoint sets of the compact set $g^{-1}(z)$. Consequently, the above sum is in fact finite.

Proof. As observed above, the assumptions imply that the composition gf is z-proper on M.

Assume first that z is a regular value of gf. Hence, $(gf)^{-1}(z)$ is a finite set and

$$\deg(gf,M,z) = \sum_{x\in(gf)^{-1}(z)} \operatorname{sign} D(gf)(x).$$

Since f and g are Fredholm of index zero, then z is a regular value for g and any $y\in g^{-1}(z)$ is a regular value for f. Thus,

$$\begin{aligned}\sum_{x\in(gf)^{-1}(z)} \operatorname{sign} D(gf)(x) &= \sum_{x\in(gf)^{-1}(z)} \operatorname{sign} Dg(f(x)) \operatorname{sign} Df(x)\\ &= \sum_{y\in g^{-1}(z)} \left(\sum_{x\in f^{-1}(y)} \operatorname{sign} Df(x)\right) \operatorname{sign} Dg(y)\\ &= \sum_{y\in g^{-1}(z)} \operatorname{sign} Dg(y) \deg(f,M,y).\end{aligned}$$

Since, by assumption, $z\in Z\setminus g(C)$, by considering the family $\mathscr{V}$ of the components of $N\setminus C$, we can write

$$\begin{aligned}\sum_{y\in g^{-1}(z)} \operatorname{sign} Dg(y) \deg(f,M,y) &= \sum_{V\in\mathscr{V}} \sum_{y\in g^{-1}(z)\cap V} \operatorname{sign} Dg(y) \deg(f,M,y)\\ &= \sum_{V\in\mathscr{V}} \sum_{y\in g^{-1}(z)\cap V} \operatorname{sign} Dg(y) \deg(f,M,V)\\ &= \sum_{V\in\mathscr{V}} \deg(g,V,z) \deg(f,M,V).\end{aligned}$$

Thus, the multiplicativity formula for the degree is proved in the case when z is, in addition, a regular value of gf.

Consider now the general case and take any $z\notin(g(C)\cup\partial g)$. As a consequence of (3) of Proposition 2.3, $Z\setminus(g(C)\cup\partial g)$ is an open set, so that, if $\bar z$ is any regular value for gf in the component of $Z\setminus(g(C)\cup\partial g)$ containing z, there exists a continuous path joining z and $\bar z$ and having image K contained in $Z\setminus(g(C)\cup\partial g)$. Clearly, K is compact and $K\cap\partial(gf)=\emptyset$. Hence, $(gf)^{-1}(K)$ is compact, so that, by the homotopy invariance of the degree, one has

$$\deg(gf,M,z) = \deg(gf,M,\bar z).$$

Therefore, by the first part of the proof,

$$\deg(gf,M,\bar z) = \sum_{V\in\mathscr{V}} \deg(g,V,\bar z) \deg(f,M,V).$$

On the other hand, by taking into account again that $K\subset Z\setminus(g(C)\cup\partial g)$, we obtain that $g^{-1}(K)$ is compact and contained in $N\setminus C$. Moreover, since any $V\in\mathscr{V}$, being a component on $N\setminus C$, is closed, the set $g^{-1}(K)\cap V$ is compact. Consequently, again by the homotopy invariance of the degree, it follows

$$\deg(g,V,\bar z) = \deg(g,V,z), \quad \forall V\in\mathscr{V}.$$

This completes the proof. □

In many situations, the maps f and g above turn out to be proper. Hence, if this is the case, ∂f and ∂g are empty. Therefore, by taking $C=\emptyset$, we obtain the following simplified version of Theorem 3.1.

