Retarded functional differential equations on manifolds and applications to motion problems for forced constrained systems

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Abstract

We prove an existence result for T-periodic retarded functional differential equations of the type $x'(t) = f(t, x_t)$, where f is a T-periodic functional tangent vector field on a smooth manifold. As an application we show that any constrained system acted on by a periodic force, possibly with delay, admits a forced oscillation provided that the constraint is a topologically nontrivial compact manifold and the frictional coefficient is nonzero. We conjecture that the same assertion holds true even in the frictionless case.

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1 Introduction

Let $M \subseteq \mathbb{R}^k$ be a smooth manifold, possibly with boundary, and let $f : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$ be a continuous map such that

$$f(t, \varphi) \in T_{\varphi(0)}M\,, \quad \forall\, (t, \varphi) \in \mathbb{R} \times C((-\infty, 0], M),$$

where, given $p \in M$, $T_pM \subseteq \mathbb{R}^k$ denotes the tangent space of M at p. Any map with this property will be called a *functional tangent vector field* (or, briefly, a *functional field*) on M.

Assume, in addition, that f is T-periodic in the first variable, with bounded image, and inward along the boundary (in a sense to be explained). Our main result, Theorem 3.1 below, states that the retarded functional differential equation (RFDE) on M

$$x'(t) = f(t, x_t) \tag{1.1}$$

admits a T-periodic solution, provided that M is compact with nonzero Euler–Poincaré characteristic, and that f verifies a suitable Lipschitz-type assumption. Here, given $t \in \mathbb{R}$, we adopt the standard notation $x_t : (-\infty, 0] \to M$ for the function $x_t : \theta \mapsto x(t + \theta)$.

To prove this result we apply the classical fixed point index theory for locally compact maps on ANRs to a sort of Poincaré T-translation operator acting in the Banach space $C([-T, 0], \mathbb{R}^k)$. The idea of considering $C([-T, 0], \mathbb{R}^k)$ instead of the metrizable space $C((-\infty, 0], M)$ of the initial conditions became apparent thinking about a fruitful conversation about delay equations with Matteo Franca in which he observed that one knows the entire past of a T-periodic function if one knows its recent past. We are grateful to Matteo for his precious hint.

This paper is strictly related to our recent ones [1, 3, 4] in which we study constant time lag equations of the type

$$x'(t) = \widetilde{f}(t, x(t), x(t-1)), \tag{1.2}$$

with $\widetilde{f} : \mathbb{R} \times M \times M \to \mathbb{R}^k$ continuous, T-periodic in the first variable, and tangent to M in the second one; i.e.

$$\widetilde{f}(t + T, p, q) = \widetilde{f}(t, p, q) \in T_pM\,, \quad \forall\, (t, p, q) \in \mathbb{R} \times M \times M.$$

The equation (1.2) is clearly a special case of the RFDE (1.1). In fact, given $\widetilde{f} : \mathbb{R} \times M \times M \to \mathbb{R}^k$ as above, one can define $f : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$ by

$$f(t, \varphi) = \widetilde{f}(t, \varphi(0), \varphi(-1)).$$

Actually in [1, 3, 4] we do not limit ourselves to the study of the existence of T-periodic solutions, but we focus on the structure of the set of pairs (λ, x), where λ is a real parameter and x a T-periodic solution of the equation

$$x'(t) = \lambda \widetilde{f}(t, x(t), x(t-1)),$$

and we obtain global continuation results. Here we are merely concerned with existence results, leaving the study of continuation to future investigation.

We conclude the paper with an application to motion problems for forced constrained systems. Precisely, we consider the following retarded functional motion equation on a boundaryless smooth manifold $X \subseteq \mathbb{R}^s$:

$$x''_\pi(t) = F(t, x_t) - \varepsilon x'(t), \tag{1.3}$$

where

1. $x''_\pi(t)$ stands for the tangential part of the acceleration $x''(t) \in \mathbb{R}^s$ at the point $x(t) \in X$,
2. the frictional coefficient ε is a positive constant,
3. the applied force $F : \mathbb{R} \times C((-\infty, 0], X) \to \mathbb{R}^s$ is a continuous, T-periodic functional field on X.

We prove (see Theorem 4.1 below) that the equation (1.3) admits at least one forced oscillation, provided that the constraint X is compact with nonzero Euler–Poincaré characteristic and the functional field F is bounded and verifies a suitable Lipschitz-type assumption. Such a result is obtained by applying Theorem 3.1 to the first order RFDE on the tangent bundle $TX \subseteq \mathbb{R}^{2s}$ which is equivalent to the second order equation (1.3). The presence of a nonzero frictional coefficient allows us to restrict the search of forced oscillations inside a compact manifold whose boundary is obtained by cutting the noncompact tangent bundle TX with an appropriate limiting energy.

Theorem 4.1 generalizes results given in [2] and [4] for equations with constant time lag (see also [10] for the undelayed case). As far as we know, when the frictional coefficient ε is zero, the problem of the existence of forced oscillations of (1.3) is still open, even in the undelayed case. An affirmative answer, in the undelayed situation, regarding the special constraint $X = S^2$ (the spherical pendulum) can be found in [11] (see also [12] for the extension to the case $X = S^{2n}$).

Among the wide bibliography on RFDEs in Euclidean spaces we cite the works of Gaines and Mawhin [13], Nussbaum [23, 24] and Mallet-Paret, Nussbaum and Paraskevopoulos [19]. For RFDEs on manifolds we cite the papers of Oliva [25, 26]. For general reference we suggest the monograph by Hale and Verduyn Lunel [16].

