

SOBOLEV INEQUALITIES IN 2-D HYPERBOLIC SPACE: A BORDERLINE CASE

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ABSTRACT. Sobolev inequalities in two-dimensional hyperbolic space \mathbb{H}^2 are dealt with. Here \mathbb{H}^2 is modeled on the upper Euclidean half-plane equipped with the Poincaré-Bergman metric. Some borderline inequalities, where the leading exponent equals the dimension, are focused. The technique involves rearrangements of functions, and tools from calculus of variations and ordinary differential equations.

1. Introduction.

1.1. In a paper where fluid mechanics blends with real and functional analysis, [Fr], L.E. Fraenkel supplied a proof that if $2 \leq q < \infty$ some constant A exists such that

$$\left\{ \int_{\mathbb{R}_+^2} |\varphi|^q y^{-q/2-2} dx dy \right\}^{1/q} \leq A \left\{ \int_{\mathbb{R}_+^2} (\varphi_x^2 + \varphi_y^2) y^{-1} dx dy \right\}^{1/2} \quad (1.1)$$

for every test function φ . Throughout we let \mathbb{R}_+^2 be

$$\{(x, y) : -\infty < x < \infty, 0 < y < \infty\},$$

the upper Euclidean half-plane, and let *test* qualify any smooth real-valued function defined in \mathbb{R}_+^2 whose value at (x, y) approaches zero fast enough as either y approaches 0 or $x^2 + y^2$ approaches infinity. (For technical reasons, our notations and terminology differ slightly from those adopted in [Fr].)

Though chiefly concerned with an existence theory for a partial differential equation, Fraenkel had an eye to the sharp form of his inequality, and detected indeed such a form in the case where $q = 2$ and $q = 10/3$. If $q = 2$, the smallest constant A , which renders (1.1) true for every test function φ , is exactly 1. If $q = 10/3$, let

$$A = 2^{6/5} \cdot 15^{-1/2} \cdot \pi^{-1/5}; \quad [=0.471802666130\dots];$$

then (1.1) holds if φ is any test function, and becomes an equality if φ is specified by

$$\varphi(x, y) = y^2(1 + x^2 + y^2)^{-3/2}.$$

The former statement is a straightforward consequence of the familiar Hardy's inequality, the latter rests upon the following change of variable

$$u(x_1, x_2, x_3, x_4, x_5) = (x_2^2 + x_3^2 + x_4^2 + x_5^2)^{-1} \varphi \left(x_1, \sqrt{x_2^2 + x_3^2 + x_4^2 + x_5^2} \right)$$

1991 *Mathematics Subject Classification.* 26D10, 45J95, 58D106.

Key words and phrases. Sobolev inequalities, hyperbolic space, calculus of variations.

and a Sobolev inequality in the Euclidean space of dimension 5.

Two moves help to disentangle the matter. One, which did appear in [Fr] and turns (1.1) into

$$A^{-2} \left\{ \int_{\mathbb{R}_+^2} |u|^q \frac{dx dy}{y^2} \right\}^{2/q} \leq \int_{\mathbb{R}_+^2} (u_x^2 + u_y^2) dx dy + \frac{3}{4} \int_{\mathbb{R}_+^2} u^2 \frac{dx dy}{y^2} \quad (1.2)$$

is making the change of variable

$$\varphi(x, y) = \sqrt{y} \cdot u(x, y).$$

Another is realizing that (1.2) falls under *Sobolev inequalities* in the *hyperbolic* (or *Poincaré half-plane*).

The present paper is the second of a series, devoted precisely to these inequalities. It continues [MT], which we refer to for preparatory results and a bibliography, and focuses some *borderline* Sobolev inequalities — instances, such as (1.2), where the leading exponent equals the dimension.

A motive here is to point out that certain lineaments of the hyperbolic half-plane — the Riemannian length and area, the geodesic polar coordinates, the isoperimetric theorem — and the theory of rearrangements outlined in [Bae] and [MT] are a key to Fraenkel's inequality. A feature will emerge indeed: if a point (a, b) is fixed in \mathbb{R}_+^2 ad libitum, then the test functions that really count in (1.1) — those rendering

$$\left\{ \int_{\mathbb{R}_+^2} (\varphi_x^2 + \varphi_y^2) y^{-1} dx dy \right\} \times \left\{ \int_{\mathbb{R}_+^2} |\varphi|^q y^{-q/2-2} dx dy \right\}^{-2/q}$$

a minimum — have this special form

$$\varphi(x, y) = \sqrt{y} \cdot v(s).$$

Here $s =$ *Riemannian area* of a *geodesic disk* whose radius equals the *Riemannian distance* between (a, b) and (x, y) ; v is a smooth real-valued function that is defined in $[0, \infty[$, decays fast enough at ∞ and makes

$$\left\{ \int_0^\infty s(s + 4\pi)(v') ds + \frac{3}{4} \int_0^\infty v^2 ds \right\} \times \left\{ \int_0^\infty |v|^q ds \right\}^{-2/q}$$

a minimum. In conclusion, Fraenkel's inequality amounts de facto to a variational problem for functions of a *single* variable, which can be treated by simple tools of the calculus of variations and the theory of ordinary differential equations.

1.2. Let \mathbb{H}^2 be \mathbb{R}_+^2 equipped with

$$y^{-2} [(dx)^2 + (dy)^2], \quad (1.3)$$

the *Poincaré-Bergman* metric. \mathbb{H}^2 is a Riemannian manifold that models the *two-dimensional hyperbolic space* and has the following properties — see [Bi, Chapter 14], [McC, Section 15], [SG, Section 9.5] and [Sie, Section 2.2], for example.

The Riemannian *length* of a tangent vector to \mathbb{H}^2 at a point (x, y) equals $y \times$ (the Euclidean length). The *geodesics* of \mathbb{H}^2 are the half-lines and the half-circles orthogonal to the x -axis. The Riemannian *distance* between two points (x_1, y_1) and (x_2, y_2) is the length of the geodesic arc joining (x_1, y_1) and (x_2, y_2) , and obeys

$$\cosh(\text{distance}) = \frac{1}{2y_1y_2} [(x_1 - x_2)^2 + y_1^2 + y_2^2]. \quad (1.4)$$

The Riemannian *area* on \mathbb{H}^2 , \mathcal{M} , is given by

$$d\mathcal{M} = \frac{dx dy}{y^2} \quad (1.5)$$

The *geodesic circle* in \mathbb{H}^2 , center (a, b) and radius r , has equation

$$(x - a)^2 + (y - b)^2 = [2 \sinh(r/2)]^2 by \quad (1.6a)$$

— hence coincides with the Euclidean circle whose center is $(a, b \cosh r)$ and whose radius is $b \sinh r$. The Riemannian radius, area and perimeter of a *geodesic disk* in \mathbb{H}^2 obey

$$\text{radius} = \log [1 + (2\pi)^{-1}(\text{area} + \text{perimeter})], \quad (1.6b)$$

$$\text{area} = \pi [2 \sinh(\text{radius}/2)]^2 \quad (1.6c)$$

$$\text{perimeter} = [(\text{area})^2 + 4\pi(\text{area})]^{1/2} = 2\pi \sinh(\text{radius}). \quad (1.6d)$$

The *Laplace-Beltrami* operator on \mathbb{H}^2 , Δ , is given by

$$\Delta = y^2 \cdot \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \quad (1.7)$$

The *curvature* of \mathbb{H}^2 is -1 . The following statements are closely related to the last mentioned one and germane to the present context — the former appears in [McK]; the latter, known as the *isoperimetric theorem* in \mathbb{H}^2 , appears in [BuZa, Section 10] and [Cha, Chapter 6], for example.

The *spectrum* of $-\Delta : L^2(\mathbb{H}^2) \mapsto L^2(\mathbb{H}^2)$ is exactly $[1/4, \infty[$.

If E is any sufficiently smooth subset of \mathbb{H}^2 , then the Riemannian *perimeter* and the Riemannian *area* of E , obey the following inequality. If the area of E is finite, then

$$\text{perimeter} \geq \sqrt{\text{area}(\text{area} + 4\pi)}; \quad (1.8)$$

moreover, equality holds in (1.8) if, and only if, E is a disk.