Corollary 3.2. *Let $f:M\to N$ and $g:N\to Z$ be two proper oriented maps. Let $\mathscr{V}$ be the family of the connected component of N. Then, for any connected and open subset W of Z, one has*

$$\deg(gf,M,W)=\sum_{V\in\mathscr{V}}\deg(g,V,W)\deg(f,M,V).$$

In the case of finite-dimensional oriented manifolds, one can clearly extend to not necessarily proper maps the notion of Brouwer degree for triples (f,M,y) with $f:M\to N$ continuous and $y\notin\partial f$. This extended notion of degree will be denoted by $\deg_B(f,M,y)$. In this context, by using a standard smooth approximation of continuous maps, from Theorem 3.1 we obtain the following extension of the usual version of the multiplicativity property for Brouwer degree (see, e.g., [3,15,17,19]).

Theorem 3.3. *Let M,N, and Z be oriented finite-dimensional manifolds, $f:M\to N$ and $g:N\to Z$ be continuous maps, C a closed subset of N containing ∂f. Then, for any $z\notin g(C)\cup\partial g$ one has*

$$\deg_B(gf,M,z)=\sum_{V\in\mathscr{V}}\deg_B(g,V,z)\deg_B(f,M,V),$$

where $\mathscr{V}$ denotes the family of the components of $N\backslash C$. Therefore, if W is any connected open subset of $Z\backslash(g(C\cup\partial g)$ one has

$$\deg_B(gf,M,W)=\sum_{V\in\mathscr{V}}\deg_B(g,V,W)\deg_B(f,M,V).$$

An immediate consequence of Theorem 3.3 is the following well known product formula (see [12,13,16]).

Corollary 3.4. *Let $f:M\to N$ and $g:N\to Z$ be two continuous maps between compact, connected and oriented finite-dimensional manifolds. Then*

$$\deg_B gf=\deg_B g\deg_B f.$$

Now, let Ω be a not necessarily bounded, open subset of a Banach space E, and let $f:\Omega\to E$ be a compact vector field. Take $y\notin\partial(f,\Omega)$. Hence, $f^{-1}(y)$ is a compact subset of Ω. Consequently, it makes sense to define the Leray–Schauder degree of f in Ω with respect to y as follows:

$$\deg_{LS}(f,\Omega,y):=\deg_{LS}(f,\Omega_1,y),$$

where Ω_1 is any bounded open subset of Ω such that $\bar{\Omega}_1\subset\Omega$ and $f^{-1}(y)\subset\Omega_1$. Clearly, the excision property of the Leray–Schauder degree guarantees that the above definition is independent of Ω_1. In particular, if Ω is bounded and f is defined on $\bar{\Omega}$, then, as already observed, f is proper on $\bar{\Omega}$ and, by (2) of Proposition 2.3, $\partial(f,\Omega) = f(\partial\Omega)$. Thus, as usual, we obtain that the degree is defined for $y \notin f(\partial\Omega)$. More generally, if f is defined only on $\partial\Omega$ and $y \notin f(\partial\Omega)$, then again we will use the notation $\deg_{\mathrm{LS}}(f,\Omega,y)$ to indicate the degree of any compact vector field defined on $\bar{\Omega}$ and coinciding with f on $\partial\Omega$. This makes sense because of the boundary dependence property of the Leray–Schauder degree.

In the context of compact vector fields in Banach spaces, the analog of Theorem 3.1 is the following result, which is an extension of the classical Leray Product Theorem (see, e.g., [3,15,17,19]). We point out that this extension cannot be considered a corollary of Theorem 3.1, since the maps are not necessarily of class C^1. Nevertheless, the proof is similar and will be omitted.