2 Preliminaries

2.1 RFDE

Let, for the moment, M be an arbitrary subset of $\mathbb{R}^k$. We recall the notions of tangent cone and tangent space of M at a given point p in the closure $\overline{M}$ of M. The definition of tangent cone is equivalent to the classical one introduced by Bouligand in [6].

Definition 2.1 A vector $v \in \mathbb{R}^k$ is said to be *inward* to M at $p \in \overline{M}$ if there exist two sequences $\{\alpha_n\}$ in $[0, +\infty)$ and $\{p_n\}$ in M such that

$$p_n \to p \quad \text{and} \quad \alpha_n(p_n - p) \to v.$$

The set C_pM of the inward vectors to M at p is called the *tangent cone* of M at p. The *tangent space* T_pM of M at p is the vector subspace of $\mathbb{R}^k$ spanned by C_pM. A vector v of $\mathbb{R}^k$ is said to be *tangent* to M at p if $v \in T_pM$.

To simplify some statements and definitions we put $C_pM = T_pM = \emptyset$ whenever $p \in \mathbb{R}^k$ does not belong to $\overline{M}$ (this can be regarded as a consequence of Definition 2.1 if one replaces the assumption $p \in \overline{M}$ with $p \in \mathbb{R}^k$). Observe that T_pM is the trivial subspace $\{0\}$ of $\mathbb{R}^k$ if and only if p is an isolated point of M. In fact, if p is a limit point, then, given any $\{p_n\}$ in $M \backslash \{p\}$ such that $p_n \to p$, the sequence $\big\{\alpha_n(p_n - p)\big\}$, with $\alpha_n = 1/\|p_n - p\|$, admits a convergent subsequence whose limit is a unit vector.

One can show that in the special and important case when M is a smooth manifold with (possibly empty) boundary ∂M (a *∂-manifold* for short), this definition of tangent space is equivalent to the classical one (see for instance [20], [15]). Moreover, if $p \in \partial M$, C_pM is a closed half-space in T_pM (delimited by $T_p\partial M$), while $C_pM = T_pM$ if $p \in M \backslash \partial M$.

Let, as above, M be a subset of $\mathbb{R}^k$. We denote by D a nontrivial closed real interval with $\max D = 0$; that is, D is either $(-\infty, 0]$ or $[-r, 0]$ with $r > 0$. By $C(D, M)$ we mean the metrizable space of the M-valued continuous functions defined on D with the topology of the uniform convergence on compact subintervals of D.

Given a continuous function $x : J \to M$, defined on a real interval J, and given $t \in \mathbb{R}$ such that $t + D \subseteq J$, we adopt the standard notation $x_t : D \to M$ for the function defined by $x_t(\theta) = x(t + \theta)$.

Let $h : \mathbb{R} \times C(D, M) \to \mathbb{R}^k$ be a continuous map. As in the Introduction, we say that h is a *functional (tangent vector) field on M* if $h(t, \varphi) \in T_{\varphi(0)}M$ for all $(t, \varphi) \in \mathbb{R} \times C(D, M)$. In particular, h will be said *inward* (to M) if $h(t, \varphi) \in C_{\varphi(0)}M$ for all (t, φ). If M is a closed subset of a boundaryless smooth manifold $N \subseteq \mathbb{R}^k$, we will say that h is *away from* $N \backslash M$ if $h(t, \varphi) \notin C_{\varphi(0)}(N \backslash M)$ for all $(t, \varphi) \in \mathbb{R} \times C(D, M)$. Notice that this condition is satisfied whenever $\varphi(0)$, which is a point of M, is not in the topological boundary of M relative to N since, in that case, $C_{\varphi(0)}(N \backslash M) = \emptyset$.

In this paper we are interested in retarded functional differential equations (*RFDE* for short) of the type

$$x'(t) = h(t, x_t), \tag{2.1}$$

where $h : \mathbb{R} \times C(D, M) \to \mathbb{R}^k$ is a functional field on M.

By a *solution* of (2.1) we mean a continuous function $x : J \to M$, defined on a real interval J with $\inf J = -\infty$, which verifies eventually the equality $x'(t) = h(t, x_t)$. That is, x is a solution of (2.1) if there exists $\bar{t}$, with $-\infty \le \bar{t} < \sup J$, such that x is C^1 on the subinterval $(\bar{t}, \sup J)$ of J and verifies $x'(t) = h(t, x_t)$ for all $t \in J$ with $t > \bar{t}$.

Observe that, when $D = [-r, 0]$, there is a one-to-one correspondence between our notion of solution and the classical one which can be found e.g. in [16] (see also [25]). The correspondence is the one that assigns to any solution of (2.1) its restriction to the interval $[\bar{t} - r, \sup J)$.

Remark 2.1 Any equation of the form (2.1) with $D = [-r, 0]$ can be regarded as an equation of the same type with $D = (-\infty, 0]$, in the sense that to any equation (2.1) with $D = [-r, 0]$ can be associated an equivalent equation of the same type with $D = (-\infty, 0]$. In other words, given a functional field $h : \mathbb{R} \times C([-r, 0], M) \to \mathbb{R}^k$, there exists a functional field $g : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$ such that the equation

$$x'(t) = g(t, x_t) \tag{2.2}$$

has the same set of solutions as (2.1). To see this, it is enough to define $g : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$ by

$$g(t, \varphi) = h(t, \varphi|_{[-r,0]}),$$

for any $(t, \varphi) \in \mathbb{R} \times C((-\infty, 0], M)$.

