DELAY DIFFERENTIAL EQUATIONS ON MANIFOLDS AND APPLICATIONS TO MOTION PROBLEMS FOR FORCED CONSTRAINED SYSTEMS

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Abstract. We prove a global bifurcation result for T-periodic solutions of the delay T-periodic differential equation $x'(t) = \lambda f(t, x(t), x(t-1))$ on a smooth manifold (λ is a nonnegative parameter). The approach is based on the asymptotic fixed point index theory for C^1 maps due to Eells–Fournier and Nussbaum. As an application, we prove the existence of forced oscillations of motion problems on topologically nontrivial compact constraints. The result is obtained under the assumption that the frictional coefficient is nonzero, and we conjecture that it is still true in the frictionless case.

1. Introduction

Let $M \subseteq \mathbb{R}^k$ be a smooth manifold, possibly with boundary ∂M, and let $f : \mathbb{R} \times M \times M \to \mathbb{R}^k$ be a continuous map which is T-periodic in the first variable and tangent to M in the second one; meaning that

$$f(t+T, p, q) = f(t, p, q) \in T_pM\,, \quad \forall\,(t, p, q) \in \mathbb{R} \times M \times M,$$

where $T_pM \subseteq \mathbb{R}^k$ denotes the tangent space of M at p. Consider the following delay differential equation on M, depending on a parameter $\lambda \geq 0$:

$$x'(t) = \lambda f(t, x(t), x(t-1)). \tag{1.1}$$

A pair (λ, x), with $\lambda \geq 0$ and $x : \mathbb{R} \to M$ a T-periodic solution of (1.1) corresponding to λ, is called a *T-periodic pair* of (1.1). The set of T-periodic pairs is regarded as a subset of $[0, +\infty) \times C_T(M)$, where $C_T(M)$ is the set of continuous T-periodic maps from $\mathbb{R}$ to M with the metric induced by the Banach space $C_T(\mathbb{R}^k)$ of continuous T-periodic $\mathbb{R}^k$-valued maps (with the standard supremum norm). A T-periodic pair of the type $(0, x)$ is said to be *trivial*. In this case x is a constant M-valued map and, therefore, one may think of M as the set of trivial T-periodic pairs.

In two recent papers, [1] and [3], we investigated the structure of the set of *T-periodic pairs* of (1.1). In the first one we tackled the case when the period T is not smaller than the delay, that, without loss of generality, we supposed to be 1. We also assumed that M is compact, possibly with boundary, with nonzero Euler–Poincaré characteristic, and that f satisfies a natural inward condition along ∂M. Under these assumptions, we proved the existence of an unbounded (with respect to λ) connected branch of nontrivial T-periodic pairs whose closure intersects the set of the trivial T-periodic pairs in the so-called *set of bifurcation points*. Thus, this result extends an analogous one of the last two authors for the undelayed case (see [8] and [9]). The approach followed in [1] consists in applying to a Poincaré-type T-translation operator, acting on the space $C([-1, 0], M)$, the fixed point index theory for locally compact maps on ANRs. For this purpose, the assumption $T \geq 1$ is crucial, since otherwise the compactness of the Poincaré operator fails.

In [3] we dealt with the case of arbitrary period $T > 0$, and we proved a global bifurcation result as in [1], but with the additional assumption that $\partial M = \emptyset$. This extra condition is due to the fact that, when $0 < T < 1$, the Poincaré operator is not locally compact and, consequently, we applied the fixed point index theory of Eells–Fournier (see [6]) and Nussbaum (see [18]) instead of the classical one. This theory regards eventually condensing C^1 maps on C^1-ANRs and cannot be applied when $\partial M \neq \emptyset$. In fact, in this case M is not a C^1 retract of any of its neighborhoods and, consequently, the argument used in [5] to prove that $C([-1,0], M)$ is a C^1-ANR fails. Our purpose here is to overcome this difficulty, in order to give a complete extension of the results in [1] and [3]. Since $M \backslash \partial M$ is a boundaryless manifold which is not compact when $\partial M \neq \emptyset$, unless otherwise stated we will assume that M is boundaryless, but not necessarily compact. In this context, we prove a global bifurcation result, Theorem 3.13, whose consequence, Corollary 3.16, provides the desired extension of the results in [1] and [3].

Since we do not assume $T \geq 1$, as in [3] our approach is based on the fixed point index theory by Eells–Fournier and Nussbaum.

We conclude the paper with an application to motion problems for forced constrained systems. Precisely, we consider the following second order delay differential equation on a boundaryless manifold $X \subseteq \mathbb{R}^s$:

$$x''_\pi(t) = F(t, x(t), x(t-1)) - \varepsilon x'(t), \tag{1.2}$$

where, regarding (1.2) as a motion equation,

(1) $x''_\pi(t)$ stands for the tangential part of the acceleration $x''(t) \in \mathbb{R}^s$ at the point $x(t) \in X$,
(2) the frictional coefficient ε is a positive constant,
(3) the applied force $F : \mathbb{R} \times X \times X \to \mathbb{R}^s$ is continuous, T-periodic in the first variable and tangent to X in the second one.

Theorem 4.1 asserts that, whatever is $T > 0$, the equation (1.2) admits at least one *forced oscillation* (i.e. a T-periodic solution) provided that the constraint X is compact with nonzero Euler–Poincaré characteristic. Such a result generalizes a theorem of the last two authors regarding the undelayed case (see [10]) as well as a theorem given in [2] in which the period T is not less than the delay. To get Theorem 4.1 we apply Corollary 3.15 to a first order equation on the noncompact tangent bundle $TX \subseteq \mathbb{R}^{2s}$ which is equivalent to (1.2).

As far as we know, when the frictional coefficient ε is zero, the problem of the existence of forced oscillations of (1.2) is still open, even in the undelayed case. An affirmative answer, in the undelayed situation, regarding the special constraint $X = S^2$ (the spherical pendulum) can be found in [11] (see also [13] for the extension to the case $X = S^{2n}$).

2. Preliminaries

Throughout the paper M will be a boundaryless smooth manifold embedded in $\mathbb{R}^k$.

Let $g : \mathbb{R} \times M \times M \to \mathbb{R}^k$ be a continuous map. We say that g is *tangent to M in the second variable* or, for short, that g is a *vector field on M* if $g(t,p,q) \in T_pM$ for all $(t,p,q) \in \mathbb{R} \times M \times M$, where $T_pM \subseteq \mathbb{R}^k$ denotes the tangent space of M at p. In this paper we are interested in delay differential equations of the type

$$x'(t) = g(t, x(t), x(t-1)), \tag{2.1}$$

where $g : \mathbb{R} \times M \times M \to \mathbb{R}^k$ is a vector field on M. We will regard (2.1) as a particular case of the equation

$$x'(t) = g(t, x(t), x(t-s)), \tag{2.2}$$

where the (constant) time lag s belongs to $[0, 1]$. For technical reasons, no matter what is the delay s, by a *solution* of (2.2) we shall mean a continuous function $x : J \to M$, defined on a (possibly unbounded) interval with length greater than 1, which is of class C^1 on the subinterval $(\inf J + 1, \sup J)$ of J and verifies $x'(t) = g(t, x(t), x(t-s))$ for all $t \in J$ with $t > \inf J + 1$.

Given a continuous map $\varphi : [-1, 0] \to M$ and $s \in [0, 1]$, consider the following initial value problem:

$$\begin{cases} x'(t) = g(t, x(t), x(t-s)), & \\ x(t) = \varphi(t), & t \in [-1, 0]. \end{cases} \tag{2.3}$$

A *solution* of this problem is a solution $x : J \to M$ of (2.2) such that $J \supset [-1, 0]$ and $x(t) = \varphi(t)$ for all $t \in [-1, 0]$.

Consider a compact subset K of M and let $x : J \to M$ be a maximal solution of problem (2.3). As in the ODE case, one can prove that, if the image of x is contained in K, then $\sup J = +\infty$. In particular, when the manifold M is compact, then problem (2.3) admits a solution defined (at least) on $[-1, +\infty)$.

The following technical lemma regards the uniqueness and the continuous dependence on data of solutions of problem (2.3) in the case when g is of class C^1. The proof is standard in the theory of ODEs (see e.g. [4] and [16]) and can be adapted to the delay case. Therefore, it will be omitted.

Lemma 2.1. *Let g be a C^1 vector field on M. Then, problem (2.3) has a unique solution which depends continuously on data. More precisely, let $\{g_n\}$ be a sequence of C^1 vector fields on M which converges uniformly to g, and $\{\varphi_n\}$ a sequence of continuous maps from $[-1, 0]$ to M which converges uniformly to φ. Denote by $x_n(\cdot)$ the maximal solution of the initial value problem*

$$\begin{cases} x'(t) = g_n(t, x(t), x(t-s)), & t > 0, \\ x(t) = \varphi_n(t), & t \in [-1, 0]. \end{cases}$$

Let I be a compact interval contained in the domain of the maximal solution $x_0(\cdot)$ of (2.3). Then, for n sufficiently large, $x_n(\cdot)$ is defined on I and $x_n(t)$ converges to $x_0(t)$ uniformly on I.