Theorem 3.5. *Let E be a Banach space and Ω an open subset of E. Let $f:\Omega \to E$ and $g:\tilde{\Omega} \to E$ be (continuous) compact vector fields, where $\tilde{\Omega}$ is an open subset of E containing $f(\Omega)$. Then, if C is a closed subset of $\tilde{\Omega}$ containing $\partial(f,\Omega)$ and $z \notin g(C) \cup \partial(g,\tilde{\Omega})$, one has*

$$\deg_{\mathrm{LS}}(gf,\Omega,z) = \sum_{V \in \mathscr{V}} \deg_{\mathrm{LS}}(g,V,z)\deg_{\mathrm{LS}}(f,\Omega,V),$$

where $\mathscr{V}$ denotes the family of the components of $\tilde{\Omega}\setminus C$. Therefore, if W is any connected open subset of $E\setminus(g(C) \cup \partial(g,\tilde{\Omega}))$ one has

$$\deg_{\mathrm{LS}}(gf,\Omega,W) = \sum_{V \in \mathscr{V}} \deg_{\mathrm{LS}}(g,V,W)\deg_{\mathrm{LS}}(f,\Omega,V).$$

From Theorem 3.5, we obtain the following well known multiplicativity formula for Leray–Schauder degree. In the proof of Corollary 3.6, we will make use of the following fact, which we recall here for completeness. If $f:\Omega \to E$ is a compact vector field on a bounded open subset Ω of E, then there exists a proper compact vector field $\hat{f}:E \to E$ extending f. To see this, suppose f of the form $f(x) = x - \varphi(x)$, with $\varphi:\Omega \to E$ compact and recall that, since $\varphi(\Omega)$ is relatively compact, by Dugundji extension theorem (see [6]) there exists $\hat{\varphi}:E \to E$ coinciding with φ in Ω and with image contained in $\operatorname{co}\varphi(\Omega)$, the convex hull of $\varphi(\Omega)$. By Mazur theorem, $\operatorname{co}\varphi(\Omega)$ is relatively compact (see, e.g., [7]). Thus, $\hat{f}(x) = x - \hat{\varphi}(x)$ is a compact vector field extending f and, as already observed in the introduction, proper since $\|\hat{f}(x)\| \to \infty$ as $\|x\| \to \infty$.

Corollary 3.6. *Let E be a Banach space and Ω be a bounded open subset of E. Let $f:\partial\Omega \to E$ and $g:f(\partial\Omega) \to E$ be compact vector fields. Then, if $z \notin gf(\partial\Omega)$, one has*

$$\deg_{\mathrm{LS}}(gf,\Omega,z) = \sum_{V \in \mathscr{V}, V \neq V_\infty} \deg_{\mathrm{LS}}(g,V,z)\deg_{\mathrm{LS}}(f,\Omega,V),$$

where $\mathscr{V}$ denotes the family of the components of $E\setminus f(\partial\Omega)$ and V_∞ is the unbounded component.

Proof. Let $\hat{f}:E \to E$ and $\hat{g}:E \to E$ be proper compact vector fields extending f and g, respectively. Hence, $\partial\hat{g} = \emptyset$ and, by (2) of Proposition 2.3, $\partial(\hat{f},\Omega) = f(\partial\Omega)$. Therefore, by applying to $\hat{f}$ and $\hat{g}$ Theorem 3.5 with $\tilde{\Omega} = E$ and $C = f(\partial\Omega)$, we obtain, for any $z \notin \hat{g}\hat{f}(\partial\Omega)$,

$$\deg_{\mathrm{LS}}(\hat{g}\hat{f},\Omega,z) = \sum_{V \in \mathscr{V}} \deg_{\mathrm{LS}}(\hat{g},V,z)\deg_{\mathrm{LS}}(\hat{f},\Omega,V),$$

where $\mathscr{V}$ denotes the family of the components of $E \setminus f(\partial\Omega)$.

Observe now that one can restrict the above sum only to the bounded components of $E \setminus f(\partial\Omega)$ since $\deg_{\mathrm{LS}}(\hat{f},\Omega,V_\infty) = 0$. To see this, it is enough to compute $\deg_{\mathrm{LS}}(\hat{f},\Omega,y)$ with $y \notin \hat{f}(\bar{\Omega})$. Moreover, if V is any bounded component of $E \setminus f(\partial\Omega)$, since $\partial V \subset f(\partial\Omega)$, then $\hat{g} = g$ on ∂V. Hence, by recalling that the degree depends only on the restriction of a map to the boundary of an open bounded set, the above equality becomes

$$\deg_{\mathrm{LS}}(gf,\Omega,z) = \sum_{V \in \mathscr{V}, V \neq V_\infty} \deg_{\mathrm{LS}}(g,V,z)\deg_{\mathrm{LS}}(f,\Omega,V),$$

as claimed. □

The following more general version of Theorem 3.1 can be obtained by the same proof as that given above.