As a consequence of Remark 2.1, it is not restrictive to study the broader class of RFDE's of the type

$$x'(t) = g(t, x_t), \tag{2.3}$$

where $g : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$ is a functional field on M. Therefore, from now on we will focus on this kind of equations.

2.2 Initial value problem

We are now interested in the following initial value problem:

$$\begin{cases} x'(t) = g(t, x_t), & t > 0, \\ x(t) = \eta(t), & t \le 0, \end{cases} \tag{2.4}$$

where M is a subset of $\mathbb{R}^k$, $g : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$ is a functional field on M, and $\eta : (-\infty, 0] \to M$ is a continuous map.

A *solution* of problem (2.4) is a solution $x : J \to M$ of (2.3) such that $\sup J > 0$, $x'(t) = g(t, x_t)$ for $t > \bar{t} = 0$, and $x(t) = \eta(t)$ for $t \le 0$.

The following technical lemma regards the existence of a persistent solution of problem (2.4).

Lemma 2.1 *Let M be a compact subset of a boundaryless smooth manifold $N \subseteq \mathbb{R}^k$, and g a functional field on M which is away from $N \backslash M$. Suppose that g is bounded. Then problem (2.4) admits at least one solution defined on the whole real line.*

Proof. We define a suitable extension $\widetilde{g} : \mathbb{R} \times C((-\infty, 0], \mathbb{R}^k) \to \mathbb{R}^k$ of g. Let $U \subseteq \mathbb{R}^k$ be a tubular neighborhood of N and let $\rho : U \to N$ be the associated retraction (if N is an open subset of $\mathbb{R}^k$, then $U = N$ and ρ is the identity). Fix

$\delta > 0$ such that $M_\delta = \{p \in U : \operatorname{dist}(p, M) \le \delta\}$ is a compact neighborhood of M in U.

We extend g to a functional field $\widetilde{g} : \mathbb{R} \times C((-\infty, 0], \mathbb{R}^k) \to \mathbb{R}^k$ with the following properties:

i) $\widetilde{g}$ is bounded;

ii) $\widetilde{g}(t, \varphi) = 0$ if $\operatorname{dist}(\varphi(0), M) \ge \delta$;

iii) $\widetilde{g}(t, \varphi) \in T_{\rho(\varphi(0))} N$ for all $(t, \varphi) \in \mathbb{R} \times C((-\infty, 0], \mathbb{R}^k)$ such that $\varphi(0) \in M_\delta$.

Observe that the existence of a map $\widetilde{g}$ satisfying the first two properties is ensured by the Tietze Extension Theorem. In fact, $C((-\infty, 0], M)$ and the set $\{\varphi \in C((-\infty, 0], \mathbb{R}^k) : \operatorname{dist}(\varphi(0), M) \ge \delta\}$ are two disjoint closed subsets of the metrizable space $C((-\infty, 0], \mathbb{R}^k)$. Moreover, we may assume that $\widetilde{g}$ has the additional property *iii)*. In fact, if this is not the case, it is sufficient to consider the orthogonal projection of $\widetilde{g}(t, \varphi)$ onto the space $T_{\rho(\varphi(0))} N$.

Now, consider the following auxiliary problem depending on $n \in \mathbb{N}$:

$$\begin{cases} x'(t) = \widetilde{g}(t, x_{t-\frac{1}{n}}), & t > 0, \\ x(t) = \eta(t), & t \le 0. \end{cases} \tag{2.5}$$

Clearly problem (2.5) has a solution defined on $(-\infty, 1/n]$ and, given a solution on $(-\infty, \beta]$, one can extend it to the interval $(-\infty, \beta + 1/n]$. Thus, problem (2.5) has a global solution $x^n : \mathbb{R} \to \mathbb{R}^k$.

From Ascoli's Theorem it follows that there exists a subsequence of $\{x^n(t)\}$ that converges to a continuous function $x(t)$, uniformly on compact subintervals of $\mathbb{R}$. Let us assume, without loss of generality, that, as $n \to \infty$, $\{x^n(t)\}$ converges (uniformly on compact subintervals of $\mathbb{R}$ to a continuous function $x(t)$).

Observe that problem (2.5) is equivalent to the following integral equation:

$$x(t) = \eta(0) + \int_0^t \widetilde{g}(s, x_{s-\frac{1}{n}})\, ds, \quad t \ge 0.$$

Moreover, for any given $t > 0$, the sequence $\{\widetilde{g}(t, x^n_{t-\frac{1}{n}})\}$ converges to $\widetilde{g}(t, x_t)$. Thus, $\widetilde{g}$ being bounded, from Lebesgue's Dominated Convergence Theorem we get

$$x(t) = \eta(0) + \int_0^t \widetilde{g}(s, x_s)\, ds, \quad t \ge 0.$$

Therefore, $x'(t) = \widetilde{g}(t, x_t)$ for all $t > 0$, and the assertion follows if we prove that $x(t)$ lies entirely in M.