As said in the Introduction, our approach to the study of equation (2.2) is based on the fixed point index theory for eventually compact C^1 maps between C^1-ANRs. This index has been defined independently by Eells–Fournier in [6] and Nussbaum in [18] for the more general class of eventually condensing maps.

Recall that a Banach manifold X is a C^r-ANR ($r \in \mathbb{N} \cup \{\infty\}$) if there exist an embedding j of class C^r of X into a Banach space E, an open neighborhood W of $j(X)$ in E, and a retraction of W onto $j(X)$ of class C^r.

Recall also that, given a topological space Y and a subset A of Y, a continuous map $k : A \to Y$ is said to be *eventually compact* if for some $n \in \mathbb{N}$ the n-th iterate k^n is defined on A and $k^n(A)$ is contained in a compact subset of Y. Moreover, given a compact interval I, a continuous map $H : I \times A \to Y$ is called an *eventually compact homotopy* if the map

$$(\lambda, x) \mapsto (\lambda, H(\lambda, x)), \quad (\lambda, x) \in I \times A,$$

is eventually compact.

Let X be a C^1-ANR and consider an eventually compact map $k : W \to X$, defined on an open subset W of X, and of class C^1. Given an open subset V of W, if the set of fixed points of k in V is compact, the pair (k, V) is called *admissible*. Then (as proved in [6] and [18]) it is possible to associate to any admissible pair (k, V) an integer $\mathrm{ind}_X(k, V)$ - the *fixed point index* of k in V - which satisfies the classical properties of the fixed point index theory. Obviously, in this new theory, the continuity assumption of homotopies is strengthened by assuming the C^1 regularity, and the compactness is weakened by supposing the eventual compactness.

As far as we know, whether or not the above theory holds for the merely C^0 case is still an open problem.

To conclude these preliminaries, let us recall some basic notions on degree theory for tangent vector fields on differentiable manifolds. Let $v : M \to \mathbb{R}^k$ be a continuous (autonomous) tangent vector field on M, and let U be an open subset of M. We say that the pair (v, U) is *admissible* (or, equivalently, that v is admissible on U) if $v^{-1}(0) \cap U$ is compact. In this case one can assign to the pair (v, U) an integer, $\deg(v, U)$, called the *degree* (or *index*, or *Euler characteristic*, or *rotation*) of the tangent vector field v on U which, roughly speaking, counts algebraically the number of zeros of v in U (for general references see e.g. [14, 15, 17, 19]). Notice that the condition for $v^{-1}(0) \cap U$ to be compact is clearly satisfied if U is a relatively compact open subset of M and $v(p) \neq 0$ for all p in the boundary of U.

As a consequence of the Poincaré–Hopf theorem, when M is compact, $\deg(v, M)$ equals $\chi(M)$, the Euler–Poincaré characteristic of M. In the particular case when U is an open subset of $\mathbb{R}^k$, $\deg(v, U)$ is just the classical Brouwer degree, $\deg(v, U, 0)$, of the map v on U with respect to zero.

All the standard properties of the Brouwer degree for continuous maps on open subsets of Euclidean spaces, such as homotopy invariance, excision, additivity, existence, still hold in the more general context of differentiable manifolds. To see this, one can use an equivalent definition of degree of a tangent vector field based on the fixed point index theory as presented in [9] and [12]. Let us point out that no orientability of M is required for the degree of a tangent vector field to be defined.

Observe that, if (v, U) is admissible, then

$$\deg(v, U) = (-1)^m \deg(-v, U), \tag{2.4}$$

where m denotes the dimension of M.

We recall that, when v is a C^1 tangent vector field on M, a zero $p \in M$ of v is said to be *nondegenerate* if $v'(p) : T_pM \to \mathbb{R}^k$ is one-to-one. Since the condition $v(p) = 0$ implies that $v'(p)$ maps T_pM into itself (see e.g. [17]), then $v'(p)$ is actually an isomorphism of T_pM. Thus, the determinant $\det(v'(p))$ is nonzero and its sign is called the *index* of v at p.

In the particular case when an admissible pair (v, U) is *regular* (i.e. v is smooth with only nondegenerate zeros), one can show that $\deg(v, U)$ coincides with the sum of the indices at the zeros of v in U. This makes sense, since $v^{-1}(0) \cap U$ is compact (v being admissible in U) and discrete; therefore, the sum is finite.

3. Branches of periodic solutions

From now on we will adopt the following notation. By $C([-1, 0], M)$ we mean the metric space of the M-valued continuous functions defined on $[-1, 0]$ with the metric induced by the Banach space $C([-1, 0], \mathbb{R}^k)$. Given $T > 0$, by $C_T(\mathbb{R}^k)$ we denote the Banach space of the continuous T-periodic maps $x : \mathbb{R} \to \mathbb{R}^k$ (with the standard supremum norm) and by $C_T(M)$ we mean the metric subspace of $C_T(\mathbb{R}^k)$ of the M-valued maps.

We point out that the metric spaces $C([-1, 0], M)$ and $C_T(M)$ need not be complete, unless M is closed in $\mathbb{R}^k$. However, due to the fact that M is locally compact, one can prove that $C([-1, 0], M)$ and $C_T(M)$ are locally complete.

It is known that $C([-1, 0], M)$ is a smooth infinite dimensional manifold (see e.g. [5]), and it is not difficult to prove (see e.g. [6]) that it is a C^1-ANR as well. In fact, it is a C^1 retract of the open subset $C([-1, 0], U)$ of $C([-1, 0], \mathbb{R}^k)$, $U \subseteq \mathbb{R}^k$ being a tubular neighborhood of M.

Let $f : \mathbb{R} \times M \times M \to \mathbb{R}^k$ be a vector field on M which is T-periodic in the first variable. Consider the following delay differential equation depending on a parameter $\lambda \geq 0$:

$$x'(t) = \lambda f(t, x(t), x(t-1)). \tag{3.1}$$

We will say that $(\lambda, x) \in [0, +\infty) \times C_T(M)$ is a *T-periodic pair* (of (3.1)) if $x : \mathbb{R} \to M$ is a T-periodic solution of (3.1) corresponding to λ. A T-periodic pair of the type $(0, x)$ is said to be *trivial*. Notice that in this case the function x is constant.

A pair $(\lambda, \varphi) \in [0, +\infty) \times C([-1, 0], M)$ will be called a *T-starting pair* (of (3.1)) if there exists $x \in C_T(M)$ such that $x(t) = \varphi(t)$ for all $t \in [-1, 0]$ and (λ, x) is a T-periodic pair. A T-starting pair of the type $(0, \varphi)$ will be called *trivial*. In this case the map φ is constant, being the restriction of a constant map defined on $\mathbb{R}$.

Clearly, the map $\rho : (\lambda, x) \mapsto (\lambda, \varphi)$ which associates to a T-periodic pair (λ, x) the corresponding T-starting pair (λ, φ) is continuous, φ being the restriction of x to the interval $[-1, 0]$. Moreover, if f is C^1, from Lemma 2.1 it follows that ρ is actually a homeomorphism between the set $\Gamma \subseteq [0, +\infty) \times C_T(M)$ of the T-periodic pairs and the set $\Sigma \subseteq [0, +\infty) \times C([-1, 0], M)$ of the T-starting pairs.

It is not difficult to see that the set Γ is closed in $[0, +\infty) \times C_T(M)$ (and locally closed in $[0, +\infty) \times C_T(\mathbb{R}^k)$). It is consequently locally complete, as a closed subset of a locally complete space. Moreover, using Ascoli's Theorem, one can show that Γ is actually a locally compact space, and this fact will turn out to be useful in order to get our main result.

Given $p \in M$, we denote by $\bar{p} \in C_T(M)$ the constant map $t \mapsto p$, $t \in \mathbb{R}$, and by $\hat{p} \in C([-1, 0], M)$ the constant map $t \mapsto p$, $t \in [-1, 0]$. With this notation, a trivial T-periodic pair is of the form $(0, \bar{p})$, and the corresponding trivial T-starting pair is $(0, \hat{p})$; that is, $\rho(0, \bar{p}) = (0, \hat{p})$. Clearly, M can be identified in a natural way both with the set of the trivial T-periodic pairs $\{0\} \times \{\bar{p} : p \in M\} \subseteq \Gamma$ and the set of the trivial T-starting pairs $\{0\} \times \{\hat{p} : p \in M\} \subseteq \Sigma$. In other words, the restriction of the map ρ to $\{0\} \times \{\bar{p} : p \in M\} \subseteq [0, +\infty) \times C_T(M)$ as domain and to $\{0\} \times \{\hat{p} : p \in M\} \subseteq [0, +\infty) \times C([-1, 0], M)$ as codomain can be regarded as the identity on M.

An element $p_0 \in M$ will be called a *bifurcation point* of the equation (3.1) if every neighborhood of $(0, \bar{p}_0)$ in $[0, +\infty) \times C_T(M)$ contains a nontrivial T-periodic pair (i.e. a T-periodic pair (λ, x) with $\lambda > 0$). Roughly speaking, p_0 is a bifurcation point if, for $\lambda > 0$ sufficiently small, there are T-periodic orbits of (3.1) that rotate arbitrarily close to p_0.