1.3. The architecture of a Sobolev inequality on a Riemannian manifold is affected by the underlying curvature. A typical *Sobolev inequality in hyperbolic*

space \mathbb{H}^2 claims that if p, q and R lie in a suitable range — $p \geq 1$, $q \geq p$, $1/q \geq 1/p - 1/2$ and $R \leq p^{-2}$ — then some positive constant exists such that

$$\left\{ \int_{\mathbb{R}_+^2} y^p (u_x^2 + u_y^2)^{p/2} \frac{dx dy}{y^2} \right\}^{2/p} \geq (\text{Constant}) \left\{ \int_{\mathbb{R}_+^2} |u|^q \frac{dx dy}{y^2} \right\}^{2/q} + R \left\{ \int_{\mathbb{R}_+^2} |u|^p \frac{dx dy}{y^2} \right\}^{2/p} \quad (1.9)$$

for every test function u .

Observe that

$$\left\{ \int_{\mathbb{R}_+^2} |u|^p \frac{dx dy}{y^2} \right\}^{1/p} = \left\{ \int_{\mathbb{H}^2} |u|^p d\mathcal{M} \right\}^{1/p},$$

the norm of u in Lebesgue space $L^p(\mathbb{H}^2)$, and similarly

$$\left\{ \int_{\mathbb{R}_+^2} |u|^q \frac{dx dy}{y^2} \right\}^{1/q} = \left\{ \int_{\mathbb{H}^2} |u|^q d\mathcal{M} \right\}^{1/q},$$

the norm of u in $L^q(\mathbb{H}^2)$. On the other hand,

$$\left\{ \int_{\mathbb{R}_+^2} y^p (u_x^2 + u_y^2)^{p/2} \frac{dx dy}{y^2} \right\}^{1/p} = \left\{ \int_{\mathbb{H}^2} |\nabla u|^p d\mathcal{M} \right\}^{1/p},$$

the norm of ∇u in $(L^p(\mathbb{H}^2))^2$. Proof: If u is a smooth scalar field on \mathbb{H}^2 then ∇u , the covariant derivative of u , is the tangent vector field to \mathbb{H}^2 whose components are u_x and u_y , and whose Riemannian length, $|\nabla u|$, equals $y\sqrt{u_x^2 + u_y^2}$.

In conclusion, inequality (1.9) reads

$$\|\nabla u\|_{(L^p(\mathbb{H}^2))^2}^2 \geq (\text{Constant}) \|u\|_{L^q(\mathbb{H}^2)}^2 + R \|u\|_{L^p(\mathbb{H}^2)}^2.$$

Let

$$C(p, q, R) = \text{the largest constant} \quad (1.10)$$

such that inequality (1.9) holds for every test function u . A theorem from [MT] — which was derived there as a consequence of the isoperimetric theorem in \mathbb{H}^2 , and implies the latter — results in the following equations:

$$C(p, q, R) = p^{-2} \quad (1.11a)$$

if $1 \leq p < \infty$, $q = p$ and $R = 0$;

$$C(p, q, R) = 4\pi \quad (1.11b)$$

if $p = 1$, $q = 2$ and $-\infty < R \leq 1$;

$$C(p, q, R) = \left(\frac{4\pi}{q}\right)^2 \left(\frac{q}{2} - 1\right)^{2/q} \left[\sin\left(\frac{2\pi}{q}\right)\right]^{-1} \quad (1.11c)$$

if $1 < p < 2$, $q = 2p/(2-p)$ and $-\infty < R \leq p^{-2}$;

$$C(p, q, R) = (4\pi)^{2/p} \left[\Gamma\left(\frac{p}{2(p-1)}\right)\right]^{2-2/p} \left[\Gamma\left(\frac{p-2}{2(p-1)}\right) \Gamma\left(\frac{1}{p-1}\right)\right]^{2/p-2} \quad (1.11d)$$

if $2 < p < \infty$, $q = \infty$ and $R = 0$.

1.4. In view of observations made in subsection 1.1, the smallest constant A which makes Fraenkel's inequality (1.1) true for every test function φ is given by

$$A^{-2} = C(p, q, -3/4)$$

— here $p = 2$ and $2 < q < \infty$, a case *not* covered by equations (1.11).

In the present paper we look precisely into such a case, and investigate $C(2, q, R)$.

By definition,

$$C(2, q, R) = \inf \frac{\int_{\mathbb{R}_+^2} (u_x^2 + u_y^2) dx dy - R \int_{\mathbb{R}_+^2} u^2 y^{-2} dx dy}{\left\{ \int_{\mathbb{R}_+^2} |u|^q y^{-2} dx dy \right\}^{2/q}} \quad (1.12)$$

under the conditions: u is a test function, $u \not\equiv 0$.

Alternatively, observe that

$$\int (u_x^2 + u_y^2) dx dy = - \int (u_{xx} + u_{yy}) u dx dy$$

if u is sufficiently well behaved — the left-hand side is the standard *Dirichlet integral*, the right-hand side is the *scalar product* of $(-\Delta u)$ and u in $L^2(\mathbb{H}^2)$. Deduce that $C(2, q, R)$ coincides with the largest constant such that

$$((-\Delta - R)u, u)_{L^2(\mathbb{H}^2)} \geq (\text{Constant}) \cdot \|u\|_{L^q(\mathbb{H}^2)}^2 \quad (1.13)$$

for every test function u .

Geodesic polar coordinates in \mathbb{H}^2 are introduced in the proof of Lemma 3.2 together with an instrumental variant. Arguments of dimensional analysis based upon these coordinates, spectral properties of the Laplace-Beltrami operator in \mathbb{H}^2 , formula (1.12) and inequality (1.13) show that

$$C(2, q, R) = \begin{cases} -\infty & \text{if } R > 1/4, \\ 0 & \text{if } q < 2 \text{ and } R \leq 1/4. \end{cases} \quad (1.14)$$

2. Main results.

Theorem 2.1. *Assume $2 < q < \infty$ and $-\infty < R < 1/4$.*

A test function u exists such that $u \not\equiv 0$ and

$$C(2, q, R) = \frac{\int_{\mathbb{R}_+^2} (u_x^2 + u_y^2) dx dy - R \int_{\mathbb{R}_+^2} u^2 y^{-2} dx dy}{\left\{ \int_{\mathbb{R}_+^2} |u|^q y^{-2} dx dy \right\}^{2/q}}; \quad (2.1)$$

both $C(2, q, R)$ and u are provided by the following recipe.

A smooth real-valued function defined in $[0, +\infty[$, v , exists such that:

(i) v satisfies the following differential equation

$$\frac{d}{ds} (s(s + 4\pi)v'(s)) + Rv(s) + |v(s)|^{q-2} \cdot v(s) = 0 \quad (2.2)$$

for $0 < s < \infty$.

(ii) v satisfies the following boundary conditions

$$-4\pi v'(0) = Rv(0) + |v(0)|^{q-2} \cdot v(0), \quad v(\infty) = 0, \quad (2.3)$$

and decays at infinity in such a way that

$$\int_0^\infty (sv'(s))^2 ds < \infty. \quad (2.4)$$

(iii) v is strictly decreasing.

(2.5)

Let (a, b) be any point in \mathbb{H}^2 .

The following equations hold:

$$C(2, q, R) = \left\{ \int_0^\infty |v(s)|^q ds \right\}^{1-2/q}, \quad (2.6)$$

and

$$u(x, y) = v \left(\frac{\pi}{by} ((x - a)^2 + (y - b)^2) \right) \quad (2.7)$$

for every (x, y) from \mathbb{H}^2 — in other terms,

$$u(x, y) = v(s), \quad (2.8a)$$

$$s = \pi [2 \sinh(r/2)]^2 = \text{Riemannian area of a geodesic disk of radius } r, \quad (2.8b)$$

$$r = \text{Riemannian distance between } (a, b) \text{ and } (x, y). \quad (2.8c)$$

Remarks. (i) Equations (2.7) and (2.8) inform that u is *radial and radially decreasing* — the value of u at any point (x, y) depends only upon the Riemannian distance between (a, b) and (x, y) , and decreases monotonically as such a distance increases.