Let us show first that $x(t) \in N$ for all $t \ge 0$ (this could be false if $\widetilde{g}$ were an arbitrary continuous extension of g). Clearly $x(t) \in M_\delta$ for all $t \ge 0$ (recall that $\widetilde{g}(t, \varphi) = 0$ if $\varphi(0) \notin M_\delta$). Thus, the C^1 function

$$\sigma(t) = \|x(t) - \rho(x(t))\|^2$$

is well defined for $t \ge 0$ and verifies $\sigma(0) = 0$. Assume, by contradiction, that $x(t) \notin N$ for some $t > 0$. This means that $\sigma(t) > 0$ for some $t > 0$ and, consequently, its derivative must be positive at some $\tau > 0$. That is,

$$\sigma'(\tau) = 2\big\langle x(\tau) - \rho(x(\tau)),\ \widetilde{g}(\tau, x_\tau) - w(\tau)\big\rangle > 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^k$, and $w(\tau)$ is the derivative at $t = \tau$ of the curve $t \mapsto \rho(x(t))$. This is a contradiction since both the vectors $\widetilde{g}(\tau, x_\tau)$ and $w(\tau)$ are tangent to N at $\rho(x(\tau))$ and, consequently, orthogonal to $x(\tau) - \rho(x(\tau))$.

It remains to show that $x(t) \in M$ for all $t > 0$. Let $s = \inf\{t > 0 : x(t) \in N\backslash M\}$, and assume by contradiction $s < +\infty$ (here we adopt the convention $\inf \emptyset = +\infty$). Note that $x(s) \in M$ since M is compact. Let $\{t_n\}$ be a sequence converging to s and such that $x(t_n) \in N\backslash M$. We have $t_n > s$ for all n and

$$\lim_{n\to\infty} \frac{x(t_n) - x(s)}{t_n - s} = x'(s) = \widetilde{g}(s, x_s) \in C_{x(s)}(N\backslash M).$$

Now, the function x_s takes values in M and, consequently, we have $g(s, x_s) = \widetilde{g}(s, x_s) \in C_{x(s)}(N\backslash M)$, contradicting the fact that the functional field g is away from $N\backslash M$. $\square$

From now on M will be a compact ∂-manifold in $\mathbb{R}^k$. In this case one may regard M as a subset of a smooth boundaryless manifold N of the same dimension as M (see e.g. [17], [21]). It is not hard to show that a functional field g on M is away from the complement $N\backslash M$ of M if and only if it is *strictly inward*; meaning that g is inward and $g(t, \varphi) \notin T_{\varphi(0)}\partial M$ for all $(t, \varphi) \in \mathbb{R} \times C((-\infty, 0], M)$ such that $\varphi(0) \in \partial M$.

We say that a subset Q of $C((-\infty, 0], M)$ is a *brush* if there exists $\sigma \leq 0$ such that $\varphi(\theta) = \psi(\theta)$ for all $\varphi, \psi \in Q$ and $\theta \leq \sigma$. We will make the following assumption:

(H) Given $\delta > 0$ and any compact brush Q of $C((-\infty, 0], M)$, there exists $L \geq 0$ such that
$$\|g(t, \varphi) - g(t, \psi)\| \leq L \sup_{s \leq 0} \|\varphi(s) - \psi(s)\| \tag{2.6}$$
for all $t \in [0, \delta]$ and $\varphi, \psi \in Q$.

Remark 2.2 Assumption (H) extends the one given in [16]. Indeed, in that monograph the authors study equations of the type

$$x'(t) = h(t, x_t),$$

where $h : \mathbb{R} \times C([-r, 0], \mathbb{R}^k) \to \mathbb{R}^k$ is Lipschitz in the second variable in each compact subset of $\mathbb{R} \times C([-r, 0], \mathbb{R}^k)$. Now, define $g : \mathbb{R} \times C((-\infty, 0], \mathbb{R}^k) \to \mathbb{R}^k$ by

$$g(t, \varphi) = h(t, \varphi|_{[-r,0]})$$

and observe that the functional field g clearly verifies (H).

We will use the following folk result, whose proof is given for the sake of completeness.

Lemma 2.2 *Let $\alpha : [0, b] \to \mathbb{R}^k$ be a C^1 function such that $\alpha(0) = 0$ and*

$$\|\alpha'(t)\| \leq c \sup_{0 \leq s \leq t} \|\alpha(s)\|, \quad t \in [0, b]$$

for some constant $c \geq 0$. Then, $\alpha(t) = 0$ for all $t \in [0, b]$.

Proof. Let $0 < \delta \leq b$ be such that $\delta c < 1$. Let $\tau \in [0, \delta]$ be such that $\|\alpha(\tau)\| = \max_{0 \leq s \leq \delta} \|\alpha(s)\|$. We have

$$\|\alpha(\tau)\| = \|\alpha(\tau) - \alpha(0)\| \leq \tau \sup_{0 \leq s \leq \tau} \|\alpha'(s)\| \leq \delta c \|\alpha(\tau)\|.$$

Being $\delta c < 1$, this inequality is verified if and only if $\alpha(\tau) = 0$. Thus $\alpha(t) = 0$ for any $t \in [0, \delta]$, and the assertion follows in a finite number of steps. □

The following proposition regards existence and uniqueness of solutions of problem (2.4) in the case when g is inward, bounded, and verifies (H).