Let $w : M \to \mathbb{R}^k$ be the mean value tangent vector field given by

$$w(p) = \frac{1}{T} \int_0^T f(t, p, p)\, dt.$$

Throughout the paper, the tangent vector field w will play a crucial role in obtaining bifurcation results for equation (3.1).

The following result provides a necessary condition for $p_0 \in M$ to be a bifurcation point. The easy proof is given for the sake of completeness.

Theorem 3.1. *Assume that $p_0 \in M$ is a bifurcation point of the equation* (3.1)*. Then, the tangent vector field w vanishes at p_0.*

Proof. By assumption there exists a sequence $\{(\lambda_n, x_n)\}$ of T-periodic pairs such that $\lambda_n > 0$, $\lambda_n \to 0$, and $x_n(t) \to p_0$ uniformly on $\mathbb{R}$. Given $n \in \mathbb{N}$, since $x_n(T) = x_n(0)$ and $\lambda_n \neq 0$, we get

$$\int_0^T f(t, x_n(t), x_n(t-1))\, dt = 0,$$

and the assertion follows passing to the limit. □

Let Ω be an open subset of $[0, +\infty) \times C_T(M)$. Our main result (Theorem 3.13 below) provides a sufficient condition for the existence of a bifurcation point p in M with $(0, \bar{p}) \in \Omega$. More precisely, we give conditions which ensure the existence of a connected subset of Ω of nontrivial T-periodic pairs of equation (3.1) whose closure in Ω is not compact and intersects the set of trivial T-periodic pairs contained in Ω (a *global bifurcating branch* for short). Notice that, because of Ascoli's Theorem, a global bifurcating branch either is unbounded or, if bounded, must contain points which are arbitrarily close to the boundary of Ω in $[0, +\infty) \times C_T(M)$.

From now on we will adopt the following notation. Given a subset Y of a metric space Z, we will denote by $\overline{Y}$, $\mathring{Y}$ and $\operatorname{Fr} Y$ the closure, the interior and the boundary of Y, respectively.

Given a subset A of M, we denote by $\tilde{A}$ the subset of $C([-1,0], M)$ of all maps with values in A; i.e. $\tilde{A} = C([-1,0], A)$, and by $\hat{A}$ the subset of $\tilde{A}$ of constant maps; i.e. the set $\{\hat{p} : p \in A\}$. In particular, for simplicity, throughout the paper $\tilde{M}$ will stand for $C([-1,0], M)$. Notice that the closure, $\overline{\tilde{A}}$, of $\tilde{A}$ in $\tilde{M}$ coincides with $\tilde{\overline{A}}$. Moreover,

$$\operatorname{Fr} \tilde{A} = \big\{\varphi \in \tilde{\overline{A}} : \varphi(t) \in \operatorname{Fr} A \text{ for some } t \in [-1,0]\big\}.$$

Suppose, for the moment, that f is C^1 (this assumption will be removed in Theorem 3.13). Given $\lambda \geq 0$ and $\varphi \in \tilde{M}$, consider in M the following delay differential (initial value) problem:

$$\begin{cases} x'(t) = \lambda f(t, x(t), x(t-1)), & t > 0, \\ x(t) = \varphi(t), & t \in [-1,0]. \end{cases} \tag{3.2}$$

Given a positive integer n, define

$$D^n = \big\{(\lambda, \varphi) \in [0, +\infty) \times \tilde{M} : \text{the maximal solution of (3.2) is defined (at least) on } [-1, nT]\big\}.$$

With an argument analogous to that given in [16] for the ODE case, one can show that D^n is open in $[0, +\infty) \times \tilde{M}$. Moreover, $D^{n+1} \subseteq D^n$ for any n and, when the manifold M is compact, any set D^n coincides with $[0, +\infty) \times \tilde{M}$.

Notice that, when $\lambda = 0$, the solution of problem (3.2) is eventually constant and, when (λ, φ) is a T-starting pair of equation (3.1), the maximal solution of problem (3.2), being T-periodic, is defined on the whole real line. In other words, both $\{0\} \times \tilde{M}$ and the set of the T-starting pairs of (3.1) are contained in D^n for any n.

Given $(\lambda, \varphi) \in D^1$, denote by $x_{(\lambda,\varphi)}$ the maximal solution of problem (3.2). Consider the Poincaré-type operator

$$P : D^1 \to \tilde{M}$$

defined as $P(\lambda, \varphi)(t) = x_{(\lambda,\varphi)}(t + T)$, $t \in [-1, 0]$. The following lemma regards an important property of the operator P. The proof is standard and will be omitted.

Lemma 3.2. *The fixed points of $P(\lambda, \cdot)$ correspond to the T-periodic solutions of the equation* (3.1) *in the following sense: φ is a fixed point of $P(\lambda, \cdot)$ if and only if it is the restriction to $[-1, 0]$ of a T-periodic solution.*

We point out that D^n is the natural domain of definition of the n-th iterate of the map from D^1 to $[0,+\infty) \times \tilde{M}$ defined by $(\lambda, \varphi) \mapsto (\lambda, P(\lambda, \varphi))$.

In what follows we will denote by ν the smallest integer such that $\nu T \geq 1$, and we will consider P defined just on D^ν.

The proof of the next lemma can be carried out as in Lemmas 3.3 and 3.4 of [3].

Lemma 3.3. *The map $P : D^\nu \to \tilde{M}$ is of class C^1 and the ν-th iterate of the map from D^ν to $[0,+\infty) \times \tilde{M}$, defined as $(\lambda, \varphi) \mapsto (\lambda, P(\lambda, \varphi))$, is locally compact.*

The following theorem will be crucial in the proof of our main results. We recall that w is the mean value tangent vector field associated with f.

Theorem 3.4. *Let U be a relatively compact open subset of M. Assume that there are no zeros of w on the boundary of U. Then, there exists $\varepsilon > 0$ such that, for any $0 < \lambda \leq \varepsilon$, $\mathrm{ind}_{\tilde{M}}(P(\lambda, \cdot), \tilde{U})$ is well defined and equals* $\deg(-w, U)$.

The proof of Theorem 3.4 is divided in a number of intermediate results: from Lemma 3.6 to Lemma 3.10 below.

Given $\lambda \geq 0$ and $\varphi \in \tilde{M}$, consider in M the initial value problem

$$\begin{cases} x'(t) = \lambda f(t, x(t), x(t-s)), & t > 0, \\ x(t) = \varphi(t), & t \in [-1, 0], \end{cases} \tag{3.3}$$

depending on $s \in [0,1]$, and regard problem (3.2) as a particular case of this one.

As before, let ν be the smallest integer such that $\nu T \geq 1$ and (recalling the definition of solution of the above problem given in Section 2) define the following subset of D^ν:

$$\Delta = \big\{(\lambda, \varphi) \in [0,+\infty) \times \tilde{M} : \text{the maximal solution of (3.3) is defined on } [-1, \nu T] \text{ for all } s \in [0,1]\big\}.$$

One can prove that Δ is open in D^ν and that, when M is compact, $\Delta = [0,+\infty) \times \tilde{M}$.

Given $(\lambda, \varphi, s) \in \Delta \times [0,1]$, denote by $\xi = \xi_{(\lambda,\varphi,s)}$ the maximal solution of problem (3.3). Consider the Poincaré-type operator

$$\Pi : \Delta \times [0,1] \to \tilde{M}$$

defined as $\Pi(\lambda, \varphi, s)(t) = \xi(t+T)$, $t \in [-1,0]$. Notice that $\Pi(\cdot,\cdot,1)$ coincides with P, while $\Pi(\cdot,\cdot,0)$ turns out to be the (infinite dimensional) Poincaré-type operator associated with the undelayed problem

$$\begin{cases} x'(t) = \lambda f(t, x(t), x(t)), & t > 0, \\ x(t) = \varphi(t), & t \in [-1, 0]. \end{cases}$$

We point out that even in this undelayed case we adopt the definition of solution given in Section 2.

Remark 3.5. Analogously to $P(\lambda, \cdot)$, one can show that the fixed points of $\Pi(\lambda, \cdot, s)$ correspond to the T-periodic solutions of the equation

$$x'(t) = \lambda f(t, x(t), x(t-s)),$$

in the sense that φ is a fixed point of $\Pi(\lambda, \cdot, s)$ if and only if it is the restriction to $[-1,0]$ of a T-periodic solution.

The next lemma, which is similar to Lemma 3.3, regards some properties of Π.

Lemma 3.6. *The map $\Pi : \Delta\times[0,1] \to \tilde{M}$ is of class C^1 and the map from $\Delta\times[0,1]$ to $[0,+\infty)\times\tilde{M}\times[0,1]$, defined by $(\lambda,\varphi,s) \mapsto (\lambda,\Pi(\lambda,\varphi,s),s)$, is eventually locally compact. Consequently, the set $\{(\lambda,\varphi,s) \in \Delta\times[0,1] : \Pi(\lambda,\varphi,s) = \varphi\}$ is locally compact.*

The following result shows in particular that, if V is a relatively compact open subset of M, then the map $P(\lambda,\cdot)$ is well defined on $\overline{\tilde{V}}$ for $\lambda > 0$ sufficiently small.