An appropriate use of geodesic polar coordinates gives

$$\Delta u(x, y) = \frac{d}{ds} [s(s + 4\pi)v'(s)], \quad (2.9)$$

as detailed in the proof of Lemma 3.2 below. Plugging (2.9) in (2.2) results in

$$\Delta u + Ru + |u|^{q-2} \cdot u = 0 \quad (2.10)$$

— in the language of the calculus of variations, (2.10) is exactly the *Euler equation* implied by (1.12), (2.1) and a suitable normalization.

(ii) The *curvature* of \mathbb{H}^2 is a reason why the leading coefficient in equation (2.2) is a polynomial of degree two with two distinct roots. In fact, the genesis of such equation reveals that the coefficient in question = $4\pi s - (\text{curvature})s^2$.

Though harder than equations like

$$(4\pi s \cdot v')' + (\text{lower order terms}) = 0$$

— which appear in [Ka, Sections 6.73 to 6.76] and when borderline Sobolev inequalities in the *Euclidean* plane are considered — equation (2.2) has certain particular solutions available in *closed form*. The function defined by

$$v(s) = (q-2)^{-2/(q-2)} \left(1 + \frac{s}{4\pi}\right)^{-1/(q-2)} \quad (2.11)$$

happens to satisfy (2.2) if $q > 2$ and $R = (q-3)(q-2)^{-2}$. The function defined by

$$v(s) = \left[\frac{2q}{(q-2)^2}\right]^{1/(q-2)} \left(1 + \frac{s}{2\pi}\right)^{-2/(q-2)} \quad (2.12)$$

satisfies (2.2) if $q > 2$ and $R = 2(q-4)(q-2)^{-2}$.

(iii) Some *radial* solutions to equation (2.10) result from the previous remarks. The function defined by

$$u(x, y) = \left[\frac{2}{(q-2)}\right]^{2/(q-2)} \left[\frac{by}{(x-a)^2 + (y+b)^2}\right]^{1/(q-2)} \quad (2.13)$$

satisfies (2.10) in the case where $q > 2$ and $R = (q-3)(q-2)^{-2}$; the function defined by

$$u(x, y) = \left[\frac{4q}{(q-2)^2}\right]^{1/(q-2)} \left[\frac{by}{(x-a)^2 + y^2 + b^2}\right]^{2/(q-2)} \quad (2.14)$$

satisfies (2.10) in the case where $q > 2$ and $R = 2(q-4)(q-2)^{-2}$.

Constant factors apart, some of these solutions appear in theorem 2.2 below, and one of them appeared between the lines in subsection 1.1.

(iv) The function defined by (2.11) satisfies boundary conditions (2.3). Property (2.4) holds if in addition $q \leq 4$, whereas (2.5) holds plainly. The function defined by (2.12) does the same if $2 < q \leq 6$.

Coupling (2.6), (2.11) and (2.12) would give

$$C(2, q, R) = (2\pi)^{1-2/q} (q-2)^{-1-2/q} \quad (2.15)$$

in the case where

$$-\infty < R \leq 1/4, \quad q = 2 + 2 \cdot \left(1 + \sqrt{1 - 4R}\right)^{-1},$$

and

$$C(2, q, R) = (2\pi)^{1-2/q} (q-2)^{-1-2/q} \cdot 2q(q+2)^{-1+2/q} \quad (2.16)$$

in the case where

$$-\infty < R \leq 1/4, \quad q = 2 + 4 \cdot \left(1 + \sqrt{1 - 4R}\right)^{-1}.$$

One might wonder whether formulas (2.15) and (2.16) are correct. Fraenkel's result tells us that (2.16) holds if $q = 10/3$ and $R = -3/4$. Theorem 2.2 below shows that (2.15) and (2.16) identify $C(2, q, R)$ properly in the case where R is zero.

(v) If q is larger than 2 (but is not too large) and R is below $1/4$, then $C(2, q, R)$ can be accurately computed by an algorithm. Such algorithm consists in: (i) solving equation (2.2) subject to conditions (2.3) and (2.4) via an appropriate shooting method; (ii) selecting a solution satisfying condition (2.5); (iii) checking the uniqueness of this solution via an appropriate test; (iv) using formula (2.6).

Details can be found in [Mu], sample values are displayed in the table below.

q	$C(2, q, 0)$	q	$C(2, q, 0)$
3.0	1.8452701	8.0	3.6479107
10/3	2.2655626	8.5	3.6096511
3.5	2.4484193	9.0	3.5635303
4.0	2.8944050	9.5	3.5119626
4.5	3.2085021	10.0	3.4567447
5.0	3.4207189	11.0	3.3403841
5.5	3.5569171	12.0	3.2216028
6.0	3.6372463	13.0	3.1043799
6.5	3.6767932	14.0	2.9908876
7.0	3.6866772	15.0	2.8823095
7.5	3.6750545		

(2.17)

Theorem 2.2. (i) $C(2, 3, 0) = (2\pi)^{1/3}$, i.e. the following inequality

$$\int_{\mathbb{R}_+^2} (u_x^2 + u_y^2) dx dy \geq (2\pi)^{1/3} \left\{ \int_{\mathbb{R}_+^2} |u|^3 \frac{dx dy}{y^2} \right\}^{2/3} \quad (2.18)$$

holds for every test function u and is sharp. Equality takes place in (2.18) if

$$u(x, y) = \frac{y}{x^2 + (y + 1)^2}.$$

(ii) $C(2, 4, 0) = (8\pi/3)^{1/2}$, i.e. the following inequality

$$\int_{\mathbb{R}_+^2} (u_x^2 + u_y^2) dx dy \geq \left\{ \frac{8\pi}{3} \int_{\mathbb{R}_+^2} u^4 \frac{dx dy}{y^2} \right\}^{1/2} \quad (2.19)$$

holds for every test function u and is sharp. Equality takes place in (2.19) if

$$u(x, y) = \frac{y}{x^2 + y^2 + 1}.$$

3. Key lemmas.

The lemmas from this section, culminating in Lemma 3.5 below, show that the Sobolev inequalities in hand amount to a variational problem in dimension one.

Lemma 3.1. If u is any test function, then a real-valued function defined in $[0, \infty[$, u^* , exists such that:

- (i) $u^*(\infty) = 0$;
- (ii) u^* is decreasing;
- (iii) u^* is equidistributed with u ;
- (iv) u^* is absolutely continuous, and the following inequality holds

$$\int_{\mathbb{R}_+^2} (u_x^2 + u_y^2) dx dy \geq \int_0^\infty s(s + 4\pi) \left[\frac{du^*}{ds}(s) \right]^2 ds.$$

Proof. Let μ be the decreasing right-continuous map from $[0, \infty[$ into $[0, \infty]$ defined by the following formula

$$\mu(t) = \text{Riemannian area of } \{(x, y) \in \mathbb{H}^2 : |u(x, y)| > t\}, \quad (3.1)$$

and let u^* the map from $[0, \infty[$ into $[0, \infty]$ defined by

$$u^*(s) = \min \{t \geq 0 : \mu(t) \leq s\}. \quad (3.2)$$

(These definitions mimic those introduced by Hardy and Littlewood and elaborated by several authors: μ is the *distribution function*, u^* is the *decreasing rearrangement* of u — see [HLP, Chapter 10], [Kw], [Ko], [Bae], [Ta], and the references quoted therein.)

Clearly, formulas (3.1) and (3.2) imply properties (i) and (ii).

The same formulas and a brief reflection yield $\{s \geq 0 : u^*(s) > t\} = [0, \mu(t)[$ for every nonnegative t . We deduce that the following equation

$$\text{length of } \{s \geq 0 : u^*(s) > t\} = \text{Riemannian area of } \{(x, y) \in \mathbb{H}^2 : |u(x, y)| > t\} \quad (3.3)$$

holds for every nonnegative t . Property (iii) is demonstrated.

A version of a theorem, which is central to the present context and is offered in [Bae, Sections 3 and 4] and [MT, Section 2], implies property (iv). \square

Lemma 3.2. Suppose v is a real-valued function defined in $[0, \infty[$; suppose v is smooth and decays fast enough at infinity. Let (a, b) be any point in \mathbb{H}^2 , and let u be defined by

$$u(x, y) = v \left(\frac{\pi}{by} ((x-a)^2 + (y-b)^2) \right). \quad (3.4)$$

Then u is a test function and the following properties hold:

- (i) u and v are *equidistributed*;
- (ii) $\int_{\mathbb{R}_+^2} (u_x^2 + u_y^2) dx dy = \int_0^\infty s(s + 4\pi)(v'(s))^2 ds$.

