FORCED OSCILLATIONS FOR DELAY MOTION EQUATIONS ON MANIFOLDS

PIERLUIGI BENEVIERI, ALESSANDRO CALAMAI, MASSIMO FURI, AND MARIA PATRIZIA PERA

Abstract. We prove an existence result for T-periodic solutions of a T-periodic second order delay differential equation on a boundaryless compact manifold with nonzero Euler-Poincaré characteristic. The approach is based on an existence result recently obtained by the authors for first order delay differential equations on compact manifolds with boundary.

1. Introduction

Let $M \subseteq \mathbb{R}^k$ be a smooth boundaryless manifold and let

$$f : \mathbb{R} \times M \times M \to \mathbb{R}^k$$

be a continuous map which is T-periodic in the first variable and tangent to M in the second one; that is,

$$f(t+T, q, \tilde{q}) = f(t, q, \tilde{q}) \in T_qM, \quad \forall\, (t, q, \tilde{q}) \in \mathbb{R} \times M \times M,$$

where $T_qM \subseteq \mathbb{R}^k$ denotes the tangent space of M at q. Consider the following second order delay differential equation on M:

$$x''_\pi(t) = f(t, x(t), x(t-\tau)) - \varepsilon x'(t), \tag{1.1}$$

where, regarding (1.1) as a motion equation,

(1) $x''_\pi(t)$ stands for the tangential part of the acceleration $x''(t) \in \mathbb{R}^k$ at the point $x(t)$;
(2) the frictional coefficient ε is a positive real constant;
(3) $\tau > 0$ is the delay.

In this paper we prove that equation (1.1) admits at least one forced oscillation, provided that the constraint M is compact with nonzero Euler–Poincaré characteristic and that $T \geq \tau$. This generalizes a theorem of the last two authors regarding the undelayed case (see [3]). Our result will be deduced from an existence theorem for first order delay equations on compact manifolds with boundary recently obtained by the authors (see [1, Theorem 4.6]). The possibility of reducing (1.1) to the first order equation treated in [1] is due to the fact that any second order differential equation on M is equivalent to a first order system on the tangent bundle TM of M. The difficulty arising from the noncompactness of TM will be removed by restricting the search for T-periodic solutions to a convenient compact manifold with boundary contained in TM. The choice of such a manifold is suggested by *a priori* estimates on the set of all the possible T-periodic solutions of equation (1.1).

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These estimates are made possible by the compactness of M and the presence of the positive frictional coefficient ε.

We ask whether or not the existence of forced oscillations holds true even in the frictionless case, provided that the constraint M is compact with nonzero Euler–Poincaré characteristic. We believe the answer to this question is affirmative; but, as far as we know, this problem is still unsolved even in the undelayed case.

An affirmative answer regarding the special case $M = S^2$ (the spherical pendulum) can be found in [4] (see also [5] for the extension to the case $M = S^{2n}$).

We point out that the assumption $T \geq \tau$ is crucial in this paper, since our result is deduced from Theorem 2.1 below, whose proof, given in [1], is based on the fixed point index theory for locally compact maps applied to a Poincaré-type T-translation operator which is a locally compact map if and only if $T \geq \tau$. In a forthcoming paper we will tackle the case $0 < T < \tau$, in which this operator is not even locally condensing.

2. Second order delay differential equations on manifolds

Let, as before, M be a compact smooth boundaryless manifold in $\mathbb{R}^k$. Given $q \in M$, let T_qM and $(T_qM)^\perp$ denote, respectively, the tangent and the normal space of M at q. Since $\mathbb{R}^k = T_qM \oplus (T_qM)^\perp$, any vector $u \in \mathbb{R}^k$ can be uniquely decomposed into the sum of the *parallel* (or *tangential*) *component* $u_\pi \in T_qM$ *of* u *at* q and the *normal component* $u_\nu \in (T_qM)^\perp$ *of* u *at* q. By

$$TM = \left\{(q,v) \in \mathbb{R}^k \times \mathbb{R}^k :\ q \in M,\ v \in T_qM\right\}$$

we denote the *tangent bundle of* M, which is a smooth manifold containing a natural copy of M via the embedding $q \mapsto (q,0)$. The natural projection of TM onto M is just the restriction (to TM as domain and to M as codomain) of the projection of $\mathbb{R}^k \times \mathbb{R}^k$ onto the first factor.

Given, as in the Introduction, a continuous map $f : \mathbb{R} \times M \times M \to \mathbb{R}^k$ which is T-periodic in the first variable and tangent to M in the second one, consider the following delay motion equation on M:

$$x''_\pi(t) = f(t, x(t), x(t-\tau)) - \varepsilon x'(t), \tag{2.1}$$

where

i) $x''_\pi(t)$ stands for the parallel component of the acceleration $x''(t) \in \mathbb{R}^k$ at the point $x(t)$;
ii) the frictional coefficient ε and the delay τ are positive real constants.

