SOBOLEV INEQUALITIES IN 2-DIMENSIONAL HYPERBOLIC SPACE

Francesco Mugelli & Giorgio Talenti

Abstract. A sharp form of some Sobolev-type inequalities in 2-dimensional hyperbolic space $\mathbb{H}^2$ is discussed. Here $\mathbb{H}^2$ is modeled on the upper Euclidean half-plane endowed with the Poincaré-Bergman metric. The proof rests upon rearrangements of functions, the isoperimetric theorem in $\mathbb{H}^2$, and inequalities in the calculus of variations.

1. Main result.

Theorem 1. *Let $\mathbb{R}^2_+ = \{(x,y) : -\infty < x < \infty, 0 < y < \infty\}$, the upper Euclidean half-plane, and let u be a real-valued function defined in $\mathbb{R}^2_+$. Suppose u is sufficiently smooth (e.g. Lipschitz-continuous) and decays well (e.g. the support of u is bounded and is bounded away from the x-axis). The following inequalities hold and are sharp.*

$$(i) \quad \left\{\int_{\mathbb{R}^2_+} |u| \, \frac{dxdy}{y^2}\right\}^2 + 4\pi \int_{\mathbb{R}^2_+} |u|^2 \, \frac{dxdy}{y^2} \le \left\{\int_{\mathbb{R}^2_+} y\sqrt{u_x^2+u_y^2}\, \frac{dxdy}{y^2}\right\}^2. \tag{1.1}$$

(ii) If $1 < p < 2$ and $q = 2p/(2-p)$ then

$$p^{-2}\left\{\int_{\mathbb{R}^2_+} |u|^p \, \frac{dxdy}{y^2}\right\}^{2/p} + \frac{(4\pi/q)^2(q/2-1)^{2/q}}{sin(2\pi/q)}\left\{\int_{\mathbb{R}^2_+} |u|^q \, \frac{dxdy}{y^2}\right\}^{2/q} \le \left\{\int_{\mathbb{R}^2_+} y^p(u_x^2+u_y^2)^{p/2}\, \frac{dxdy}{y^2}\right\}^{2/p}. \tag{1.2}$$

(iii) If $p > 2$ then

$$\sup|u| \le (4\pi)^{-1/p}\left\{\frac{\Gamma\left(\frac{p-2}{2(p-1)}\right)\Gamma\left(\frac{1}{p-1}\right)}{\Gamma\left(\frac{p}{2(p-1)}\right)}\right\}^{1-1/p} \times \left\{\int_{\mathbb{R}^2_+} y^p(u_x^2+u_y^2)^{p/2}\, \frac{dxdy}{y^2}\right\}^{1/p}. \tag{1.3}$$

(iv) If $1 \le p < \infty$ then

$$\left\{\int_{\mathbb{R}^2_+} |u|^q \, \frac{dxdy}{y^2}\right\}^{2/q} \le p^p \int_{\mathbb{R}^2_+} y^p(u_x^2+u_y^2)^{p/2}\, \frac{dxdy}{y^2}. \tag{1.4}$$

Typeset by $\mathcal{AMS}$-TEX

Theorem 1 can be interpreted as follows. The quadratic form

$$y^{-2}\left[(dx)^2+(dy)^2\right], \tag{1.5a}$$

sometimes called *Poincaré-Bergman metric*, makes $\mathbb{R}^2_+$ a Riemannian manifold that models the *2-dimensional hyperbolic space*, is denoted by $\mathbb{H}^2$ throughout, and has the properties listed below. (Relevant references are [Bi, Chapter 14], [Bl, Sections 74 and 75], [SG, Section 9.5] and [Sie, Section 2.2], for example.)

Any differentiable mapping from $\mathbb{H}^2$ into $\mathbb{H}^2$ that leaves the Riemannian metric of $\mathbb{H}^2$ *invariant* is given either by a rational function of order 1 of the complex variable $x+iy$ — a Möbius transformation — having real coefficients, or by the conjugate of such a function. The Riemannian *angle* between two tangent vectors to $\mathbb{H}^2$ coincides with the Euclidean angle; the Riemannian *lenght* of a tangent vector to $\mathbb{H}^2$ at a point (x,y) equals $y\times$(the Euclidean length). The *geodesics* of $\mathbb{H}^2$ are the half-lines and the half-circles orthogonal to the x-axis; the Riemannian *distance* between two points (x_1,y_1) and (x_2,y_2) equals

$$\log\left[\frac{\sqrt{(x_2-x_1)^2+(y_2+y_1)^2}+\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}{\sqrt{(x_2-x_1)^2+(y_2+y_1)^2}-\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}\right]. \tag{1.5b}$$

The Riemannian *2-dimensional measure* on $\mathbb{H}^2$, $\mathcal{M}$, is given by

$$d\mathcal{M}=y^{-2}dxdy. \tag{1.5c}$$

The *Laplace-Beltrami operator* on $\mathbb{H}^2,\Delta$, obeys $\Delta=y^2\left((\partial/\partial x)^2+(\partial/\partial y)^2\right)$; the *curvature* of $\mathbb{H}^2$ is identically -1.