Lemma 3.7. *Let K be a compact subset of M. Then, there exists $\varepsilon > 0$ such that $[0,\varepsilon]\times\tilde{K} \subseteq \Delta$.*

Proof. Since M is locally compact, there exists $\delta > 0$ such that $K_\delta = \{p \in M : \operatorname{dist}(p,K) \le \delta\}$ is a compact subset of M. Put $c = \max\{\|f(t,p,q)\| : t \in \mathbb{R}, p,q \in K_\delta\}$ and fix a positive ε such that $\varepsilon c\nu T < \delta$. Given $\lambda \in [0,\varepsilon]$, $\varphi \in \tilde{K}$, and $s \in [0,1]$, let ξ be the maximal solution of problem (3.3). Let us show that ξ is defined (at least) on $[-1,\nu T]$. If the image of ξ is contained in the compact set K_δ, then the domain of ξ contains $[-1,+\infty)$, and the assertion is true. Suppose therefore that there exists t_0 such that $\xi(t) \in K_\delta$ for any $t \in [-1,t_0]$ and $\xi(t_0) \in \operatorname{Fr} K_\delta$. We claim that $t_0 \ge \nu T$. Indeed, assume by contradiction $t_0 < \nu T$. Then, as $\|\xi'(t)\| \le \varepsilon c$ for any $t \in (0,t_0]$, we have $\operatorname{dist}(\xi(t_0),K) \le \varepsilon c\nu T < \delta$, which is a contradiction since $\xi(t_0) \in \operatorname{Fr} K_\delta$. This completes the proof. □

Lemma 3.8. *For $\lambda > 0$ sufficiently small, $\operatorname{ind}_{\tilde{M}}(\Pi(\lambda,\cdot,s),\tilde{U})$ does not depend on s.*

Proof. First notice that, by Lemma 3.7, there exists $\varepsilon > 0$ such that $[0,\varepsilon]\times\overline{\tilde{U}} \subseteq \Delta$. Let us show that, if $\lambda > 0$ is sufficiently small, then

$$A_\lambda = \big\{(\varphi,s) \in \tilde{U}\times[0,1] : \Pi(\lambda,\varphi,s) = \varphi\big\}$$

is a compact subset of $\tilde{U}\times[0,1]$. For this purpose it is enough to prove that, for λ small, A_λ coincides with

$$B_\lambda = \big\{(\varphi,s) \in \overline{\tilde{U}}\times[0,1] : \Pi(\lambda,\varphi,s) = \varphi\big\},$$

which is compact by Ascoli's Theorem. In other words, we have to show that B_λ does not intersect $\operatorname{Fr}\tilde{U}\times[0,1]$ for $\lambda > 0$ small. Recall that

$$\operatorname{Fr}\tilde{U} = \big\{\varphi \in \overline{\tilde{U}} : \varphi(t) \in \operatorname{Fr} U \text{ for some } t \in [-1,0]\big\}$$

and assume by contradiction that there exist sequences $\{s_n\}$ in $[0,1]$, $\{\lambda_n\}$ in $(0,\varepsilon]$, $\{\varphi_n\}$ in $\overline{\tilde{U}}$, $\{t_n\}$ in $[-1,0]$ such that

i) $\varphi_n(t_n) \in \operatorname{Fr} U$;
ii) $\lambda_n \to 0$;
iii) $\Pi(\lambda_n,\varphi_n,s_n) = \varphi_n$.

By the compactness of the sets B_λ, $\lambda \in (0,\varepsilon]$, it is not restrictive to assume that there exists $(\varphi_0,s_0) \in \overline{\tilde{U}}\times[0,1]$ such that $s_n \to s_0$ and $\varphi_n(t) \to \varphi_0(t)$ uniformly on $[-1,0]$.

Given $n \in \mathbb{N}$, let ξ_n be the maximal solution of the following problem:

$$\begin{cases} x'(t) = \lambda_n f(t,x(t),x(t-s_n)), & t > 0, \\ x(t) = \varphi_n(t), & t \in [-1,0]. \end{cases}$$

Then,

$$\xi_n(t) = \varphi_n(0) + \lambda_n \int_0^t f(\tau,\xi_n(\tau),\xi_n(\tau-s_n))\,d\tau, \quad t \in [0,\nu T]. \tag{3.4}$$

Moreover, since $\Pi(\lambda_n,\varphi_n,s_n) = \varphi_n$, Remark 3.5 implies that ξ_n is a T-periodic solution of the equation

$$x'(t) = \lambda_n f(t,x(t),x(t-s_n))$$

whose restriction to $[-1,0]$ coincides with φ_n. Thus, $\xi_n(T) = \xi_n(0) = \varphi_n(0)$. Since $\lambda_n \neq 0$ for any n, from equality (3.4) we get

$$\int_0^T f(\tau, \xi_n(\tau), \xi_n(\tau - s_n))\, d\tau = 0. \tag{3.5}$$

On the other hand, since $\lambda_n \to 0$, we have $\xi_n'(t) \to 0$ uniformly on $\mathbb{R}$. Therefore, $\xi_n(t) \to p_0$ uniformly on $\mathbb{R}$, where $p_0 = \varphi_0(0)$. Consequently, $\varphi_n(t) \to p_0$ uniformly on $[-1,0]$, and the assumption $\varphi_n(t_n) \in \operatorname{Fr} U$ implies that $p_0 \in \operatorname{Fr} U$. Passing to the limit in equality (3.5), we get $w(p_0) = 0$, which contradicts the assumption $w(p) \neq 0$ on $\operatorname{Fr} U$ in Theorem 3.4. Hence, if $\lambda > 0$ is sufficiently small, the index $\operatorname{ind}_{\tilde{M}}(\Pi(\lambda, \cdot, s), \tilde{U})$ is well defined and the homotopy invariance property of the fixed point index implies that $\operatorname{ind}_{\tilde{M}}(\Pi(\lambda, \cdot, s), \tilde{U})$ does not depend on $s \in [0,1]$. □

We will need a consequence of the commutativity property of the fixed point index (Lemma 3.9 below). Let $g : \mathbb{R} \times M \to \mathbb{R}^k$ be a C^1 map which is T-periodic in the first variable and tangent to M in the second one. Given $p \in M$, consider in M the following (initial value) problem:

$$\begin{cases} x'(t) = g(t, x(t)), \quad t > 0, \\ x(0) = p. \end{cases} \tag{3.6}$$

Define the following open subset of M:

$$R = \big\{ p \in M : \text{the maximal solution of problem (3.6) is defined on } [0,T] \big\}.$$

Given $p \in R$, denote by ζ_p the maximal solution of problem (3.6) and let

$$\Phi_0 : R \to M$$

be the Poincaré T-translation operator defined as $\Phi_0(p) = \zeta_p(T)$.

Let W be a relatively compact open subset of M such that $\overline{W} \subseteq R$. Define $k : \tilde{W} \to M$ by $k(\varphi) = \varphi(0)$ and $h : W \to \tilde{M}$ by $h(p)(t) = \zeta_p(t+T)$, $t \in [-1,0]$. Notice that $k^{-1}(W) = \tilde{W}$ and

$$h^{-1}(\tilde{W}) = \{ p \in W : \zeta_p(t+T) \in W \text{ for all } t \in [-1,0] \}.$$

Observe that the composition $kh : h^{-1}(\tilde{W}) \to M$ coincides with the restriction to $h^{-1}(\tilde{W})$ of the (finite dimensional) Poincaré operator Φ_0, and the composition $hk : \tilde{W} \to \tilde{M}$ coincides with the restriction to $\tilde{W}$ of the infinite dimensional Poincaré-type operator associated with the undelayed problem

$$\begin{cases} x'(t) = g(t, x(t)), \quad t > 0, \\ x(0) = \varphi(0), \end{cases}$$

that we will denote by Φ_1.

Assume now that the set of fixed points $\big\{ \varphi \in \tilde{W} : \Phi_1(\varphi) = \varphi \big\}$ is compact. Consequently, the set $\big\{ p \in h^{-1}(\tilde{W}) : \Phi_0(p) = p \big\}$ is compact as well, and the commutativity property of the index implies that

$$\operatorname{ind}_{\tilde{M}}(\Phi_1, \tilde{W}) = \operatorname{ind}_{\tilde{M}}(hk, k^{-1}(W)) = \operatorname{ind}_M(kh, h^{-1}(\tilde{W})) = \operatorname{ind}_M(\Phi_0, h^{-1}(\tilde{W})).$$

We summarize this argument in the following lemma.

Lemma 3.9. *Let W be a relatively compact open subset of M such that $\overline{W} \subseteq R$. If $\operatorname{ind}_{\tilde{M}}(\Phi_1, \tilde{W})$ is defined, then $\operatorname{ind}_M(\Phi_0, h^{-1}(\tilde{W}))$ is defined as well and $\operatorname{ind}_{\tilde{M}}(\Phi_1, \tilde{W}) = \operatorname{ind}_M(\Phi_0, h^{-1}(\tilde{W}))$.*

We are now in the position to state and prove the last lemma needed in the proof of Theorem 3.4.