Proposition 2.1 *Let $M \subseteq \mathbb{R}^k$ be a compact ∂-manifold and g an inward functional field on M. Suppose that g is bounded. Then, problem (2.4) admits a solution defined on the whole real line. Moreover, if g verifies (H), then the solution is unique.*

Proof. As already pointed out, we may regard M as a subset of a smooth boundaryless manifold N of the same dimension as M. Define $\nu : M \to \mathbb{R}^k$ as follows. Given $p \in \partial M$, let $\mu(p)$ be the unique unit vector belonging to $C_p M \cap (T_p \partial M)^\perp$. Then, extend $\mu : \partial M \to \mathbb{R}^k$ by Tietze's Theorem to a map from M to $\mathbb{R}^k$ and, for any $p \in M$, consider its orthogonal projection $\nu(p)$ onto the space $T_p M$. Given any $n \in \mathbb{N}$, define the strictly inward functional field $g_n : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$ by $g_n(t, \varphi) = g(t, \varphi) + \frac{1}{n}\nu(\varphi(0))$, and let $x^n : \mathbb{R} \to M$ be a solution of the initial value problem

$$\begin{cases} x'(t) = g_n(t, x_t), & t > 0, \\ x(t) = \eta(t), & t \leq 0, \end{cases}$$

whose existence is ensured by Lemma 2.1. As in the proof of that lemma, one can show that $\{x^n(t)\}$ has a subsequence which converges uniformly on compact subintervals of $\mathbb{R}$ to a solution of problem (2.4) defined on the whole real line.

Assume now that g verifies (H). Let $x^1, x^2 : \mathbb{R} \to M$ be two solutions of problem (2.4), and let $b > 0$. Then, the family of functions

$$\{x_t^i \in C((-\infty, 0], M) : \; t \in [0, b], i = 1, 2\},$$

which is clearly a brush, is a compact set, since it is the image of two curves in $C((-\infty, 0], M)$ both defined on the compact interval $[0, b]$. Thus, there exists $L \geq 0$ such that for any $t \in [0, b]$ we have

$$\begin{aligned} \|g(t, x_t^2) - g(t, x_t^1)\| \leq L \sup\nolimits_{s \leq 0} \|x_t^2(s) - x_t^1(s)\| &= L \sup\nolimits_{s \leq t} \|x^2(s) - x^1(s)\| \\ &= L \sup\nolimits_{0 \leq s \leq t} \|x^2(s) - x^1(s)\|. \end{aligned}$$

Now putting $y = x^2 - x^1$, we get $\|y(t)\| = 0$ for $t \leq 0$ and

$$\|y'(t)\| = \|g(t, x_t^2) - g(t, x_t^1)\| \leq L \sup_{0 \leq s \leq t} \|y(s)\|$$

for $t \in [0, b]$. Hence, the assertion follows from Lemma 2.2. □

2.3 Fixed point index

Here we summarize the main properties of the fixed point index in the context of absolute neighborhood retracts (ANRs). Let X be a metric ANR and consider a locally compact (continuous) X-valued map k defined on a subset $\mathcal{D}(k)$ of X. Given an open subset U of X contained in $\mathcal{D}(k)$, if the set of fixed points of k in U is compact, the pair (k, U) is called *admissible*. It is known that to any admissible pair (k, U) we can associate an integer $\mathrm{ind}_X(k, U)$ - the *fixed point index* of k in U - that satisfies properties which are analogous to those of the classical Leray–Schauder degree [18]. The reader can see for instance [5], [14], [22] or [24] for a comprehensive presentation of the index theory for ANRs. As regards connections with algebraic topological concepts we cite the textbooks [7] and [27].

For the reader's convenience we summarize the main properties of the index.

i) (*Existence*) If $\mathrm{ind}_X(k, U) \neq 0$, then k admits at least one fixed point in U.

ii) (*Normalization*) If X is compact, then $\mathrm{ind}_X(k, X) = \Lambda(k)$, where $\Lambda(k)$ denotes the Lefschetz number of k.

iii) (*Additivity*) Given two disjoint open subsets U_1, U_2 of U such that any fixed point of k in U is contained in $U_1 \cup U_2$, then $\mathrm{ind}_X(k, U) = \mathrm{ind}_X(k, U_1) + \mathrm{ind}_X(k, U_2)$.

iv) (*Excision*) Given an open subset U_1 of U such that k has no fixed points in $U \backslash U_1$, then $\mathrm{ind}_X(k, U) = \mathrm{ind}_X(k, U_1)$.

v) (*Commutativity*) Let X and Y be metric ANRs. Suppose that U and V are open subsets of X and Y respectively and that $k : U \to Y$ and $h : V \to X$ are locally compact maps. Assume that one of the sets of fixed points of hk in $k^{-1}(V)$ or kh in $h^{-1}(U)$ is compact. Then the other set is compact as well and $\mathrm{ind}_X(hk, k^{-1}(V)) = \mathrm{ind}_Y(kh, h^{-1}(U))$.

vi) (*Homotopy invariance*) Let $H : [0,1] \times U \to X$ be a locally compact map such that the set $\{(\lambda, x) \in [0,1] \times U : H(\lambda, x) = x\}$ is compact. Then $\mathrm{ind}_X(H(\lambda, \cdot), U)$ is independent of λ.

3 Existence of periodic solutions

From now on we will adopt the following notation. By M we mean a compact ∂-manifold in $\mathbb{R}^k$. Given $T > 0$, by $C_0([-T, 0], M)$ we mean the (complete) metric space of the continuous functions $\varphi : [-T, 0] \to M$ such that $\varphi(-T) = \varphi(0)$, endowed with the metric induced by the Banach space $C([-T, 0], \mathbb{R}^k)$.

Since M is an ANR, it is not difficult to show (see e.g. [8]) that the metric space $C_0([-T, 0], M)$ is an ANR as well. For the sake of simplicity, from now on, the metric space $C_0([-T, 0], M)$ will be denoted by $\widetilde{M_0}$.