Theorem 3.7 (Generalized multiplicativity). *Let M, N and Z be Banach manifolds and let $f:M \to N$ and $g:N \to Z$ be oriented maps. Given $z \in Z \setminus (g(\partial f) \cup \partial g)$, let $\mathscr{V}$ be a family of pairwise disjoint open subsets of $N \setminus \partial f$ such that*

(i) $g^{-1}(z) \subset \bigcup_{V \in \mathscr{V}} V$,

(ii) *for any $V \in \mathscr{V}$ and for any $y_1, y_2 \in V$,* $\deg(f,M,y_1) = \deg(f,M,y_2)$.

Then,

$$\deg(gf,M,z) = \sum_{V \in \mathscr{V}} \deg(g,V,z)\deg(f,M,V).$$

By taking in Theorem 3.7 the map f to be the identity with the natural orientation recalled in Section 2, we have $\deg(f,M,V) = 1$. Consequently, we immediately obtain the following generalized additivity formula for the degree.

Theorem 3.8 (Generalized additivity). *Let $g:N \to Z$ be an oriented map and let $z \in Z \setminus \partial g$. Let $\mathscr{V}$ be a family of pairwise disjoint open subsets of N such that $g^{-1}(z) \subset \bigcup_{V \in \mathscr{V}} V$. Then,*

$$\deg(g,N,z) = \sum_{V \in \mathscr{V}} \deg(g,V,z).$$

Another nice consequence of Theorem 3.7 is the following formula.

Corollary 3.9. *Let $f:M \to N$ and $g:N \to Z$ be two oriented maps. For any $k \in \mathbf{Z}$, let V_k denote the open subset of N given by $V_k = \{y \in N \setminus \partial f:$ $\deg(f,M,y) = k\}$.*

Then, given $z \in Z \setminus (g(\partial f) \cup \partial g)$, *one has*

$$\deg(gf, M, z) = \sum_{k \in \mathbf{Z}} k \deg(g, V_k, z).$$

4. Some Jordan-like theorems

We apply now the multiplicativity formulas obtained in the previous section to deduce some homotopic versions of Jordan's theorem.

Let X and Y be two subsets of a Banach space E. We say that X and Y have the same *homotopy type with respect to compact vector fields* if there exist two compact vector fields $f : X \to Y$ and $g : Y \to X$ such that gf and fg are homotopic to I_X (the identity on X) and I_Y, respectively, through homotopies which are completely continuous perturbations of the identity.

Theorem 4.1 is a consequence of the infinite-dimensional version of Alexander–Pontriagin duality due to Gęba–Granas (see [11] and references therein). Here we give a simple proof based on degree theory in the outline of the argument due to Leray in [14], where he assumes that the two sets are homeomorphic.

Theorem 4.1. *Let E be a Banach space and let X and Y be two bounded closed subsets of E having the same homotopy type with respect to compact vector fields. Then, $E \setminus X$ and $E \setminus Y$ have the same number of components.*

Proof. By assumption there exist two compact vector fields $f : X \to Y$ and $g : Y \to X$ and two homotopies $H : X \times [0,1] \to X$ and $K : Y \times [0,1] \to Y$ of the form $H(x,\lambda) = x - h(x,\lambda)$, $K(y,\lambda) = y - k(y,\lambda)$, respectively, where $h : X \times [0,1] \to E$ and $k : Y \times [0,1] \to E$ are compact maps such that $h(\cdot,0) = 0$, $h(\cdot,1) = I_X - gf$, $k(\cdot,0) = 0$, $k(\cdot,1) = I_Y - fg$. As already observed in the above section, f and g can be extended to proper compact vector fields on E, say $\hat{f} : E \to E$ and $\hat{g} : E \to E$, respectively, such that $I - \hat{f}$ and $I - \hat{g}$ have relatively compact image.