Lemma 3.10. *For $\lambda > 0$ sufficiently small, $\operatorname{ind}_{\tilde{M}}(\Pi(\lambda, \cdot, 0), \tilde{U}) = \deg(-w, U)$.*

Proof. Let us apply Lemma 3.9 with $(t,p) \mapsto \lambda f(t,p,p)$ in place of $(t,p) \mapsto g(t,p)$ and $\lambda > 0$ small. Given $p \in M$, consider in M the following (initial value) problem:

$$\begin{cases} x'(t) = \lambda f(t, x(t), x(t)), & t > 0, \\ x(0) = p. \end{cases} \tag{3.7}$$

Define

$$O = \big\{(\lambda, p) \in [0, +\infty) \times M : \text{the maximal solution of problem (3.7) is defined on } [0,T]\big\}.$$

Observe that O is an open subset of $[0,+\infty) \times M$ containing $\{0\} \times M$. Given $(\lambda, p) \in O$, denote by $y_{(\lambda,p)}$ the maximal solution of problem (3.7) and consider the map

$$Q : O \to M$$

defined as $Q(\lambda, p) = y_{(\lambda,p)}(T)$.

Choose ε so small that $[0,\varepsilon] \times \overline{U} \subseteq O$. Given $0 < \lambda \le \varepsilon$, define $h_\lambda : U \to \tilde{M}$ by $h_\lambda(p)(t) = y_{(\lambda,p)}(t+T)$, $t \in [-1, 0]$. Notice that $h_\lambda^{-1}(\tilde{U}) = \big\{p \in U : y_{(\lambda,p)}(t+T) \in U \text{ for all } t \in [-1,0]\big\}$.

As in the proof of Lemma 3.8, one can show that, when λ is sufficiently small, $\big\{\varphi \in \tilde{U} : \Pi(\lambda, \varphi, 0) = \varphi\big\}$ is a compact subset of $\tilde{U}$. Thus, by Lemma 3.9, we get

$$\mathrm{ind}_{\tilde{M}}(\Pi(\lambda, \cdot, 0), \tilde{U}) = \mathrm{ind}_M(Q(\lambda, \cdot), h_\lambda^{-1}(\tilde{U})).$$

Let now V be a relatively compact open subset of M containing the compact set $\{p \in U : w(p) = 0\}$ and such that $\overline{V} \subseteq U$. Let us show that for $\lambda > 0$ sufficiently small the following properties hold: V is contained in $h_\lambda^{-1}(\tilde{U})$ and contains the fixed points of $Q(\lambda, \cdot)$ in $h_\lambda^{-1}(\tilde{U})$. Indeed, assume by contradiction that there exist sequences $\{\lambda_n\}$ in $(0, \varepsilon]$ and $\{p_n\}$ in $\overline{U} \backslash V$ such that

i) $p_n \in h_{\lambda_n}^{-1}(\tilde{U})$;
ii) $\lambda_n \to 0$;
iii) $Q(\lambda_n, p_n) = p_n$.

By arguing as in the proof of Lemma 3.8, we get the existence of a point $p_0 \in \overline{U} \backslash V$ with $w(p_0) = 0$, which is a contradiction since V contains the set $\{p \in U : w(p) = 0\}$.

Then, by the excision property of the fixed point index, for $\lambda > 0$ small we get

$$\mathrm{ind}_M(Q(\lambda, \cdot), h_\lambda^{-1}(\tilde{U})) = \mathrm{ind}_M(Q(\lambda, \cdot), V).$$

Finally, as shown in [9], if $\lambda > 0$ is sufficiently small we get

$$\mathrm{ind}_M(Q(\lambda, \cdot), V) = \deg(-w, V)$$

and, by the excision property of the degree,

$$\deg(-w, V) = \deg(-w, U).$$

Hence, for $\lambda > 0$ small,

$$\mathrm{ind}_{\tilde{M}}(\Pi(\lambda, \cdot, 0), \tilde{U}) = \deg(-w, U),$$

and this completes the proof. □

Proof of Theorem 3.4. Since $P(\lambda, \cdot) = \Pi(\lambda, \cdot, 1)$, Lemma 3.8 above implies that $\mathrm{ind}_{\tilde{M}}(P(\lambda, \cdot), \tilde{U})$ is well defined and equals $\mathrm{ind}_{\tilde{M}}(\Pi(\lambda, \cdot, 0), \tilde{U})$. Moreover, by Lemma 3.10, if $\lambda > 0$ is sufficiently small we have

$$\mathrm{ind}_{\tilde{M}}(P(\lambda, \cdot), \tilde{U}) = \deg(-w, U),$$

which is the assertion. □

From now on, given a subset X of $[0,+\infty)\times\tilde M$ and $\lambda\geq 0$, we will denote by X_λ the slice $\{\varphi\in\tilde M : (\lambda,\varphi)\in X\}$ and by $\check X_0$ the subset $\{p\in M : (0,\hat p)\in X\}$ of M.

Let G be an open subset of D^ν, and assume that $\deg(w,\check G_0)$ is different from zero. Lemma 3.12 below regards the existence of a noncompact subset of G which is the closure (in G) of a connected set of nontrivial T-starting pairs for equation (3.1) and which, in some sense, emanates from the set of zeros of w in $\check G_0$ (recall that a T-starting pair $(\lambda,\varphi)\in[0,+\infty)\times\tilde M$ is nontrivial when $\lambda>0$).

The following topological lemma is needed.

Lemma 3.11 ([12])**.** *Let Z be a compact subset of a locally compact metric space Y. Assume that any compact subset of Y containing Z has nonempty boundary. Then $Y\backslash Z$ contains a connected set whose closure is not compact and intersects Z.*

Lemma 3.12. *Let G be an open subset of D^ν. Assume that* $\deg(w,\check G_0)$ *is different from zero. Then, the equation* (3.1) *admits a connected branch of nontrivial T-starting pairs whose closure in G is not compact and intersects the set $\{(0,\hat p)\in G : p\in M,\, w(p)=0\}$.*

Proof. Denote

$$S=\{p\in\check G_0 : w(p)=0\}\quad\text{and}\quad \Sigma_G^+=\{(\lambda,\varphi)\in G:\lambda>0,(\lambda,\varphi)\text{ is a }T\text{-starting pair of (3.1)}\},$$

and define $Z=\{0\}\times\hat S=\{(0,\hat p)\in G : p\in M,\, w(p)=0\}$ and $Y=Z\cup\Sigma_G^+$.

Notice that Σ_G^+ is locally compact, as the intersection of the open set $\{(\lambda,\varphi)\in G:\lambda>0\}$ with the set $\{(\lambda,\varphi)\in D^\nu : P(\lambda,\varphi)=\varphi\}$, which is locally compact since, by Lemma 3.3, the map P is eventually locally compact. Consequently, Y is locally compact as well, being the union of Σ_G^+ and the compact set $Z=\{0\}\times\hat S$ (observe that S is compact since, by assumption, $\deg(w,\check G_0)$ is defined).

We apply Lemma 3.11 to the metric spaces Y and Z. Assume, by contradiction, that there exists a compact set $Y'\subseteq Y$ containing Z and with empty boundary in Y. Thus, Y' is also open in Y. Hence, there exists a bounded open subset A of G such that $Y'=A\cap Y$. Since Y' is compact, the generalized homotopy invariance property of the fixed point index implies that $\mathrm{ind}_{\tilde M}(P(\lambda,\cdot),A_\lambda)$ does not depend on $\lambda>0$. Moreover, the slice $Y'_\lambda=A_\lambda\cap Y_\lambda$ is empty for some positive λ. This implies that $\mathrm{ind}_{\tilde M}(P(\lambda,\cdot),A_\lambda)=0$ for any $\lambda>0$.

Notice that A_0 is an open subset of G_0 containing the compact set $\hat S$. Let now U be a relatively compact open subset of M containing S and such that $\overline U\subseteq\check A_0$. Let us show that there exists $\varepsilon>0$ with the following properties: $Y'_\lambda\subseteq\tilde U$ for any $0<\lambda\leq\varepsilon$ and $[0,\varepsilon]\times\tilde{\overline U}\subseteq A$. Indeed, if this is not the case there exist sequences $\{\lambda_n\}$ in $(0,+\infty)$, $\{\varphi_n\}$ in $\tilde{\overline U}$, $\{t_n\}$ in $[-1,0]$ such that

i) $\varphi_n(t_n)\in\mathrm{Fr}\,U$;
ii) $\lambda_n\to 0$;
iii) $P(\lambda_n,\varphi_n)=\varphi_n$.

Analogously to the proof of Lemma 3.10, we follow the ideas in the proof of Lemma 3.8 obtaining the existence of a point $p_0\in\mathrm{Fr}\,U$ with $w(p_0)=0$, and this cannot happen because of the choice of U.

Now, by Theorem 3.4, by taking into account the excision properties of the fixed point index and the degree, and by recalling equality (2.4), for $\lambda>0$ small we get

$$0=\mathrm{ind}_{\tilde M}(P(\lambda,\cdot),A_\lambda)=\mathrm{ind}_{\tilde M}(P(\lambda,\cdot),\tilde U)=\deg(-w,U)=\deg(-w,\check G_0)\neq 0,$$

which is a contradiction. Therefore, because of Lemma 3.11, there exists a connected subset of Y whose closure in Y intersects Z and is not compact. This implies our assertion. □

We are finally in the position to present the main result of this paper: Theorem 3.13 below, which deals with T-periodic pairs instead of T-starting pairs (as in Lemma 3.12). Assuming that f is merely continuous and given an open subset Ω of $[0,+\infty) \times C_T(M)$, the result regards a sufficient condition for the existence in Ω of a global bifurcating branch of nontrivial T-periodic pairs of (3.1).