Let $f : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$ be an inward functional field on M which is T-periodic in the first variable. Assume that f is bounded and verifies (H). We are interested in the existence of a T-periodic solution of the RFDE

$$x'(t) = f(t, x_t).$$

Given $\varphi \in \widetilde{M}_0$, we will denote by $\widehat{\varphi}$ the unique element of $C((-\infty,0],M)$ obtained by considering the T-periodic backward extension of the function φ; i.e. $\widehat{\varphi}$ is defined as follows:

$$\widehat{\varphi}(\theta) = \varphi(\theta + nT) \quad \text{if } \theta \in [-(n+1)T, -nT],\ n \in \mathbb{N}.$$

Let us observe that $\widetilde{M}_0$ is bounded and closed as a subset of the Banach space $C([-T,0],\mathbb{R}^k)$. Hence, $\widetilde{M}_0$ being an ANR, there exist a bounded open subset U of $C([-T,0],\mathbb{R}^k)$ containing $\widetilde{M}_0$ and a retraction ρ of U onto $\widetilde{M}_0$.

Now, given $\lambda \in [0,+\infty)$ consider the operator

$$P_\lambda : U \to C([-T,0],\mathbb{R}^k)$$

defined as $P_\lambda(\psi)(s) = x(s+T)$, where x is the unique solution, ensured by Proposition 2.1, of the following initial value problem:

$$\begin{cases} x'(t) = \lambda f(t,x_t), & t > 0, \\ x(t) = \widehat{\rho(\psi)}(t), & t \le 0. \end{cases} \tag{3.1}$$

The following two propositions regard some crucial properties of P_λ. The proof of Proposition 3.1 is straightforward and, therefore, it is omitted.

Proposition 3.1 *The set of fixed points of P_λ is contained in $\widetilde{M}_0$. Moreover, the fixed points of P_λ correspond to the T-periodic solutions of the equation*

$$x'(t) = \lambda f(t,x_t)$$

in the following sense: ψ is a fixed point of P_λ if and only if it is the restriction to $[-T,0]$ of a T-periodic solution.

Proposition 3.2 *The map $P : [0,1] \times U \to C([-T,0],\mathbb{R}^k)$, defined by $(\lambda,\psi) \mapsto P_\lambda(\psi)$, is continuous with relatively compact image.*

Proof. To show that P is continuous, let $\{\psi_n\}$ be a sequence in U which converges to ψ, and let $\{\lambda_n\}$ be a sequence in $[0,1]$ converging to λ. Since ρ is continuous, we have $\rho(\psi_n) \to \rho(\psi)$. Thus, $\widehat{\rho(\psi_n)}(\theta) \to \widehat{\rho(\psi)}(\theta)$ uniformly for $\theta \in (-\infty,0]$.

Now, let $x^n : \mathbb{R} \to M$ be the unique solution (ensured by Proposition 2.1) of the initial value problem

$$\begin{cases} x'(t) = \lambda_n f(t,x_t), & t > 0, \\ x(t) = \widehat{\rho(\psi_n)}(t), & t \le 0. \end{cases}$$

As in the proof of Lemma 2.1, one can show that every subsequence of $\{x^n(t)\}$ has a subsequence which converges uniformly on compact subintervals of $\mathbb{R}$ to the unique solution $x(t)$ of problem (3.1). Therefore, $x^n(t) \to x(t)$ uniformly on compact subintervals of $\mathbb{R}$ and, consequently, $P(\lambda_n,\psi_n) \to P(\lambda,\psi)$. This shows that the map P is continuous.

The compactness of P follows from Ascoli's Theorem. $\square$

We are now ready to establish our existence result.

Theorem 3.1 *Let M be a compact ∂-manifold with nonzero Euler–Poincaré characteristic, and $f : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$ an inward functional field on M which is T-periodic in the first variable. Suppose that f is bounded and verifies (H). Then, the equation*

$$x'(t) = f(t, x_t)$$

admits a T-periodic solution.

Proof. First we observe that, by Propositions 3.1 and 3.2, the set $\{(\lambda, \psi) \in [0,1] \times U : P(\lambda, \psi) = \psi\}$ is a compact subset of $[0,1] \times \widetilde{M}_0$. Hence, the fixed point index $\mathrm{ind}_E(P_\lambda, U)$, where $E = C([-T, 0], \mathbb{R}^k)$, is well defined and independent of $\lambda \in [0,1]$.

Now, if $\lambda = 0$, given $\psi \in U$, problem (3.1) becomes

$$\begin{cases} x'(t) = 0, & t > 0, \\ x(t) = \widehat{\rho(\psi)}(t), & t \le 0. \end{cases}$$

Any solution of this problem for $t \ge 0$ is constantly equal to $\rho(\psi)(0)$. It follows that

$$P_0(\psi)(s) = \rho(\psi)(0), \quad s \in [-T, 0].$$

Hence, P_0 sends U into the subset of the constant M-valued functions (which can be identified with M), and its restriction $P_0|_M : M \to M$ coincides with the identity I_M of M. By the commutativity and normalization properties of the fixed point index we get

$$\mathrm{ind}_E(P_0, U) = \mathrm{ind}_M(P_0, M) = \Lambda(I_M) = \chi(M) \neq 0.$$

Let us observe, to help the reader, that the first equality in the above formula follows from the commutativity property of the fixed point index, recalled in the above section, with $P_0|_U : U \to M$ in place of k and $P_0|_M : M \to E$ in place of h.