Proof. There is a notational convenience and no loss of generality in assuming that $a = 0$ and $b = 1$. The following equations

$$x = \frac{\sinh r \cdot \sin \vartheta}{\cosh r - \sinh r \cdot \cos \vartheta}, \quad y = \frac{1}{\cosh r - \sinh r \cdot \cos \vartheta}, \quad (3.5a)$$

the following others

$$\begin{aligned} \cos \vartheta &= \frac{x^2 + y^2 - 1}{\sqrt{x^2 + (y-1)^2} \cdot \sqrt{x^2 + (y+1)^2}}, \\ \sin \vartheta &= \frac{2x}{\sqrt{x^2 + (y-1)^2} \cdot \sqrt{x^2 + (y+1)^2}}, \\ \cosh r &= \frac{1}{2y} (1 + x^2 + y^2), \end{aligned} \quad (3.5b)$$

and the following constraints

$$0 \leq r < \infty, \quad -\pi < \vartheta < \pi \quad (3.5c)$$

define a system of *geodesic polar coordinates* in \mathbb{H}^2 . Here r stands for the *Riemannian distance* between $(0, 1)$ and (x, y) ; a line where $\vartheta = \text{Constant}$ is a *geodesic arc* whose origin is $(0, 1)$ and whose angle with the y -axis is ϑ .

Define

$$s = \pi [2 \sinh(r/2)]^2, \quad (3.6a)$$

the *Riemannian area* of a geodesic disk of radius r . In other terms,

$$\begin{aligned} 2\pi \cdot \cosh r &= 2\pi + s, & 2\pi \cdot \sinh r &= \sqrt{s(s + 4\pi)}, \\ s &= \frac{\pi}{y} (x^2 + (y-1)^2). \end{aligned} \quad (3.6b)$$

The system of curvilinear coordinates made up by s and ϑ is especially convenient in the present context — such a system makes a *dimensional analysis* of Sobolev inequalities possible, by the way.

The Poincaré-Bergman metric obeys

$$\begin{aligned} y^{-2} [(dx)^2 + (dy)^2] &= (dr)^2 + (\sinh r)^2 (d\vartheta)^2 \\ &= \frac{(ds)^2}{s(s+4\pi)} + \frac{s(s+4\pi)}{4\pi^2} (d\vartheta)^2. \end{aligned} \quad (3.7)$$

Formulas (3.7) and customary rules of differential geometry tell us that

$$d(\text{Riemannian area}) = (\sinh r) dr d\vartheta = \frac{1}{2\pi} ds d\vartheta, \quad (3.8)$$

and that any sufficiently smooth function f obeys

$$\begin{aligned} |\nabla f|^2 &= \left(\frac{\partial f}{\partial r} \right)^2 + (\sinh r)^{-2} \left(\frac{\partial f}{\partial \vartheta} \right)^2 \\ &= s(s+4\pi) \left(\frac{\partial f}{\partial s} \right)^2 + \frac{4\pi^2}{s(s+4\pi)} \left(\frac{\partial f}{\partial \vartheta} \right)^2. \end{aligned} \quad (3.9)$$

Incidentally, observe the following formula

$$\begin{aligned} \Delta f &= (\sinh r)^{-1} \frac{\partial}{\partial r} \left(\sinh r \frac{\partial f}{\partial r} \right) + (\sinh r)^{-2} \frac{\partial^2 f}{\partial \vartheta^2} \\ &= \frac{\partial}{\partial s} \left[s(s+4\pi) \frac{\partial f}{\partial s} \right] + \frac{4\pi^2}{s(s+4\pi)} \frac{\partial^2 f}{\partial \vartheta^2} \end{aligned} \quad (3.10)$$

Owing to equation (3.4), one may check that u is smooth — in particular,

$$|\nabla u(x, y)| = O(r) \text{ as } r \rightarrow 0.$$

Moreover, $u(x, y)$ approaches zero fast as r approaches infinity.

Equations (3.4) and (3.8) imply that

$$\text{Riemannian area of } \{(x, y) \in \mathbb{H}^2 : |u(x, y)| > t\} = \text{length of } \{s \geq 0 : |v(s)| > t\}$$

for every nonnegative t — property (i) is demonstrated.

Equations (3.4), (3.8) and (3.9) give

$$\text{the Dirichlet integral of } u = \int_0^\infty s(s+4\pi)(v'(s))^2 ds$$

— property (ii) is demonstrated. \square

A couple of definitions are involved in Lemma 3.3 and sections 4 and 5 below.

Definition 3.3. J is the functional defined by

$$J(v) = \frac{\int_0^\infty s(s+4\pi)(v'(s))^2 ds - R \int_0^\infty (v(s))^2 ds}{\left\{ \int_0^\infty |v(s)|^q ds \right\}^{2/q}}. \quad (3.11)$$

Definition 3.4. \mathfrak{V} denotes the collection of those real-valued functions v defined in $]0, \infty[$ such that:

- (i) $v(\infty) = 0$;
- (ii) v is absolutely continuous, and both $\int_0^\infty s(v'(s))^2 ds$ and $\int^\infty (sv'(s))^2 ds$ converge.

Lemma 3.5. Suppose $0 < q < \infty$ and $-\infty < R < \infty$. Then the following holds:

- (i) If l is any real number, then either

$$\{v : 0 \neq v \in \mathfrak{V}, J(v) \leq l\} = \emptyset$$

or

$$\{v : 0 \neq v \in \mathfrak{V}, J(v) \leq l\} \cap \{v : v \in \mathfrak{V}, v \text{ decreases}\} \neq \emptyset.$$

- (ii) $\inf \{J(v) : 0 \neq v \in \mathfrak{V}\} = C(2, q, R)$ (3.12)

Proof. Combine Lemma 3.1 and Lemma 3.2. \square

4. Estimating $C(2, q, 0)$.

Theorem 4.1. *The following inequality*

$$C(2, q, 0) \geq \left\{ \frac{8\pi q}{(q-2)^2} \frac{\left(\Gamma\left(\frac{q}{q-2}\right)\right)^2}{\Gamma\left(\frac{2q}{q-2}\right)} \right\}^{1-2/q} \quad (4.1)$$

holds if $2 < q < \infty$.

Proof. Let $2 < q < \infty$ and $R = 0$. Since

$$s(s+4\pi) \geq 4^{1-3/q} \cdot q \cdot \left(\frac{\pi}{q-2}\right)^{1-2/q} s^{1+2/q}$$

for every positive s , we have

$$J(v) \geq 4^{1-3/q} \cdot q \cdot \left(\frac{\pi}{q-2}\right)^{1-2/q} \int_0^\infty s^{1+2/q} (v'(s))^2 ds \times \left\{ \int_0^\infty |v(s)|^q ds \right\}^{-2/q} \quad (4.2)$$

if $0 \neq v \in \mathfrak{V}$. Applying Lemma 4.2 below we deduce

$$J(v) \geq \left\{ \frac{8\pi q}{(q-2)^2} \frac{\left(\Gamma\left(\frac{q}{q-2}\right)\right)^2}{\Gamma\left(\frac{2q}{q-2}\right)} \right\}^{1-2/q} \quad (4.3)$$

if $0 \neq v \in \mathfrak{V}$. Applying (ii), Lemma 3.5, concludes the proof. \square

Lemma 4.2. Let $2 < q < \infty$; let v be a real-valued absolutely continuous function defined in $]0, \infty[$ such that $v(\infty) = 0$. Then

$$\int_0^\infty s^{1+2/q} (v'(s))^2 ds \geq 2q^{-2/q} \left\{ \frac{\left(\Gamma\left(\frac{q}{q-2}\right) \right)^2}{(q-2)\Gamma\left(\frac{2q}{q-2}\right)} \right\}^{1-2/q} \left\{ \int_0^\infty |v(s)|^q ds \right\}^{2/q}, \quad (4.4)$$

and equality holds if v is specified by $v(s) = (1 + s^{1-2/q})^{-2/(q-2)}$.