Let $\mathscr{U}$ and $\mathscr{V}$ denote the family of the components of $E \setminus X$ and $E \setminus Y$, respectively. Observe first that, for any $U \in \mathscr{U}$ and $V \in \mathscr{V}$, $\deg_{\mathrm{LS}}(\hat{f}, U, V)$ and $\deg_{\mathrm{LS}}(\hat{g}, V, U)$ are defined. In fact, take for instance any $V \in \mathscr{V}$ and $y \in V$. Since $\hat{f}$ is proper in E (and, thus, in $\bar{U}$), by (2) of Proposition 2.3, $\partial(\hat{f}, U) = f(\partial U)$. On the other hand, since $\partial U \subset X$, it follows $f(\partial U) \subset Y$. Hence, $\partial(\hat{f}, U) \subset Y$, so that $\deg_{\mathrm{LS}}(\hat{f}, U, y)$ makes sense. A similar argument holds for $\deg_{\mathrm{LS}}(\hat{g}, V, x), x \in U$.

Let G_1 and G_2 be the free abelian groups generated by $\mathscr{U}$ and $\mathscr{V}$, respectively. Define the homomorphisms $\varphi : G_1 \to G_2$ and $\psi : G_2 \to G_1$ by

$$\varphi(U) = \sum_{V \in \mathscr{V}} \deg_{\mathrm{LS}}(\hat{f}, U, V)V, \quad U \in \mathscr{U}$$

and

$$\psi(V) = \sum_{U \in \mathscr{U}} \deg_{\mathrm{LS}}(\hat{g}, V, U)U, \quad V \in \mathscr{V}.$$

The result follows if we show that φ and ψ are isomorphisms.

One has

$$\psi\varphi(U) = \psi\left(\sum_{V\in\mathscr{V}} \deg_{\mathrm{LS}}(\hat{f}, U, V)V\right) = \sum_{V\in\mathscr{V}} \deg_{\mathrm{LS}}(\hat{f}, U, V)\psi(V)$$

$$= \sum_{V\in\mathscr{V}}\left(\deg_{\mathrm{LS}}(\hat{f}, U, V)\sum_{W\in\mathscr{U}} \deg_{\mathrm{LS}}(\hat{g}, V, W)W\right)$$

$$= \sum_{W\in\mathscr{U}}\left(\sum_{V\in\mathscr{V}} \deg_{\mathrm{LS}}(\hat{f}, U, V)\deg_{\mathrm{LS}}(\hat{g}, V, W)\right)W.$$

Our aim now is to apply to $\hat{f}$ and $\hat{g}$ the multiplicativity Theorem 3.5 with $\Omega = U$, $\tilde{\Omega} = E$, $C = Y$. As observed above, Y is a closed subset of E containing $\partial(\hat{f}, U)$. Moreover, since $\hat{g}(Y)\subset X$, the connected open subset W of $E\setminus X$ is in fact a connected open subset of $E\setminus\hat{g}(Y)$. Therefore, by Theorem 3.5, we obtain

$$\sum_{W\in\mathscr{U}}\left(\sum_{V\in\mathscr{V}} \deg_{\mathrm{LS}}(\hat{f}, U, V)\deg_{\mathrm{LS}}(\hat{g}, V, W)\right)W = \sum_{W\in\mathscr{U}} \deg_{\mathrm{LS}}(\hat{g}\hat{f}, U, W)W.$$