Finally, $\mathrm{ind}_E(P_1, U) \neq 0$ and the existence property implies that the operator P_1 has a fixed point. The assertion follows from Proposition 3.1. □

We close this section with the following example which illustrates how Theorem 3.1 can be applied.

Example 3.1 Let $g : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ be continuous, T-periodic in the first variable, and locally Lipschitz in the second one. Let $\mu : (-\infty, 0] \to \mathbb{R}$ be a function of bounded variation and consider the following RFDE containing a Riemann–Stieltjes integral:

$$x'(t) = g(t, x(t)) + \int_{-\infty}^{0} x(t + \theta)\, d\mu(\theta).$$

The equation is of the form $x'(t) = f(t, x_t)$, where

$$f(t, \varphi) = g(t, \varphi(0)) + \int_{-\infty}^{0} \varphi(\theta)\, d\mu(\theta)$$

is well defined for any pair (t, φ) with $t \in \mathbb{R}$ and $\varphi : (-\infty, 0] \to \mathbb{R}^k$ continuous and bounded.

Let k denote the total variation of μ in $(-\infty, 0]$ and assume there exists $c > 0$ such that the inner product $\langle g(t,x), x\rangle$ is less than $-kc^2$ for any $x \in \mathbb{R}^k$ with $\|x\| = c$. Let $M = \overline{B(0,c)}$, where $B(0,c)$ denotes the open ball in $\mathbb{R}^k$ centered at 0 with radius c. Then, $\chi(M) = 1$ since M is contractible. One can easily check that the restriction of the map f to $\mathbb{R} \times C((-\infty, 0], M)$ is a T-periodic inward functional field on the compact ∂-manifold M. Moreover, this restriction is bounded and verifies (H). Hence, Theorem 3.1 applies yielding the existence of a T-periodic solution for the given RFDE.

4 Applications to second order delay differential equations on manifolds

In this section we apply the results obtained above to some motion problems for forced constrained systems.

Let $X \subseteq \mathbb{R}^s$ be a boundaryless manifold. Given $q \in X$, let $(T_qX)^\perp \subseteq \mathbb{R}^s$ denote the normal space of X at q. Since $\mathbb{R}^s = T_qX \oplus (T_qX)^\perp$, any vector $u \in \mathbb{R}^s$ can be uniquely decomposed into the sum of the *parallel* (or *tangential*) *component* $u_\pi \in T_qX$ *of* u *at* q and the *normal component* $u_\nu \in (T_qX)^\perp$ *of* u *at* q. By

$$TX = \{(q,v) \in \mathbb{R}^s \times \mathbb{R}^s : \ q \in X, \ v \in T_qX\}$$

we denote the *tangent bundle of* X, which is a smooth manifold containing a natural copy of X via the embedding $q \mapsto (q, 0)$. The natural projection of TX onto X is just the restriction (to TX as domain and to X as codomain) of the projection of $\mathbb{R}^s \times \mathbb{R}^s$ onto the first factor.

Given a functional field $F : \mathbb{R} \times C((-\infty, 0], X) \to \mathbb{R}^s$ which is T-periodic in the first variable, consider the following retarded functional motion equation on X:

$$x''_\pi(t) = F(t, x_t) - \varepsilon x'(t), \tag{4.1}$$

where

i) $x''_\pi(t)$ stands for the parallel component of the acceleration $x''(t) \in \mathbb{R}^s$ at the point $x(t)$;

ii) the frictional coefficient ε is a positive real constant.

By a *solution* of (4.1) we mean a continuous function $x : J \to X$, defined on a real interval J with $\inf J = -\infty$, which verifies eventually the equality (4.1). That is, x is a solution of (4.1) if there exists $-\infty \le \bar{t} < \sup J$ such that x is C^2 on the subinterval $(\bar{t}, \sup J)$ of J and verifies

$$x''_\pi(t) = F(t, x_t) - \varepsilon x'(t)$$

for all $t \in J$ with $t > \bar{t}$. A *forced oscillation of (4.1)* is a solution which is T-periodic and globally defined on $J = \mathbb{R}$.

It is known that, associated with $X \subseteq \mathbb{R}^s$, there exists a unique smooth map $R : TX \to \mathbb{R}^s$, called the *reactive force* (or *inertial reaction*), with the following properties:

(a) $R(q,v) \in (T_qX)^{\perp}$ for any $(q,v) \in TX$;

(b) R is quadratic in the second variable;

(c) any C^2 curve $\gamma : (a,b) \to X$ verifies the condition

$$\gamma_{\nu}''(t) = R(\gamma(t), \gamma'(t)), \quad \forall t \in (a,b),$$

i.e., for each $t \in (a,b)$, the normal component $\gamma_{\nu}''(t)$ of $\gamma''(t)$ at $\gamma(t)$ equals $R(\gamma(t), \gamma'(t))$.

The map R is strictly related to the second fundamental form on X and may be interpreted as the reactive force due to the constraint X.

By properties (a) and (c) above, equation (4.1) can be equivalently written as

$$x''(t) = R(x(t), x'(t)) + F(t, x_t) - \varepsilon x'(t). \tag{4.2}$$

Notice that, if the above equation reduces to the so-called *inertial equation*

$$x''(t) = R(x(t), x'(t)),$$

one obtains the geodesics of X as solutions.