In the sequel we will denote by $\check{\Omega}_0$ the subset $\{p \in M : (0,\bar{p}) \in \Omega\}$ of M (recall that, given $p \in M$, $\bar{p}$ denotes the constant map $t \mapsto p$, $t \in \mathbb{R}$).

Theorem 3.13. *Let $M \subseteq \mathbb{R}^k$ be a boundaryless smooth manifold, f a vector field on M which is T-periodic in the first variable, and $w : M \to \mathbb{R}^k$ the autonomous tangent vector field given by*

$$w(p) = \frac{1}{T}\int_0^T f(t,p,p)\,dt.$$

Let Ω be an open subset of $[0,+\infty) \times C_T(M)$, and assume that $\deg(w,\check{\Omega}_0)$ is different from zero. Then, there exists a connected subset of Ω of nontrivial T-periodic pairs of equation (3.1) whose closure in Ω is not compact and intersects the set $\{(0,\bar{p}) \in \Omega : p \in M,\, w(p) = 0\}$.

Proof. The proof will be divided into two steps. In the first one f is assumed to be C^1 (so that Lemma 3.12 applies) and in the second one f is merely continuous.

Step 1. Assume that f is of class C^1. Let $\Gamma \subseteq [0,+\infty) \times C_T(M)$ denote the set of the T-periodic pairs of (3.1) and $\Sigma \subseteq [0,+\infty) \times \tilde{M}$ the set of the T-starting pairs. As already pointed out, the map $\rho : \Gamma \to \Sigma$, which associates to any T-periodic pair (λ, x) the corresponding T-starting pair (λ, φ), is a homeomorphism. Moreover, given $p \in M$, one has $\rho(0,\bar{p}) = (0,\hat{p})$. In other words, the restriction of ρ to $\{0\} \times \{\bar{p} : p \in M\} \subseteq \Gamma$ as domain and to $\{0\} \times \{\hat{p} : p \in M\} \subseteq \Sigma$ as codomain can be regarded as the identity on M (one may identify M in a natural way both with $\{0\} \times \{\bar{p} : p \in M\} \subseteq \Gamma$ and with $\{0\} \times \{\hat{p} : p \in M\} \subseteq \Sigma$).

Let now $\Sigma_\Omega = \rho(\Omega \cap \Gamma)$. Since Σ is contained in D^ν and Σ_Ω is an open subset of Σ, there exists an open subset G of D^ν such that $\Sigma_\Omega = G \cap \Sigma$. Observe that $\{0\} \times \check{G}_0 = \rho(\{0\} \times \check{\Omega}_0)$. Therefore, the two subsets $\check{\Omega}_0$ and $\check{G}_0$ of M coincide. Thus, by assumption, $\deg(w, \check{G}_0) \neq 0$, and one can apply Lemma 3.12 to the set $G \subseteq D^\nu$.

Let $C \subseteq G$ be a connected branch of nontrivial T-starting pairs as in the assertion of Lemma 3.12. Thus, the subset $B = \rho^{-1}(C)$ of $\Omega \cap \Gamma$ is connected, it is made up of nontrivial T-periodic pairs, and its closure in $\Omega \cap \Gamma$ (which is the same as in Ω) is not compact and meets the set $\{(0,\bar{p}) \in \Omega : p \in M,\, w(p) = 0\}$.

Step 2. Suppose now that f is continuous. Let

$$Z = \{(0,\bar{p}) \in \Omega : p \in M,\, w(p) = 0\} \quad \text{and} \quad Y = Z \cup (\Omega_+ \cap \Gamma),$$

where $\Omega_+ = \{(\lambda, x) \in \Omega : \lambda > 0\}$ and Γ, as in the previous step, denotes the set of the T-periodic pairs of (3.1).

We apply Lemma 3.11 to the metric spaces Y and Z. Assume, by contradiction, that there exists a compact set $Y' \subseteq Y$ containing Z and with empty boundary in Y. Thus, Y' is also open in Y and, consequently, both Y' and $Y \backslash Y'$ are closed in Ω. Hence, there exists a bounded open subset A of Ω such that $Y' \subseteq A$ and $\partial A \cap Y = \emptyset$. Moreover, since $C_T(M)$ is a locally complete metric space and Y' is compact, A can be chosen so that its closure $\overline{A}$ in $C_T(M)$ is complete and contained in Ω.

Let $\{f_n\}$ be a sequence of C^1 vector fields on M, T-periodic in the first variable, and such that $\big\{f_n(t,p,q)\big\}$ converges to $f(t,p,q)$ uniformly on $\mathbb{R}\times M\times M$. For any $n\in\mathbb{N}$, define

$$w_n(p)=\frac{1}{T}\int_0^T f_n(t,p,p)\,dt.$$

Thus, the sequence $\{w_n(p)\}$ converges to $w(p)$ uniformly on M and, consequently, there exists $\bar n\in\mathbb{N}$ such that

$$\deg(w_n,\check\Omega_0)=\deg(w,\check\Omega_0)\neq 0 \quad \text{for } n>\bar n.$$

For any $n>\bar n$, let Γ_n denote the set of the T-periodic pairs of the equation

$$x'(t)=\lambda f_n(t,x(t),x(t-1)). \tag{3.8}$$

Let

$$Z_n=\big\{(0,\bar p)\in\Omega : p\in M,\ w_n(p)=0\big\} \quad \text{and} \quad Y_n=Z_n\cup\big(\Omega_+\cap\Gamma_n\big).$$

By the previous step, for $n>\bar n$, any equation (3.8) has in Ω a connected set B_n of nontrivial T-periodic pairs whose closure in Ω is noncompact and meets the set Z_n.

Since the closure of A is a bounded and complete subset of Ω, for $n>\bar n$ the branch B_n must intersect the complement of A in Ω. This implies that for $n>\bar n$ there exists a pair $(\lambda_n,x_n)\in B_n\cap\partial A$.

We may assume $\lambda_n\to\lambda_0$ and, by Ascoli's Theorem, $x_n(t)\to x_0(t)$ uniformly. Since $\big\{\lambda_n f_n(t,p,q)\big\}$ converges to $\lambda_0 f(t,p,q)$ uniformly on $\mathbb{R}\times M\times M$, then $x_0(t)$ is a T-periodic solution of the equation

$$x'(t)=\lambda_0 f(t,x(t),x(t-1)).$$

That is, (λ_0,x_0) is a T-periodic pair of (3.1) and, consequently, (λ_0,x_0) belongs to $\partial A\cap Y$, which is a contradiction. Therefore, by Lemma 3.11, one can find a connected branch of nontrivial T-periodic pairs of (3.1) whose closure in Y (which is the same as in Ω) intersects $Z=\big\{(0,\bar p)\in\Omega : p\in M,\ w(p)=0\big\}$ and is not compact. □

Theorem 3.1 asserts that a necessary condition for $p_0\in M$ to be a bifurcation point is that the mean value vector field w vanishes at p_0. The following consequence of Theorem 3.13 provides a sufficient condition for a zero p_0 of w to be of bifurcation.

Corollary 3.14. *Let p_0 be a zero of the mean value vector field w. Assume that w is of class C^1 and $w'(p_0): T_{p_0}M\to\mathbb{R}^k$ is one-to-one. Then p_0 is a bifurcation point of the equation* (3.1).

Proof. As we already pointed out, the assumption $w(p_0)=0$ implies that $w'(p_0)$ maps $T_{p_0}M$ into itself. Consequently, $w'(p_0)$ is an automorphism of $T_{p_0}M$ and p_0 is an isolated zero of w.

Given an open isolating neighborhood U of p_0 in M, let Ω be the open set obtained by removing the "vertical" set $\big\{(0,\bar p) : p\in M\setminus U\big\}$ from the space $[0,+\infty)\times C_T(M)$. Observe that $\check\Omega_0=U$, and $\deg(w,U)=\operatorname{sign}\det(w'(p_0))\neq 0$. Thus, Theorem 3.13 implies the existence of a connected set of nontrivial T-periodic pairs of (3.1) whose closure meets the singleton $\big\{(0,\bar p) : p\in U,\ w(p)=0\big\}=\{(0,\bar p_0)\}$, and this shows that p_0 is a bifurcation point of (3.1). □

The following consequence of Theorem 3.13 will be applied to prove that any forced constrained mechanical system admits forced oscillations, provided that the constraint is compact, with nonzero Euler–Poincaré characteristic, and the frictional coefficient is strictly positive (see Theorem 4.1).

Corollary 3.15. *Let f, w and M be as in Theorem 3.13. Assume that M is closed as a subset of $\mathbb{R}^k$, and let U be an open subset of M such that* $\deg(w, U)$ *is nonzero. Then, the equation* (3.1) *admits in* $[0, +\infty) \times C_T(M)$ *a connected branch of nontrivial T-periodic pairs whose closure meets the set*

$$\big\{(0, \bar{p}) : p \in U,\, w(p) = 0\big\}$$

and satisfies at least one of the following properties:

(i) *it is unbounded;*
(ii) *it meets the set* $\big\{(0, \bar{p}) : p \in M \setminus U,\, w(p) = 0\big\}$.