Let $\bar{h}:(E\times\{0\})\cup(X\times[0,1])\cup(E\times\{1\})\to E$ be the map defined by

$$\bar{h}(x,\lambda) = \begin{cases} 0 & \text{if } \lambda = 0,\\ h(x,\lambda) & \text{if } (x,\lambda)\in X\times[0,1],\\ x - \hat{g}\hat{f}(x) & \text{if } \lambda = 1.\end{cases}$$

Clearly, $\bar{h}$ is a continuous map defined on a closed subset of $E\times[0,1]$ and it is easy to check that its image is relatively compact. Hence, as previously observed, $\bar{h}$ can be extended to a compact map $\hat{h}:E\times[0,1]\to E$. Consequently, the compact vector field $\hat{H} = I_E - \hat{h}$ is a proper homotopy joining the identity with $\hat{g}\hat{f}$ and satisfying $\hat{H}_\lambda(X) := \hat{H}(X,\lambda)\subset X$ for all $\lambda\in[0,1]$. Moreover, as above, it is easily seen that $\partial(\hat{H}_\lambda, U)\subset X$, $\forall\lambda\in[0,1]$. Therefore, $\deg_{\mathrm{LS}}(\hat{H}_\lambda, U, W)$ is defined and independent of λ. Hence,

$$\deg_{\mathrm{LS}}(I, U, W) = \deg_{\mathrm{LS}}(\hat{H}_0, U, W) = \deg_{\mathrm{LS}}(\hat{H}_1, U, W) = \deg_{\mathrm{LS}}(\hat{g}\hat{f}, U, W).$$

Thus,

$$\psi\varphi(U) = \sum_{W\in\mathscr{U}} \deg_{\mathrm{LS}}(\hat{g}\hat{f}, U, W)W = \sum_{W\in\mathscr{U}} \deg_{\mathrm{LS}}(I, U, W)W.$$

Clearly, $\deg_{\mathrm{LS}}(I, U, W)$ is equal to 1 if $W = U$ and equal to 0 if $W \neq U$. Consequently,

$$\psi\varphi(U) = U, \quad U\in\mathscr{U}.$$

By a similar argument we also obtain

$$\varphi\psi(V) = V, \quad V\in\mathscr{V}.$$

The above equalities show that the free abelian groups G_1 and G_2 have the same rank, that is, the families $\mathscr{U}$ and $\mathscr{V}$ have the same cardinality. This proves the assertion. □

Since any continuous map in $\mathbf{R}^n$ is a compact vector field, we obtain the following extension of the well known Jordan Separation Theorem. The same result can also be deduced as a consequence of the classical Alexander duality as is shown, for instance, in [4]. Another elegant proof of the Jordan theorem can be found in [5].

Theorem 4.2. *Let X and Y be compact subsets of $\mathbf{R}^n$ having the same homotopy type. Then, $\mathbf{R}^n \setminus X$ and $\mathbf{R}^n \setminus Y$ have the same number of components.*

We close the paper by noting that the multiplicativity property of the oriented degree proved in Theorem 3.1 allows us to prove a quite general version in Banach manifolds of a Jordan's like separation theorem. To this end, let M and N be Banach manifolds and let X and Y be two closed subsets of M and N, respectively. We say that (M,X) and (N,Y) have the same *proper oriented homotopy type*, provided that there exist two proper orientable maps $\hat{f}:(M,X) \to (N,Y)$ and $\hat{g}:(N,Y) \to (M,X)$ such that $\hat{g}\hat{f}$ and $\hat{f}\hat{g}$ are homotopic to the identity maps $I_{(M,X)}$ and $I_{(N,Y)}$ through proper oriented homotopies, respectively.

We close with the following result whose proof is in the outline of that of Theorem 4.1, and, therefore, will be omitted.

Theorem 4.3. *Let M and N be two Banach manifolds and let X and Y be two closed subsets of M and N, respectively. Assume (M,X) and (N,Y) have the same proper oriented homotopy type. Then $M \setminus X$ and $N \setminus Y$ have the same number of components.*

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