Proof. (4.4) is a variant of an inequality by Bliss, [Bl].

5. Further lemmas.

The lemmas from this section prepare a proof of Theorem 2.1, and describe properties of function space \mathfrak{V} and functional J . Lemma 5.2, coupled with a standard theorem of functional analysis, enables to assert that if $2 < q < \infty$ any bounded subset of \mathfrak{V} is relatively compact in $L^q(0, \infty)$. Lemma 5.5 enables to show that if $-\infty < R < 1/4$ the restriction of J to the unit sphere of $L^q(0, \infty)$ is lower semicontinuous.

Lemma 5.1. If v belongs to \mathfrak{V} , then

$$(v(s))^2 \leq \log\left(1 + \frac{\varepsilon}{s}\right) \cdot \left\{ \int_s^\infty t(v'(t))^2 dt + \varepsilon^{-1} \int_s^\infty (tv'(t))^2 dt \right\} \quad (5.1)$$

for every positive s and every positive ε . The following asymptotics hold:

$$v(s) = o\left(s^{-1/2}\right) \text{ as } s \rightarrow \infty, \quad v(s) = O\left(\sqrt{\log \frac{1}{s}}\right) \text{ as } s \rightarrow 0. \quad (5.2)$$

Proof. Since v is absolutely continuous and $v(\infty) = 0$, we have

$$v(s) = \int_s^\infty (-v'(t)) dt$$

for every positive s — thus

$$v(s) = \int_s^\infty (\varepsilon t + t^2)^{-1/2} \cdot (\varepsilon t + t^2)^{1/2} (-v'(t)) dt$$

if both s and ε are positive. Hence Schwarz inequality gives

$$(v(s))^2 \leq \int_s^\infty \frac{dt}{\varepsilon t + t^2} \times \left\{ \varepsilon \int_s^\infty t(v'(t))^2 dt + \int_s^\infty (tv'(t))^2 dt \right\}$$

if both s and ε are positive. Inequality (5.1) follows.

Properties (5.2) are an obvious consequence of (5.1). \square

Lemma 5.2. If $2 < q < \infty$, then a positive constant C exists such that

$$\int_h^\infty |v(s)|^q ds \leq C \cdot (\varepsilon + h)^{1-q/2} \cdot \left\{ \varepsilon \int_0^\infty s(v'(s))^2 ds + \int_0^\infty (sv'(s))^2 ds \right\}^{q/2} \quad (5.3)$$

and

$$\int_0^\infty |v(s+h) - v(s)|^q ds \leq C \cdot \frac{h}{1+\varepsilon h} \cdot \left\{ \int_0^\infty s(v'(s))^2 ds + \varepsilon \int_0^\infty (sv'(s))^2 ds \right\}^{q/2} \quad (5.4)$$

for every v from \mathfrak{A} , and every nonnegative h and ε .

Proof. A formula from [GR, Section 4.271] gives

$$\int_0^\infty \left[\log \left(1 + \frac{1}{s} \right) \right]^k ds = \Gamma(k+1) \zeta(k) \quad (5.5)$$

if $k > 1$.

Formula (5.5) implies

$$\int_h^\infty \left[\log \left(1 + \frac{1}{s} \right) \right]^k ds \leq \Gamma(k+1) \zeta(k) \cdot (1+h)^{1-k}$$

if $h \geq 0$ and $k > 1$, since

$$(1+h)^{k-1} \int_h^\infty \left[\log \left(1 + \frac{1}{s} \right) \right]^k ds$$

decreases monotonically as h increases from 0 to ∞ and $k > 1$. Lemma 5.1 yields

$$\int_h^\infty |v(s)|^q ds \leq \int_h^\infty \left[\log \left(1 + \frac{\varepsilon}{s} \right) \right]^{q/2} ds \times \left\{ \int_0^\infty s(v'(s))^2 ds + \varepsilon^{-1} \int_0^\infty (sv'(s))^2 ds \right\}^{q/2}$$

if h is nonnegative and ε is positive. Therefore

$$\int_h^\infty |v(s)|^q ds \leq \varepsilon \cdot \frac{\Gamma(1+q/2) \zeta(q/2)}{(1+h/\varepsilon)^{q/2-1}} \cdot \left\{ \int_0^\infty s(v'(s))^2 ds + \varepsilon^{-1} \int_0^\infty (sv'(s))^2 ds \right\}^{q/2}$$

if h is nonnegative and ε is positive. Inequality (5.3) is demonstrated.

Since $v(s+h) - v(s) = \int_s^{s+h} v'(t) dt$, we have

$$v(s+h) - v(s) = \int_s^{s+h} (t^2 + t/\varepsilon)^{-1/2} \cdot (t^2 + t/\varepsilon)^{1/2} v'(t) dt$$

if s , h and ε are positive. As

$$\int_s^{s+h} \frac{dt}{t^2 + t/\varepsilon} = \varepsilon \log \frac{1 + \frac{h}{s}}{1 + \frac{\varepsilon h}{s}} \leq \varepsilon \log \left[1 + \frac{h}{(1 + \varepsilon h)s} \right],$$

Schwarz inequality gives

$$|v(s+h) - v(s)|^2 \leq \log \left[1 + \frac{h}{(1 + \varepsilon h)s} \right] \cdot \left\{ \int_0^\infty t(v'(t))^2 dt + \varepsilon \int_0^\infty (tv'(t))^2 dt \right\}$$

if s , h and ε are positive. Owing to (5.5), we deduce

$$\int_0^\infty |v(s+h) - v(s)|^q ds \leq \frac{h}{1 + \varepsilon h} \cdot \Gamma \left(1 + \frac{q}{2} \right) \zeta \left(\frac{q}{2} \right) \cdot \left\{ \int_0^\infty s(v'(s))^2 ds + \varepsilon \int_0^\infty (sv'(s))^2 ds \right\}^{q/2}.$$

Inequality (5.4) is demonstrated. \square

Lemma 5.3. Suppose v is a real-valued function defined in $]0, \infty[$; suppose v is absolutely continuous and $v(\infty) = 0$. Then

$$\int_0^\infty (sv'(s))^2 ds \geq \frac{1}{4} \int_0^\infty (v(s))^2 ds. \quad (5.6)$$

Proof. Pass to the limit in inequality (4.4) as $q \rightarrow 2$, or see [HLP, Theorem 328]. Alternatively, use the following identity

$$v^2 + (v + 2sv')^2 = 4(sv')^2 + 2(sv^2)',$$

or Plancherel's theorem for Mellin transforms. \square

The following definition is involved in Lemma 5.5 and subsection 6.1 below.

Definition 5.4. K is the functional defined by

$$K(v) = \int_0^\infty s(s + 4\pi)(v'(s))^2 ds - R \int_0^\infty (v(s))^2 ds.$$

Lemma 5.5. If $-\infty < R < 1/4$ and $v \in \mathfrak{B}$, then

$$\sqrt{K(v)} = \sup \int_0^\infty v \left\{ -\frac{d}{ds} (s(s + 4\pi)\varphi') - R\varphi \right\} ds$$

provided the trial functions obey:

φ is smooth and behaves well near 0 and ∞ , $K(\varphi) = 1$.

Proof. Let Q be the bilinear symmetric form defined by

$$Q(v, \varphi) = \int_0^\infty s(s + 4\pi)v'(s)\varphi'(s)ds - R \int_0^\infty v(s)\varphi(s)ds.$$

Lemma 5.3 tells us that $Q(v, v) \geq 0$ for every v from \mathfrak{V} , if $R < 1/4$. We deduce

$$\sqrt{Q(v, v)} = \max \{Q(v, \varphi) : \varphi \in \mathfrak{V}, Q(\varphi, \varphi) = 1\}$$

for every v from \mathfrak{V} , if $R < 1/4$. An integration by parts shows that

$$Q(v, \varphi) = \int_0^\infty v \left\{ -\frac{d}{ds} (s(s + 4\pi)\varphi') - R\varphi \right\} ds$$

provided v is in \mathfrak{V} , φ is smooth enough and behaves well near 0 and ∞ , and $K(\varphi) = 1$.