By a *solution* of (2.1) we mean a continuous function $x : J \to M$, defined on a (possibly unbounded) real interval, with length greater than τ, which is of class C^2 on the subinterval $(\inf J + \tau, \sup J)$ of J and verifies

$$x''_\pi(t) = f(t, x(t), x(t-\tau)) - \varepsilon x'(t)$$

for all $t \in J$ with $t > \inf J + \tau$. A *forced oscillation of* (2.1) is a solution which is T-periodic and globally defined on $J = \mathbb{R}$.

It is known that, associated with $M \subseteq \mathbb{R}^k$, there exists a unique smooth map $r : TM \to \mathbb{R}^k$, called the *reactive force* (or *inertial reaction*), with the following properties:

(a) $r(q,v) \in (T_qM)^\perp$ for any $(q,v) \in TM$;
(b) r is quadratic in the second variable;

(c) any C^2 curve $\gamma : (a,b) \to M$ verifies the condition
$$\gamma''_\nu(t) = r(\gamma(t), \gamma'(t)), \quad \forall t \in (a,b),$$
i.e., for each $t \in (a,b)$, the normal component $\gamma''_\nu(t)$ of $\gamma''(t)$ at $\gamma(t)$ equals $r(\gamma(t), \gamma'(t))$.

The map r is strictly related to the second fundamental form on M and may be interpreted as the reactive force due to the constraint M.

By condition (c) above, equation (2.1) can be equivalently written as
$$x''(t) = r(x(t), x'(t)) + f(t, x(t), x(t-\tau)) - \varepsilon x'(t). \tag{2.2}$$

Notice that, if the above equation reduces to the so-called *inertial equation*
$$x''(t) = r(x(t), x'(t)),$$
one obtains the geodesics of M as solutions.

Equation (2.2) can be written as a first order differential system on TM as follows:
$$\begin{cases} x'(t) = y(t) \\ y'(t) = r(x(t), y(t)) + f(t, x(t), x(t-\tau)) - \varepsilon y(t). \end{cases}$$
This makes sense since the map
$$g : \mathbb{R} \times TM \times M \to \mathbb{R}^k \times \mathbb{R}^k, \quad g(t,(q,v),\tilde{q}) = (v, r(q,v) + f(t,q,\tilde{q}) - \varepsilon v) \tag{2.3}$$
verifies the condition $g(t,(q,v),\tilde{q}) \in T_{(q,v)}TM$ for all $(t,(q,v),\tilde{q}) \in \mathbb{R} \times TM \times M$ (see, for example, [2] for more details).

Theorem 2.1 below, which is a straightforward consequence of Theorem 4.6 in [1], will play a crucial role in the proof of our result (Theorem 2.2). Its statement needs some preliminary definitions.

Let $X \subseteq \mathbb{R}^s$ be a smooth manifold with (possibly empty) boundary ∂X. Following [1], we say that a continuous map $F : \mathbb{R} \times X \times X \to \mathbb{R}^s$ is *tangent to X in the second variable* or, for short, that F is a *vector field (on X)* if $F(t,p,\tilde{p}) \in T_pX$ for all $(t,p,\tilde{p}) \in \mathbb{R} \times X \times X$. A vector field F will be said *inward* (to X) if for any $(t,p,\tilde{p}) \in \mathbb{R} \times \partial X \times X$ the vector $F(t,p,\tilde{p})$ points inward at p. Recall that, given $p \in \partial X$, the set of the tangent vectors to X pointing inward at p is a closed half-subspace of T_pX, called *inward half-subspace* of T_pX (see e.g. [6]) and here denoted T_p^-X.

Theorem 2.1. *Let $X \subseteq \mathbb{R}^s$ be a compact manifold with (possibly empty) boundary, whose Euler–Poincaré characteristic $\chi(X)$ is different from zero. Let $\tau > 0$ and let $F : \mathbb{R} \times X \times X \to \mathbb{R}^s$ be an inward vector field on X which is T-periodic in the first variable, with $T \geq \tau$. Then, the delay differential equation*
$$x'(t) = F(t, x(t), x(t-\tau)) \tag{2.4}$$
has a T-periodic solution.

The result of this paper is the following.

Theorem 2.2. *Assume that the period T of f is not less than the delay τ and that the Euler–Poincaré characteristic of M is different from zero. Then, the equation* (2.1) *has a forced oscillation.*

Proof. As we already pointed out, the equation (2.1) is equivalent to the following first order system on TM:

$$\begin{cases} x'(t) = y(t) \\ y'(t) = r(x(t), y(t)) + f(t, x(t), x(t-\tau)) - \varepsilon y(t). \end{cases} \tag{2.5}$$

Define $F : \mathbb{R} \times TM \times TM \to \mathbb{R}^k \times \mathbb{R}^k$ by

$$F(t, (q, v), (\tilde{q}, \tilde{v})) = (v, r(q, v) + f(t, q, \tilde{q}) - \varepsilon v).$$

Notice that the map F is a vector field on TM which is T-periodic in the first variable.