Proof. Let Ω be the open subset of the space $[0, +\infty) \times C_T(M)$ obtained by removing (from this space) the "vertical" set $\big\{(0, \bar{p}) : p \in M \setminus U\big\}$. Since $\check{\Omega}_0 = U$, Theorem 3.13 implies the existence, in Ω, of a connected branch B of nontrivial T-periodic pairs of (3.1) whose closure, in Ω, is not compact and intersects the set $\big\{(0, \bar{p}) : p \in U,\, w(p) = 0\big\}$. Assume that B is bounded. Thus, because of Ascoli's Theorem, B is actually totally bounded. Consequently, its closure $\overline{B}$ in the complete metric space $[0, +\infty) \times C_T(M)$ is compact (notice that, M being closed in $\mathbb{R}^k$, $C_T(M)$ is closed in the Banach space $C_T(\mathbb{R}^k)$). Since the closure $\overline{B} \cap \Omega$ of B in Ω is not compact, $\overline{B}$ must contain an element $(0, \bar{p}_0)$ in the complement $\big\{(0, \bar{p}) : p \in M \setminus U\big\}$ of Ω. Thus, p_0 is a bifurcation point of the equation (3.1) and, because of Theorem 3.1, $w(p_0) = 0$. □

Corollary 3.16 below extends two results: one (obtained in [1]) for $T \geq 1$ and manifolds with boundary, and the other (given in [3]) for any $T > 0$ and boundaryless manifolds. Its statement needs some preliminary definitions.

Let $N \subseteq \mathbb{R}^k$ be a smooth manifold with (possibly empty) boundary ∂N, and $f : \mathbb{R} \times N \times N \to \mathbb{R}^k$ a vector field on N. Following [1], we will say that f is *inward* (resp. *outward*) if for any $(t, p, q) \in \mathbb{R} \times \partial N \times N$ the vector $f(t, p, q)$ points inward (resp. outward) at p. Recall that, given $p \in \partial N$, the set of vectors which are tangent to N at p and point inward (resp. outward) is a closed half-subspace of T_pN, called *inward* (resp. *outward*) *half-subspace* of T_pN (see e.g. [17]). We will say that f is *strictly inward* (resp. *strictly outward*) if f is inward (resp. outward) and $f(t, p, q) \notin T_p\partial N$ for any $(t, p, q) \in \mathbb{R} \times \partial N \times N$.

Let us recall that, if N is a compact manifold with boundary and $v \colon N \to \mathbb{R}^k$ is a continuous tangent vector field on N satisfying $v(p) \neq 0$ for all $p \in \partial N$, then the degree of v in N still makes sense. In fact, it suffices to observe that, in this case, v is admissible in the boundaryless manifold $M = N \setminus \partial N$. Hence, one can define $\deg(v, N)$ as the degree of the restriction of v to M. The Poincaré–Hopf theorem asserts that this degree equals the Euler–Poincaré characteristic of N, provided v points outward along ∂N.

Corollary 3.16. *Let $N \subseteq \mathbb{R}^k$ be a compact manifold with (possibly empty) boundary, whose Euler–Poincaré characteristic $\chi(N)$ is different from zero. Let $f : \mathbb{R} \times N \times N \to \mathbb{R}^k$ be an inward (or outward) vector field on N which is T-periodic in the first variable, and $w : N \to \mathbb{R}^k$ the autonomous tangent vector field given by*

$$w(p) = \frac{1}{T} \int_0^T f(t, p, p)\, dt.$$

Then, the equation

$$x'(t) = \lambda f(t, x(t), x(t-1)) \tag{3.9}$$

admits an unbounded connected branch in $[0, +\infty) \times C_T(N)$ of nontrivial T-periodic pairs whose closure meets the set $\big\{(0, \bar{p}) : p \in N,\, w(p) = 0\big\}$. In particular, since $C_T(N)$ is bounded, the equation (3.9) *has a T-periodic solution for any $\lambda \geq 0$.*

Proof. Assume first that the vector field f is strictly inward (or outward). Notice that, in this case, $\deg(w,N)$ is well defined since $w(p) \neq 0$ for all $p \in \partial N$.

If N is boundaryless, then $\deg(w,N) = \chi(N) \neq 0$. If $\partial N \neq \emptyset$ and f points strictly outward along ∂N, then w points strictly outward as well, so that again one has $\deg(w,N) = \chi(N) \neq 0$. If f is strictly inward, then the tangent vector field $-w$ is strictly outward. Therefore, by recalling that $\deg(-w,N) = (-1)^{\dim N} \deg(w,N)$, still in this case one obtains $\deg(w,N) \neq 0$. Hence, considering the boundaryless manifold $M = N \backslash \partial N$, Theorem 3.13 yields the existence of a connected branch

$$B \subseteq \Omega := [0,+\infty) \times C_T(M)$$

of nontrivial T-periodic pairs whose closure, in Ω, meets the set $\big\{(0,\bar{p}) : p \in M,\, w(p) = 0\big\}$ and is not compact. It remains to show that B is unbounded.

If $\partial N = \emptyset$, the unboundedness of B follows from Corollary 3.15 since, in this case, the manifold $M = N$ is closed in $\mathbb{R}^k$. If, otherwise, $\partial N \neq \emptyset$, the fact that the vector field f is never tangent to ∂N implies that there are no T-periodic orbits of (3.9) which hit ∂N. Therefore, the closures of B in $[0,+\infty) \times C_T(M)$ and in $[0,+\infty) \times C_T(N)$ coincide. Thus, $\overline{B}$ is a complete metric space, which cannot be bounded since, otherwise, because of Ascoli's Theorem, it would be totally bounded and, therefore, compact.

Assume now that f is merely inward (or outward). Then, one can find a sequence $\{f_n\}$ of strictly inward (or outward) vector fields on N, T-periodic in the first variable, and such that $\big\{f_n(t,p,q)\big\}$ converges to $f(t,p,q)$ uniformly on $\mathbb{R} \times N \times N$. Hence, the previous step applies to any vector field f_n, and the existence of the required branch can be deduced as in the proof of Theorem 3.13 (step 2). □

4. Applications to second order delay differential equations on manifolds

In this section we provide an application of the results obtained above to some motion problems for forced constrained systems.

Let $X \subseteq \mathbb{R}^s$ be a boundaryless manifold. Given $q \in X$, let T_qX and $(T_qX)^\perp$ denote the tangent and the normal space of X at q, respectively. Since $\mathbb{R}^s = T_qX \oplus (T_qX)^\perp$, any vector $u \in \mathbb{R}^s$ can be uniquely decomposed into the sum of the *parallel* (or *tangential*) *component* $u_\pi \in T_qX$ *of* u *at* q and the *normal component* $u_\nu \in (T_qX)^\perp$ *of* u *at* q. By

$$TX = \big\{(q,v) \in \mathbb{R}^s \times \mathbb{R}^s :\ q \in X,\ v \in T_qX\big\}$$

we denote the *tangent bundle of* X, which is a smooth manifold containing a natural copy of X via the embedding $q \mapsto (q,0)$. The natural projection of TX onto X is just the restriction (to TX as domain and to X as codomain) of the projection of $\mathbb{R}^s \times \mathbb{R}^s$ onto the first factor.

Given a vector field $F : \mathbb{R} \times X \times X \to \mathbb{R}^s$ which is T-periodic in the first variable, consider the following delay motion equation on X:

$$x''_\pi(t) = F(t,x(t),x(t-1)) - \varepsilon x'(t), \tag{4.1}$$

where

i) $x''_\pi(t)$ stands for the parallel component of the acceleration $x''(t) \in \mathbb{R}^s$ at the point $x(t)$;
ii) the frictional coefficient ε is a positive constant.

By a *solution* of (4.1) we mean a continuous function $x : J \to X$, defined on a (possibly unbounded) interval, with length greater than 1, which is of class C^2 on the subinterval $(\inf J + 1, \sup J)$ of J and verifies

$$x''_\pi(t) = F(t,x(t),x(t-1)) - \varepsilon x'(t)$$

for all $t \in J$ with $t > \inf J + 1$. A *forced oscillation of* (4.1) is a solution which is T-periodic and globally defined on $J = \mathbb{R}$.

It is known that, associated with $X \subseteq \mathbb{R}^s$, there exists a unique smooth map $r : TX \to \mathbb{R}^s$, called the *reactive force* (or *inertial reaction*), with the following properties:

(a) $r(q,v) \in (T_qX)^\perp$ for any $(q,v) \in TX$;
(b) r is quadratic in the second variable;
(c) any C^2 curve $\gamma : (a,b) \to X$ verifies the condition
$$\gamma''_\nu(t) = r(\gamma(t), \gamma'(t)), \quad \forall t \in (a,b),$$
i.e. for each $t \in (a,b)$, the normal component $\gamma''_\nu(t)$ of $\gamma''(t)$ at $\gamma(t)$ equals $r(\gamma(t), \gamma'(t))$.

The map r is strictly related to the second fundamental form on X and may be interpreted as the reactive force due to the constraint X.