The conclusion follows. \square

6. Proof of Theorem 2.1.

6.1. A member of \mathfrak{V} , v , exists such that

$$\int_0^\infty |v(s)|^q ds = 1 \tag{6.1}$$

and

$$J(v) = \inf \{J(\varphi) : 0 \neq \varphi \in \mathfrak{V}\}. \tag{6.2}$$

Proof. We have to show that the restriction of J to the unit sphere of $L^q(0, \infty)$ attains a least value. Two typical ingredients of the calculus of variations are involved here: *compactness* and *semicontinuity*.

Lemmas 5.2 and 5.3, and a standard theorem (see e.g. [DS, Theorem IV.8.20]) tell us that the set defined by the following conditions

$$v \in \mathfrak{V}, \quad \int_0^\infty |v(s)|^q ds = 1, \quad K(v) \leq \text{constant}$$

is relatively compact with respect to the topology of $L^q(0, \infty)$ — Lemma 5.3 ensures boundedness with respect to a topology of \mathfrak{V} , Lemma 5.2 ensures sufficient conditions for compactness with respect to the topology of $L^q(0, \infty)$.

Lemma 5.5 implies that K is lower semicontinuous with respect to the topology of $L^q(0, \infty)$.

We deduce that the set mentioned above is compact with respect to the topology of $L^q(0, \infty)$, and the restriction of K to such a set attains a least value.

Since J is positively homogeneous of degree zero and

$$J(v) = K(v) \times \left\{ \int_0^\infty |v(s)|^q ds \right\}^{-2/q},$$

the assertion is demonstrated. \square

6.2. Function v satisfies the following differential equation

$$\frac{d}{ds} (s(s + 4\pi)v'(s)) + Rv(s) + J(v) \cdot |v(s)|^{q-2} \cdot v(s) = 0 \quad (6.3)$$

for $0 < s < \infty$, and the following boundary condition

$$s(s + 4\pi)v'(s) \rightarrow 0 \text{ as } s \rightarrow 0. \quad (6.4)$$

Proof. The Gateaux differential of J , J' , is the map from \mathfrak{V} into the appropriate dual of \mathfrak{V} whose value at v obeys

$$J'(v)(\varphi) = \lim_{t \rightarrow 0} \frac{1}{t} [J(v + t\varphi) - J(v)]$$

for every φ from \mathfrak{V} . Let v belong to \mathfrak{V} and satisfy (6.1). An inspection shows

$$\frac{1}{2}J'(v)(\varphi) = \int_0^\infty s(s + 4\pi)v'\varphi' ds - R \int_0^\infty v \varphi ds - J(v) \int_0^\infty |v|^{q-2}v \varphi ds$$

for every φ from \mathfrak{V} . The asymptotic behavior of $v(s)$ as $s \rightarrow 0$ or $s \rightarrow \infty$ is displayed in (5.2), Lemma 5.1. Hence an integration by parts yields

$$\frac{1}{2}J'(v)(\varphi) = \int_0^\infty \left\{ s(s + 4\pi)v'(s) + R \int_0^s v(t)dt + J(v) \int_0^s |v(t)|^{q-2}v(t)dt \right\} \varphi'(s)ds$$

for every φ from \mathfrak{V} .

Property (6.2) implies $J'(v) = 0$. We deduce

$$s(s + 4\pi)v'(s) + R \int_0^s v(t)dt + J(v) \int_0^s |v(t)|^{q-2}v(t)dt = 0 \quad (6.5)$$

— correcting v on a set of measure zero guarantees that equation (6.5) holds for every positive s .

Equation (6.5) implies both (6.3) and (6.4) — and viceversa. \square

6.3. Function v satisfies the following boundary condition

$$-4\pi v'(0) = Rv(0) + J(v) \cdot |v(0)|^{q-2} \cdot v(0). \quad (6.6)$$

Proof. Equation (6.5) gives

$$4\pi \int_0^\infty |v'(s)|ds \leq \int_0^\infty \log \left(1 + \frac{4\pi}{s} \right) \{ R|v(s)| + J(v) \cdot |v(s)|^{q-1} \} ds.$$

This inequality includes the following information

$$\int_0^\infty |v'(s)|ds < \infty,$$

because of formula (5.5) and Lemma 5.1.

We deduce that v is continuous up to 0. Dividing both sides of (6.5) by s , then letting $s \rightarrow 0$ gives (6.6).

6.4. Function v satisfies

$$v(\infty) = 0 \quad \text{and} \quad \int_0^\infty (sv'(s))^2 ds < \infty \quad (6.7)$$

trivially — these conditions are included in the membership to \mathfrak{B} .

6.5. Statement (i) from Lemma 3.5 enables us to convert v into a function that simultaneously minimizes J and is decreasing. In other words, we can assume that v obeys all the properties listed above and, in addition, is decreasing.

Observe that v decreases strictly and is smooth. Indeed, any decreasing solution of equation (6.3) is either constant or strictly decreasing; any positive solution of (6.3) is infinitely differentiable.

6.6. Replace v by

$$[J(v)]^{1/(q-2)} \times v \quad (6.8)$$

— in other words, renormalize v in such a way that

$$\left\{ \int_0^\infty |v(s)|^q ds \right\}^{1-2/q} = J(v). \quad (6.9)$$

The renormalized v satisfies conditions (2.2) to (2.5). Condition (2.6) results from (6.9) and statement (ii), Lemma 3.5.

6.7. Applying Lemma 3.2 concludes the proof. \square

7. Proof of Theorem 2.2.

7.1. A function which satisfies the following differential equation

$$\frac{d}{ds} (s(s + 4\pi)v'(s)) + |v(s)|^{q-2} \cdot v(s) = 0, \quad (7.1)$$

for $0 < s < \infty$, decays at infinity in such a way that

$$v(\infty) = 0 \quad \text{and} \quad \int_0^\infty (sv'(s))^2 ds < \infty, \quad (7.2)$$

and obeys the following boundary condition

$$-4\pi v'(0) = |v(0)|^{q-2} \cdot v(0) \quad (7.3)$$

is given by

$$v(s) = \left(1 + \frac{s}{4\pi}\right)^{-1} \quad \text{or} \quad v(s) = \sqrt{2} \cdot \left(1 + \frac{s}{2\pi}\right)^{-1}$$

according to whether $q = 3$ or $q = 4$.

Therefore Theorem 2.2 results from Theorem 2.1 and the following statement: if $q \leq 3 \leq \infty$, conditions (7.1), (7.2), (7.3), plus the following one

$$v'(s) < 0 \quad \text{for } 0 < s < \infty \quad (7.4)$$

identify v *uniquely*.

The subsequent subsections step toward a proof of this statement.

7.2. Two remarks are in order:

- (i) Let $3 \leq q < \infty$; if v satisfies (7.1) and (7.2), then $s^2 v(s) \rightarrow 0$ as $s \rightarrow 0$.
- (ii) Let $2 < q < \infty$; if v satisfies (7.1) and $sv'(s) \rightarrow 0$ as $s \rightarrow 0$ then (7.3) holds.

Proof of (i). Equation (7.1) gives

$$\lim_{t \rightarrow \infty} t(t + 4\pi) \cdot v'(t) = s(s + 4\pi) \cdot v'(s) + \int_s^\infty |v(t)|^{q-2} v(t) dt$$

if s is positive; condition (7.2) and Lemmas 4.2 and 5.3 imply that

$$\int_s^\infty |v(t)|^{q-1} dt < \infty$$

if $q \geq 3$. \square

Proof of (ii). The hypotheses give

$$-s(s + 4\pi) \cdot v'(s) = \int_0^s |v(t)|^{q-2} v(t) dt$$

for every positive s , and $v(s) = o(\log(1/s))$ as $s \rightarrow 0$. The proof goes ahead as in subsection 6.4. \square

7.3. If $q \geq 3$, the change of variables defined by

$$s = \frac{4\pi}{e^{2t} - 1}, \quad t = \frac{1}{2} \log \left(1 + \frac{4\pi}{s}\right), \quad v(s) = \lambda^{1/(q-2)} u(t)$$

converts the set made up by (7.1), (7.2), (7.3) and (7.4) into the set consisting of conditions (7.5), (7.6), (7.7) and (7.8) below. In other words, our goal becomes identifying a sufficiently smooth real-valued function defined in $[0, \infty[$, u , and a nonnegative parameter, λ , such that u and λ satisfy the following equation

$$\frac{d^2 u}{dt^2}(t) + \lambda(\sinh t)^{-2} |u(t)|^{q-2} u(t) = 0 \quad (7.5)$$

for $0 < t < \infty$, and u satisfies the following conditions

$$u(0) = 0 \quad \text{and} \quad \frac{du}{dt}(0) = 1 \quad (7.6)$$

$$\frac{du}{dt}(\infty) = 0, \quad (7.7)$$

$$\frac{du}{dt}(t) > 0 \quad \text{for} \quad 0 < t < \infty. \quad (7.8)$$

7.4. If λ is positive and $2 < q < \infty$, any solution to (7.5) and (7.6) has the following properties:

(i) $\left| \frac{du}{dt}(t) \right| \leq 1$

and

$$|u(t)| \leq \min \left\{ t, \left(\frac{q}{2\lambda} \right)^{1/q} (\sinh t)^{2/q} \right\}$$

for every nonnegative t .