Given $c > 0$, set

$$X_c = (TM)_c = \big\{(q, v) \in M \times \mathbb{R}^k : v \in T_qM,\ \|v\| \le c\big\}.$$

It is not difficult to show that X_c is a compact manifold in $\mathbb{R}^k \times \mathbb{R}^k$ with boundary

$$\partial X_c = \big\{(q, v) \in M \times \mathbb{R}^k : v \in T_qM,\ \|v\| = c\big\}.$$

Observe that

$$T_{(q,v)}(X_c) = T_{(q,v)}(TM)$$

for all $(q, v) \in X_c$. Moreover, $\chi(X_c) = \chi(M)$ since X_c and M are homotopically equivalent (M being a deformation retract of TM).

We claim that, if $c > 0$ is large enough, then F is an inward vector field on X_c. To see this, let $(q, v) \in \partial X_c$ be fixed, and observe that the inward half-subspace of $T_{(q,v)}(X_c)$ is

$$T^-_{(q,v)}(X_c) = \big\{(\dot{q}, \dot{v}) \in T_{(q,v)}(TM) : \langle v, \dot{v}\rangle \le 0\big\},$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^k$. We have to show that if c is large enough then $F(t, (q, v), (\tilde{q}, \tilde{v}))$ belongs to $T^-_{(q,v)}(X_c)$ for any $t \in \mathbb{R}$ and $(\tilde{q}, \tilde{v}) \in TM$; that is,

$$\langle v, r(q, v) + f(t, q, \tilde{q}) - \varepsilon v\rangle = \langle v, r(q, v)\rangle + \langle v, f(t, q, \tilde{q})\rangle - \varepsilon\langle v, v\rangle \le 0$$

for any $t \in \mathbb{R}$ and $(\tilde{q}, \tilde{v}) \in TM$. Now, $\langle v, r(q, v)\rangle = 0$ since $r(q, v)$ belongs to $(T_qM)^\perp$. Moreover, $\langle v, v\rangle = c^2$ since $(q, v) \in \partial X_c$, and

$$\langle v, f(t, q, \tilde{q})\rangle \le \|v\|\|f(t, q, \tilde{q})\| \le K\|v\|,$$

where

$$K = \max\big\{\|f(t, q, \tilde{q})\| : (t, q, \tilde{q}) \in \mathbb{R} \times M \times M\big\}.$$

Thus,

$$\langle v, r(q, v) + f(t, q, \tilde{q}) - \varepsilon v\rangle \le Kc - \varepsilon c^2.$$

This shows that, if we choose $c > K/\varepsilon$, then F is an inward vector field on X_c, as claimed. Therefore, given $c > K/\varepsilon$, Theorem 2.1 implies that system (2.5) admits a T-periodic solution in X_c, and this completes the proof. □

It is evident from this proof that the result holds true even if we replace

$$f(t, q, \tilde{q}) - \varepsilon v$$

by a T-periodic force $g(t, (q, v), (\tilde{q}, \tilde{v})) \in T_qM$ satisfying the following assumption: *there exists* $c > 0$ *such that* $\langle g(t, (q, v), (\tilde{q}, \tilde{v})), v\rangle \le 0$ *for any*

$$(t, (q, v), (\tilde{q}, \tilde{v})) \in \mathbb{R} \times TM \times TM$$

such that $\|v\| = c$.

We point out that, in the above theorem, the condition $\chi(M) \neq 0$ cannot be dropped. Consider for example the equation

$$\theta''(t) = a - \varepsilon\theta'(t), \quad t \in \mathbb{R}, \tag{2.6}$$

where a is a nonzero constant and $\varepsilon > 0$. Equation (2.6) can be regarded as a second order ordinary differential equation on the unit circle $S^1 \subseteq \mathbb{C}$, where θ represents an angular coordinate. In this case, a solution $\theta(\cdot)$ of (2.6) is periodic of period $T > 0$ if and only if for some $k \in \mathbb{Z}$ it satisfies the boundary conditions

$$\begin{cases} \theta(T) - \theta(0) = 2k\pi, \\ \theta'(T) - \theta'(0) = 0. \end{cases}$$

Notice that the applied force a represents a nonvanishing autonomous vector field on S^1. Thus, it is periodic of arbitrary period. However, simple calculations show that there are no T-periodic solutions of (2.6) if $T \neq 2\pi\varepsilon/a$.

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Pierluigi Benevieri, Massimo Furi, and Maria Patrizia Pera,
Dipartimento di Matematica Applicata ''Giovanni Sansone''
Università degli Studi di Firenze
Via S. Marta 3
I-50139 Firenze, Italy

Alessandro Calamai
Dipartimento di Scienze Matematiche
Università Politecnica delle Marche
Via Brecce Bianche
I-60131 Ancona, Italy.

e-mail addresses:
pierluigi.benevieri@unifi.it
calamai@math.unifi.it, calamai@dipmat.univpm.it
massimo.furi@unifi.it
mpatrizia.pera@unifi.it