Equation (4.2) can be written as a RFDE on TX as follows:

$$\begin{cases} x'(t) = y(t), \\ y'(t) = R(x(t), y(t)) + F(t, x_t) - \varepsilon y(t). \end{cases}$$

This makes sense since the map $G : \mathbb{R} \times C((-\infty, 0], TX) \to \mathbb{R}^s \times \mathbb{R}^s$, defined as

$$G(t, (\varphi, \psi)) = (\psi(0), R(\varphi(0), \psi(0)) + F(t, \varphi) - \varepsilon\psi(0)), \tag{4.3}$$

is a functional field on TX. Indeed, observe that the condition

$$G(t, (\varphi, \psi)) \in T_{(\varphi(0), \psi(0))}TX$$

is verified for all $(t, (\varphi, \psi)) \in \mathbb{R} \times C((-\infty, 0], TX)$ (see, for example, [9] for more details).

Theorem 4.1 below extends two results obtained in [2] and [4]. The proof is based on Theorem 3.1 above.

Theorem 4.1 *Let $X \subseteq \mathbb{R}^s$ be a compact boundaryless manifold whose Euler–Poincaré characteristic $\chi(X)$ is different from zero, and $F : \mathbb{R} \times C((-\infty, 0], X) \to \mathbb{R}^s$ a functional field which is T-periodic in the first variable. Suppose that F is bounded and verifies (H). Then, the equation (4.1) has a forced oscillation.*

Proof. As we already pointed out, the equation (4.1) is equivalent to the following first order system on TX:

$$\begin{cases} x'(t) = y(t), \\ y'(t) = R(x(t), y(t)) + F(t, x_t) - \varepsilon y(t). \end{cases} \tag{4.4}$$

Define $G : \mathbb{R} \times C((-\infty, 0], TX) \to \mathbb{R}^s \times \mathbb{R}^s$ as in (4.3). Then, G is a T-periodic functional field on TX.

Given $c > 0$, define

$$M_c = \big\{(q, v) \in TX : \|v\| \leq c\big\}.$$

It is not difficult to show that $M_c \subseteq TX$ is a compact ∂-manifold in $\mathbb{R}^s \times \mathbb{R}^s$ with boundary

$$\partial M_c = \big\{(q, v) \in M_c : \|v\| = c\big\}.$$

Now, let G_c be the restriction of the map G to $\mathbb{R} \times C((-\infty, 0], M_c)$. Clearly, G_c is a T-periodic functional field on M_c which verifies (H). Let us show that G_c is bounded. Indeed, the map F is bounded by assumption, and the compactness of M_c implies that the restriction of the map $(q, v) \mapsto (v, R(q, v) - \varepsilon v)$ to M_c is bounded as well. Therefore G_c is bounded, being the sum of two bounded maps.

We claim that, if $c > 0$ is large enough, then G_c is inward on M_c. To see this, observe that the tangent cone of M_c at $(q, v) \in \partial M_c$ is the half subspace of $T_{(q,v)}M_c$ given by

$$C_{(q,v)}M_c = \big\{(\dot{q}, \dot{v}) \in T_{(q,v)}(TX) : \langle v, \dot{v}\rangle \leq 0\big\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^s$. Thus we have to show that, if $c > 0$ is large enough, then $G_c(t, (\varphi, \psi))$ belongs to $C_{(\varphi(0),\psi(0))}(M_c)$ for any $t \in \mathbb{R}$ and any pair $(\varphi, \psi) \in C((-\infty, 0], M_c)$ such that $(\varphi(0), \psi(0)) \in \partial M_c$. That is, we need to prove that, for any t and any pair (φ, ψ) with $\|\psi(0)\| = c$, we have

$$\begin{aligned} &\big\langle \psi(0), R(\varphi(0), \psi(0)) + F(t, \varphi) - \varepsilon\psi(0)\big\rangle = \\ &\big\langle \psi(0), R(\varphi(0), \psi(0))\big\rangle + \big\langle \psi(0), F(t, \varphi)\big\rangle - \varepsilon\big\langle \psi(0), \psi(0)\big\rangle \leq 0. \end{aligned}$$

To see this, observe that $\big\langle \psi(0), R(\varphi(0), \psi(0))\big\rangle = 0$ since $R(\varphi(0), \psi(0))$ belongs to $(T_{\varphi(0)}X)^{\perp}$. Moreover, $\big\langle \psi(0), \psi(0)\big\rangle = c^2$ since $(\varphi(0), \psi(0)) \in \partial M_c$, and

$$\big\langle \psi(0), F(t, \varphi)\big\rangle \leq \|\psi(0)\|\|F(t, \varphi)\| \leq K\|\psi(0)\|,$$

where K is such that $\|F(t, \varphi)\| \leq K$ for all $(t, \varphi) \in \mathbb{R} \times C((-\infty, 0], X)$. Thus,

$$\big\langle \psi(0), R(\varphi(0), \psi(0)) + F(t, \varphi) - \varepsilon\psi(0)\big\rangle \leq Kc - \varepsilon c^2.$$

This shows that, if we choose $c > K/\varepsilon$, then G_c is a strictly inward functional field on M_c, as claimed.

Finally, observe that $\chi(M_c) = \chi(X) \neq 0$ since M_c and X are homotopically equivalent (X being a deformation retract of TX), and $\chi(X) \neq 0$ by assumption. Therefore, given $c > K/\varepsilon$, we can apply Theorem 3.1 with $M = M_c$ and $f = G_c$, and we get that system (4.4) admits a T-periodic solution in M_c. This completes the proof. □

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