(ii) u is asymptotically linear — more precisely, two constants A and B exist such that

$$\frac{du}{dt}(t) = A + \lambda \cdot O(t^{q-1}e^{-2t}) \quad \text{and} \quad u(t) = At + B + \lambda \cdot O(t^{q-1}e^{-2t})$$

as $t \rightarrow \infty$.

(iii) u has finitely many zeroes and finitely many bend points.

(iv) u is concave if and only if u has no positive zeroes.

(v) u is increasing if and only if u has no positive zeroes.

Proof of (i). Let H be defined by

$$H(t) = \left[\frac{du}{dt}(t) \right]^2 + (2\lambda/q) \cdot (\sinh t)^{-2} \cdot |u(t)|^q.$$

Equation (7.5) gives

$$\frac{dH}{dt}(t) = (2\lambda/q) \cdot |u(t)|^q \cdot \frac{d}{dt}(\sinh t)^{-2},$$

therefore

$$\frac{dH}{dt}(t) \leq 0$$

for every positive t . Initial conditions (7.6) give

$$H(0+) = 1.$$

We deduce

$$\left[\frac{du}{dt}(t) \right]^2 + (2\lambda/q) \cdot (\sinh t)^{-2} \cdot |u(t)|^q \leq 1$$

for every positive t . Property (i) follows. \square

Proof of (ii). Equation (7.5) plus initial conditions (7.6) give

$$\frac{du}{dt}(t) = 1 - \lambda \int_0^t (\sinh s)^{-2} |u(s)|^{q-2} u(s) ds.$$

Hence we have

$$\frac{du}{dt}(t) = 1 - \lambda \int_0^\infty (\sinh s)^{-2} |u(s)|^{q-2} u(s) ds + \text{a remainder},$$

where

$$|\text{remainder}| \leq \lambda \int_t^\infty (\sinh s)^{-2} |u(s)|^{q-1} ds.$$

Property (ii) follows, since (i) gives $|u(t)| \leq t$. \square

Proof of (iii). Equation (7.5) plus initial conditions (7.6) give

$$\frac{du}{dt}(t) \geq 1 - \lambda \int_0^t |u(s)|^{q-1} (\sinh s)^{-2} ds.$$

Therefore

$$\frac{du}{dt}(t) \geq 1 - \lambda \int_0^t s^{q-1} (\sinh s)^{-2} ds$$

because of property (i), consequently

$$\frac{du}{dt}(t) \geq 1 - \frac{\lambda}{q-2} t^{q-2}$$

and

$$u(t) \geq t - \frac{\lambda}{(q-2)(q-1)} t^{q-1}.$$

We deduce that $u(t)$ increases strictly as t increases from 0 to $[(q-2)/\lambda]^{1/(q-2)}$ and is strictly positive as $0 < t < [(q-2)(q-1)/\lambda]^{1/(q-2)}$ — in other words, u has neither positive zeroes nor bend points in some neighborhood of 0.

Let z be defined by

$$z = \frac{du}{dt} \cdot \sinh t. \tag{7.9}$$

Equations (7.5) and (7.9) yield

$$z \cdot \cosh t - (dz/dt) \cdot \sinh t = \lambda |u|^{q-2} u.$$

Eliminating u between the last two equations gives

$$\frac{d^2 z}{dt^2} + \left\{ -1 + (q-1) \frac{\lambda^{1/(q-1)}}{(\sinh t)^2} \left| z \cdot \cosh t - \frac{dz}{dt} \cdot \sinh t \right|^{(q-2)/(q-1)} \right\} z = 0.$$

Therefore z obeys the following equation

$$\frac{d^2 z}{dt^2} + [-1 + Q(t)] z = 0. \quad (7.10)$$

Here

$$Q(t) = (q-1)\lambda(\sinh t)^{-2}|u(t)|^{q-2}, \quad (7.11)$$

a coefficient which will play a role in subsequent developments too — observe that property (i) yields

$$0 \leq Q(t) \leq (q-1)\lambda(\sinh t)^{-2}t^{q-2}. \quad (7.12)$$

Since the coefficient of z in equation (7.10) approaches -1 fast enough as $t \rightarrow \infty$, Sturm comparison theorem (see e.g. [Tr, Section 20], [Swa, Chapter 1]) or standard oscillation theorems (see e.g. [Swa, Chapter 2]) guarantee that equation (7.10) is nonoscillatory, i.e. the zeroes of z do not cluster at infinity.

Property (iii) follows. \square

Observe incidentally that if H is defined by

$$H(t) = \left[\frac{du}{dt}(t) \cdot \sinh t \right]^2 + (2\lambda/q) \cdot |u(t)|^q,$$

then equation (7.5) gives

$$H'(t) = \left[\frac{du}{dt}(t) \right]^2 \cdot \frac{d}{dt}(\sinh t)^2$$

— in other words, H is an increasing function. Therefore

$$|u(t_1)| < |u(t_2)| < |u(t_3)| < \dots$$

if $t_1, t_2, t_3 \dots$ are the bend points of u arranged in increasing order — compare with Sonin & Pölya theorem, [Tr, Section 19].

Properties (iv) and (v) are an immediate consequence of (7.5) and (7.6). \square

7.5. Let assume $2 < q < \infty$, and examine how the solution u to (7.5) and (7.6) depends upon parameter λ .

Clearly, $u(t) \equiv t$ if $\lambda = 0$. On the other hand, u remains an increasing function of t if λ is positive and small enough — the inequality

$$\frac{du}{dt}(t) \geq 1 - \lambda \int_0^t s^{q-1}(\sinh s)^{-2} ds$$

(derived in the previous subsection) and the formula

$$\int_0^\infty s^{q-1}(\sinh s)^{-2} ds = 2^{2-q}\Gamma(q)\zeta(q-1)$$

(appearing in [GR, Section 3.527]) tell us that $du(t)/dt$ is positive for every positive t if

$$0 \leq \lambda \leq \frac{2^{q-2}}{[\Gamma(q)\zeta(q-1)]}.$$

Let L be $\partial u/\partial \lambda$, the derivative of u with respect to λ . An inspection shows that L is given by the following formula

$$(q-2)\lambda L = w - u, \quad (7.13)$$

where w obeys

$$\frac{d^2 w}{dt^2}(t) + Q(t)w(t) = 0 \quad (7.14)$$

for $0 < t < \infty$ and

$$w(0) = 0, \quad \frac{dw}{dt}(0) = 1. \quad (7.15)$$

Coefficient Q is defined as in (7.11). The following properties are easily inferred from (7.12), (7.14) and (7.15):

- (i) $|w(t)| \leq (\text{Constant}) \cdot t$ for every positive t ;
- (ii) w is asymptotically linear, and $|w - (\text{asymptote})|$ approaches 0 exponentially fast as t approaches ∞ ;
- (iii) w has finitely many zeroes and finitely many bend points.

Lemma 7.1. Let u satisfy (7.5), (7.6), and let w satisfy (7.14) and (7.15). Then $w(t) < u(t)$ if t is positive and does not exceed the first positive zero of w .