By properties (a) and (c) above, equation (4.1) can be equivalently written as
$$x''(t) = r(x(t), x'(t)) + F(t, x(t), x(t-1)) - \varepsilon x'(t). \tag{4.2}$$

Notice that, if the above equation reduces to the so-called *inertial equation*
$$x''(t) = r(x(t), x'(t)),$$
one obtains the geodesics of X as solutions.

Equation (4.2) can be written as a first order differential system on TX as follows:
$$\begin{cases} x'(t) = y(t) \\ y'(t) = r(x(t), y(t)) + F(t, x(t), x(t-1)) - \varepsilon y(t). \end{cases}$$
This makes sense since the map
$$g : \mathbb{R} \times TX \times X \to \mathbb{R}^s \times \mathbb{R}^s, \quad g(t, (q,v), \tilde{q}) = (v, r(q,v) + F(t, q, \tilde{q}) - \varepsilon v) \tag{4.3}$$
verifies the condition $g(t, (q,v), \tilde{q}) \in T_{(q,v)}TX$ for all $(t, (q,v), \tilde{q}) \in \mathbb{R} \times TX \times X$ (see, for example, [7] for more details).

Theorem 4.1 below extends a result obtained in [2] for the case $T \geq 1$. The proof is based on Corollary 3.15 above.

Theorem 4.1. *Let $X \subseteq \mathbb{R}^s$ be a compact boundaryless manifold whose Euler–Poincaré characteristic $\chi(X)$ is different from zero. Then, the equation* (4.1) *has a forced oscillation.*

Proof. As we already pointed out, the equation (4.1) is equivalent to the following first order system on TX:
$$\begin{cases} x'(t) = y(t) \\ y'(t) = r(x(t), y(t)) + F(t, x(t), x(t-1)) - \varepsilon y(t). \end{cases} \tag{4.4}$$
It is convenient to regard system (4.4) as a particular case of the following one, depending on the additional parameter $\lambda \geq 0$:
$$\begin{cases} x'(t) = \lambda\, y(t) \\ y'(t) = \lambda \big(r(x(t), y(t)) + F(t, x(t), x(t-1)) - \varepsilon y(t) \big). \end{cases} \tag{4.5}$$
We point out that, when $\lambda \neq 1$, system (4.5) does not represent a second order equation on X; however, it is still a first order differential equation on TX.

Define $f : \mathbb{R} \times TX \times TX \to \mathbb{R}^s \times \mathbb{R}^s$ by
$$f(t, (q,v), (\tilde{q}, \tilde{v})) = (v, r(q,v) + F(t, q, \tilde{q}) - \varepsilon v).$$

Notice that the map f is a vector field on TX which is T-periodic in the first variable. Let $w : TX \to \mathbb{R}^s \times \mathbb{R}^s$ be the autonomous tangent vector field given by

$$w(q,v) = \frac{1}{T}\int_0^T f(t,(q,v),(q,v))\,dt = (v, r(q,v) + \omega(q) - \varepsilon v),$$

where

$$\omega(q) = \frac{1}{T}\int_0^T F(t,q,q)\,dt, \quad q \in X.$$

We apply Corollary 3.15 with $U = M = TX$. To this end, we need to show that $\deg(w, TX)$ is nonzero. Notice first that the zeros of w are contained in the compact subset $X \times \{0\}$ of TX. Therefore, $\deg(w, TX)$ is well defined. Given $c > 0$, consider the following subset of TX:

$$M_c = \big\{(q,v) \in X \times \mathbb{R}^s : v \in T_qX,\ \|v\| \le c\big\}.$$

It is not difficult to show that M_c is a compact manifold in $\mathbb{R}^s \times \mathbb{R}^s$ with boundary

$$\partial M_c = \big\{(q,v) \in X \times \mathbb{R}^s : v \in T_qX,\ \|v\| = c\big\}.$$

Observe that $\chi(M_c) = \chi(X)$ since M_c and X are homotopically equivalent (X being a deformation retract of TX). Since $\mathring{M}_c = M_c \backslash \partial M_c$ is an open subset of TX containing $X \times \{0\}$, the excision property of the degree implies that

$$\deg(w, TX) = \deg(w, \mathring{M}_c).$$

We claim that, if $c > 0$ is large enough, then f is a strictly inward vector field on M_c and, consequently, so is w. To see this, let $(q,v) \in \partial M_c$ be fixed, and observe that the strictly inward open half-subspace of $T_{(q,v)}M_c$ is

$$T^-_{(q,v)}M_c = \big\{(\dot q, \dot v) \in T_{(q,v)}TX : \langle v, \dot v\rangle < 0\big\},$$

where $\langle \cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^s$. We have to show that, if $c > 0$ is large enough, then $f(t,(q,v),(\tilde q,\tilde v))$ belongs to $T^-_{(q,v)}M_c$ for any $t \in \mathbb{R}$ and $(\tilde q, \tilde v) \in TX$; that is,

$$\langle v, r(q,v) + F(t,q,\tilde q) - \varepsilon v\rangle = \langle v, r(q,v)\rangle + \langle v, F(t,q,\tilde q)\rangle - \varepsilon\langle v,v\rangle < 0$$

for any $t \in \mathbb{R}$ and $(\tilde q,\tilde v) \in TX$. Now, $\langle v, r(q,v)\rangle = 0$ since $r(q,v)$ belongs to $(T_qX)^\perp$. Moreover, $\langle v,v\rangle = c^2$ since $(q,v) \in \partial M_c$, and

$$\langle v, F(t,q,\tilde q)\rangle \le \|v\|\|F(t,q,\tilde q)\| \le K\|v\|,$$

where

$$K = \max\big\{\|F(t,q,\tilde q)\| : (t,q,\tilde q) \in \mathbb{R} \times X \times X\big\}.$$

Thus,

$$\langle v, r(q,v) + F(t,q,\tilde q) - \varepsilon v\rangle \le Kc - \varepsilon c^2.$$

This shows that, if we choose $c > K/\varepsilon$, then f and thus w are strictly inward vector fields on M_c, as claimed. Therefore, given $c > K/\varepsilon$, the Poincaré–Hopf Theorem implies that $\deg(-w, \mathring{M}_c) = \chi(M_c) = \chi(X)$. Consequently,

$$\deg(w, TX) = (-1)^{2m}\deg(-w, \mathring{M}_c) = \chi(X) \ne 0,$$

where m is the dimension of X. Hence, from Corollary 3.15 it follows that the equation (4.5) admits in $[0,+\infty) \times C_T(TX)$ an unbounded connected branch of nontrivial T-periodic triples (λ, x, y), whose closure intersects the subset $\{0\} \times X \times \{0\}$ of $[0,+\infty) \times X \times \mathbb{R}^s$.

Now, to prove that system (4.4) admits a T-periodic solution, we show that a suitable *a priori* bound holds for the nontrivial T-periodic triples of (4.5). Indeed, let (λ, x, y) be a nontrivial T-periodic triple of (4.5); that is, $(x,y) \in C_T(TX)$ is a T-periodic solution of (4.5) corresponding to $\lambda > 0$. Since X is a compact manifold and $x(t) \in X$ for any $t \in \mathbb{R}$, it suffices to give *a priori* bounds on y. To this end, consider the T-periodic function $\delta(t) = \|y(t)\|^2$ and let t_0 be such that $\delta(t_0) = \max\{\delta(t) : t \in \mathbb{R}\}$. We have

$$\begin{aligned} 0 = \delta'(t_0) = 2\langle y(t_0), y'(t_0)\rangle &= 2\lambda\big(\langle y(t_0), r(x(t_0), y(t_0))\rangle + \langle y(t_0), F(t_0, x(t_0), x(t_0-1))\rangle - \varepsilon\|y(t_0)\|^2\big) \\ &\leq 2\lambda\big(K\|y(t_0)\| - \varepsilon\|y(t_0)\|^2\big) = 2\lambda\|y(t_0)\|\big(K - \varepsilon\|y(t_0)\|\big). \end{aligned}$$

Since $\lambda > 0$, this shows that, if we choose $c > K/\varepsilon$, then the T-periodic solutions of (4.5) corresponding to λ lie entirely in M_c. Hence, the obtained branch is unbounded with respect to λ and, consequently, system (4.4) admits a T-periodic solution. This implies that the second order equation (4.1) has a forced oscillation. □

We point out that, in the above theorem, the condition $\chi(X) \neq 0$ cannot be dropped. Consider for example the equation

$$\theta''(t) = a - \varepsilon\theta'(t), \quad t \in \mathbb{R}, \tag{4.6}$$

where a is a nonzero constant and $\varepsilon > 0$. Equation (4.6) can be regarded as a second order ordinary differential equation on the unit circle $S^1 \subseteq \mathbb{C}$, where θ represents the angular coordinate. In this case, a solution $\theta(\cdot)$ of (4.6) is periodic of period $T > 0$ if and only if for some $k \in \mathbb{Z}$ it satisfies the boundary conditions

$$\begin{cases} \theta(T) - \theta(0) = 2k\pi, \\ \theta'(T) - \theta'(0) = 0. \end{cases}$$

Notice that the applied force a represents a nonvanishing autonomous vector field on S^1. Thus, it is periodic of arbitrary period. However, simple calculations show that there are no T-periodic solutions of (4.6) if $T \neq 2\pi\varepsilon/a$.

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