Proof. Equation (7.5) reads $d^2 u/dt^2 + P(t)u = 0$, where $(q-1) \cdot P(t) = Q(t)$. On the other hand, u and w obey the same initial conditions. Then either the comparison theorem appearing in [Tr, Section 20] or Levin comparison theorem — see e.g. [Swa, Chapter 1, Section 7] — leads to the conclusion. \square

Lemma 7.2. Suppose u satisfies (7.5), (7.6) and (7.8). Then the solution w to (7.14) and (7.15) has *one* positive zero at most.

Although a formal proof eluded the authors, the truth of Lemma 7.2 may be reasonably inferred from the following facts:

- (i) Suppose u satisfies (7.5), (7.6) and (7.8). Then the solution w to (7.14) and (7.15) cannot have two distinct zeroes in the following set

$$\{t > 0 : t \cdot \coth t \geq q/2\}. \quad (7.16)$$

(Observe that $t \cdot \coth t$ is convex and increases strictly from 1 to ∞ as t increases from 0 to ∞ . The root to $t \cdot \coth t = q/2$ is 1.287839 if $q = 3$, is 1.915008 if $q = 4$, lies below $q/2$ and approaches $q/2$ asymptotically as q grows large.)

- (ii) Suppose u satisfies (7.5) and (7.6), assume w satisfies (7.14) and (7.15), and let a and b obey $0 \leq a < b$, $w(a) = w(b) = 0$ and $w(x) \neq 0$ for $a < x < b$. Then

$$\lambda b^{q-3}(b-a) \geq \frac{q-2}{q-1}. \quad (7.17)$$

(iii) Suppose q , λ and a neighborhood of 0 are specified. Then both $u(t)$ and $w(t)$ can be computed with any prescribed accuracy for every t from that neighborhood. Numerical tests show that no more than one zero of w occurs as long as u remains positive — relevant information can be found in [Mu].

Proof of (i). Let u be any solution to (7.5), and let z be defined by

$$z(t) = u(t) - t \cdot u'(t), \quad (7.18)$$

the height above the origin of the tangent straight line to the graph of u at $(t, u(t))$. Equations (7.5) and (7.18) give

$$t^2 \frac{d}{dt} \left(\frac{u}{t} \right) + z = 0, \quad \frac{dz}{dt} = \lambda t (\sinh t)^{-2} |u|^{q-2} u.$$

Eliminating u between the last two equations gives

$$\frac{d^2 z}{dt^2} + \left(2 \coth t - \frac{q}{t} \right) \frac{dz}{dt} + (q-1) \lambda^{1/(q-1)} (\sinh t)^{-2/(q-1)} \left| \frac{1}{t} \frac{dz}{dt} \right|^{(q-2)/(q-1)} z = 0. \quad (7.19)$$

In other terms, we have

$$\frac{d^2 z}{dt^2} + \left(2 \coth t - \frac{q}{t} \right) \frac{dz}{dt} + Q(t)z = 0, \quad (7.20)$$

provided Q is defined by (7.11).

Suppose u obeys (7.6) and (7.8) too. Then u vanishes at 0 and is concave, hence

$$z(t) > 0 \text{ and } \frac{dz}{dt}(t) > 0 \quad (7.21)$$

for every positive t . Consequently, equation (7.20) reads

$$\frac{d^2 z}{dt^2} + \left[Q(t) + \left(t \cdot \coth t - \frac{q}{2} \right) \times (\text{a positive coeff.}) \right] z = 0. \quad (7.22)$$

Equations (7.14) and (7.22), inequalities (7.21) and Sturm comparison theorem lead to the conclusion. \square

Proof of (ii). Statement (ii) follows from a variant of de la Vallée-Poussin theorem — see [Tr, Section 17]. It is an easy matter to show that if a , b and w obey

$$0 \leq a < b, \quad w(a) = w(b) = 0, \quad \text{and } w(x) \neq 0 \text{ for } a < x < b$$

then

$$\int_a^b t \left| \frac{1}{w(t)} \frac{d^2 w}{dt^2}(t) \right| dt \geq \frac{(\sqrt{b} + \sqrt{a})^2}{b - a}. \quad (7.23)$$

Therefore we have

$$\int_a^b t Q(t) dt \geq \frac{(\sqrt{b} + \sqrt{a})^2}{b - a}.$$

Since inequality (7.12) implies

$$Q(t) \leq (q-1)\lambda t^{q-4},$$

the conclusion follows. \square

7.6. Equation (7.13), and Lemmas 7.1 and 7.2 tell us that

$$L(t) < 0$$

for every positive t if u satisfies (7.5), (7.6) and (7.8). In other words, the solution u to (7.5) and (7.6) decreases steadily with respect to λ as long as u itself remains an increasing function of t .

This implies that the solution to (7.5), (7.6), (7.7) and (7.8) is unique, and concludes the proof. \square

REFERENCES

- [Bae] A. BAERNSTEIN II, *A unified approach to symmetrization*, Pages 47–91 in Partial Differential Equations of Elliptic Type (A. Alvino & E. Fabes & G. Talenti, eds.), Symposia Mathematica 35, Cambridge Univ. Press, 1994.
- [Bi] L. BIANCHI, *Lezioni di Geometria Differenziale*, vol. 1, Zanichelli, 1927.
- [Bl] G.A. BLISS, *An integral inequality*, J. London Math. Soc. **5** (1930), 40–46.
- [BuZa] YU.D. BURAGO & V.A. ZALGALLER, *Geometric inequalities*, Springer-Verlag, 1988.
- [Cha] I. CHAVEL, *Riemannian geometry: A modern introduction*, Cambridge Tracts in Mathematics 108, Cambridge University Press, 1994.
- [DS] N. DUNFORD & J. SCHWARTZ, *Linear Operators. Part I: General Theory*, Interscience Publ., 1958.
- [Fr] L.E. FRAENKEL, *On steady vortex rings with swirl and a Sobolev inequality*, Pages 13–26 in Progress in partial differential equations: calculus of variations, applications (C. Bandle & J. Bemelmans & M. Chipot & M. Grüter & J. Saint Jean Paulin, eds.), Pitman Research Notes in Math. vol. 266, 1992.
- [GR] I.S. GRADSHTEYN & I.M. RYZHIK, *Tables of Integrals, Series, and Products*, Academic Press, 1965.
- [HLP] G.H. HARDY & J.E. LITTLEWOOD & G. PÓLYA, *Inequalities*, Cambridge Univ. Press, first edition 1934, second edition 1952.
- [Ka] E. KAMKE, *Differentialgleichungen, Lösungsmethoden und Lösungen. Band I: Gewöhnliche Differentialgleichungen*, Leipzig, 1959.
- [Kw] B. KAWOHL, *Rearrangements and convexity of level sets in PDE*, Lecture Notes in Math. 1150, Springer-Verlag, 1985.
- [Ko] V.I. KOLYADA, *Rearrangements of functions and embedding theorems.*, Russian Math. Surveys **44:5** (1989), 73–117.
- [McC] J. MCCLEARY, *Geometry from a differentiable viewpoint*, Cambridge Univ. Press, 1994.
- [McK] H.P. MCKEAN, *An upper bound to the spectrum of Δ on a manifold of negative curvature*, J. Differential Geometry **4** (1970), 359–366.
- [Mu] F. MUGELLI, *Complementary techniques in Sobolev inequalities for the hyperbolic plane*, Report, Florence Univ. Math. Dept. (1997).
- [MT] F. MUGELLI & G. TALENTI, *Sobolev inequalities in 2-dimensional hyperbolic space*, To appear.
- [SG] G. SANSONE & J. GERRETSEN, *Lectures on the theory of functions of a complex variable vol. 2*, Wotters-Noordhoff, 1970.

- [Sie] C.L. SIEGEL, *Topics in Complex Functions Theory, vol. 2: Automorphic Functions and Abelian Integrals*, Interscience Tracts in Pure Appl. Math. 25, Wiley-Interscience, 1971.
- [Swa] C.A. SWANSON, *Comparison and Oscillation Theory of Linear differential Equations*, Academic Press, 1968.
- [Ta] G. TALENTI, *Inequalities in rearrangement invariant function spaces*, Pages 177-231 in *Nonlinear Analysis, Function Spaces and Applications 5* (M. Krbeč & A. Kufner & B. Opic & J. Rakosnik, eds.), Prometheus Publ, Prague, 1994.
- [Tr] F.G. TRICOMI, *Equazioni differenziali*, Boringhieri, 1961.

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