Inequalities (1.1)-(1.4) are closely related to the Riemannian structure of $\mathbb{H}^2$. In fact, if we think of u as a scalar field on $\mathbb{H}^2$ we have

$$\left\{\int_{\mathbb{R}^2_+}|u|^q\,\frac{dxdy}{y^2}\right\}^{1/q}=\left\{\int_{\mathbb{H}^2}|u|^q\,d\mathcal{M}\right\}^{1/q} \tag{1.6a}$$

the norm of u *in Lebesgue space* $L^q(\mathbb{H}^2)$. On the other hand, the covariant derivative of $u,\nabla u$, is a tangent vector field to $\mathbb{H}^2$ whose components are u_x and u_y, and whose Riemannian length, $|\nabla u|$, equals $y\sqrt{u_x^2+u_y^2}$. Therefore

$$\left\{\int_{\mathbb{R}^2_+}y^p(u_x^2+u_y^2)^{p/2}\,\frac{dxdy}{y^2}\right\}^{1/p}=\left\{\int_{\mathbb{H}^2}|\nabla u|^p\,d\mathcal{M}\right\}^{1/p}, \tag{1.6b}$$

the norm of ∇u *in* $L^p(\mathbb{H}^2)\times L^p(\mathbb{H}^2)$.

Thus, statements (i), (ii) and (iii) from Theorem 1 amount to *Sobolev inequalities* in hyperbolic space $\mathbb{H}^2$; statement (iv) is an *inequality à la Poincaré* in $\mathbb{H}^2$. Inequalities of this sort are well-established: the point is that the constants displayed in Theorem 1 are *the best possible.* (Standard references on Sobolev spaces and inequalities are [Ad], [Ma], [Zi]. An overview of sharp Sobolev inequalities in

Euclidean n-dimensional spaces can be found in [Ta2, Section 2]. Sharp forms of some Sobolev inequalities on spheres appeared in [Ta1] and [Ci]. Sobolev spaces and inequalities on Riemannian manifolds are discussed in [Au1], [Au2] and [Au3].) Remarks on Theorem 1 follow.

(i) The proof of Theorem 1 we offer in the present paper rests on the *isoperimetric theorem* for hyperbolic space $\mathbb{H}^2$, though an alternative approach briefly mentioned in Section 2 may work as well. We would like to stress that, conversely, the isoperimetric theorem for $\mathbb{H}^2$ can be *derived* from inequality (1.1) — details of such a derivation are much as in [Ta2, Theorem 2A] or [Zi, Section 2.7], for example. In other words, statement (i) from Theorem 1 is *equivalent* to the isoperimetric theorem for $\mathbb{H}^2$.

(ii) The leading constant involved in inequality (1.2) coincides with (an appropriate power of) the *Sobolev constant* for the *Euclidean* 2-dimensional space — consistently with results by Aubin, [Au1] and [Au2].

(iii) Inequality (1.4) is peculiar to hyperbolic space $\mathbb{H}^2$. It implies also that

$$\int_{\mathbb{H}^2} u^2 d\mathcal{M} \leq 4 \int_{\mathbb{H}^2} u(-\Delta u) d\mathcal{M} \tag{1.7}$$

for every test function u — an inequality already observed by McKean [Mk] showing a cognate peculiarity of $\mathbb{H}^2$: *the spectrum of* $\Delta : L^2(\mathbb{H}^2) \mapsto L^2(\mathbb{H}^2)$ *lies below* $-1/4$. As one may infer from proofs, the peculiarities in question are concerned with the *negative curvature* of $\mathbb{H}^2$.

The present work was prompted by a paper of L.E. Fraenkel [Fr], who — aiming at an existence theory for a partial differential equation in fluid mechanics — supplied a proof that (in our notations) if $q \geq 2$ then a constant A exists such that

$$\left\{ \int_{\mathbb{R}^2_+} |\varphi|^q y^{-q/2-2} dxdy \right\}^{1/q} \leq A \left\{ \int_{\mathbb{R}^2_+} \left(\varphi_x^2 + \varphi_y^2\right) \frac{dxdy}{y} \right\}^{1/2} \tag{1.8a}$$

for every real-valued compactly supported smooth function φ defined in $\mathbb{R}^2_+$. Fraenkel had an eye to the smallest constant A that renders the above inequality true, and was able to show that such a constant equals $2^{6/5} \cdot 15^{-1/2} \cdot \pi^{-1/5}$ in the special case where $q = \frac{10}{3}$.

The change of variables $\varphi(x,y) = \sqrt{y}\, u(x,y)$ and an integration by parts turn the inequality above into

$$\left\{ \int_{\mathbb{H}^2} |u|^q \, d\mathcal{M} \right\}^{1/q} \leq A \left\{ \int_{\mathbb{H}^2} |\nabla u|^2 \, d\mathcal{M} + \frac{3}{4} \int_{\mathbb{H}^2} |u|^2 \, d\mathcal{M} \right\}^{1/2} . \tag{1.8b}$$

Thus, Fraenkel's inequality can be conveniently regarded as a *borderline case* — i.e. a case where the leading exponent equals the dimension — of Sobolev inequalities in hyperbolic space $\mathbb{H}^2$. Such a case can be approached using the methods of the present paper though it is not included in Theorem 1, and will be the subject of a future paper.

2. Rearrangements.

Let u be any real-valued function defined on hyperbolic space $\mathbb{H}^2$ such that the Riemannian measure of the level set $\{(x,y) \in \mathbb{H}^2 : |u(x,y)| > t\}$ is finite for every positive t. The following definitions mimic those introduced by Hardy and Littlewood and elaborated by several authors (see [HLP, Chapter 10], [PS, Chapters 1 and 2], [Ban, Chapter 2], [Kw, Chapter 2], [Bae], [Ta2], and the references quoted therein).

The *distribution function* of u, μ, is the Riemannian measure of the level sets of u, i.e. is defined by

$$\mu(t) = \int_{\{(x,y)\in\mathbb{H}^2 : |u(x,y)|>t\}} d\mathcal{M} \tag{2.1}$$

for every nonnegative t. The *decreasing rearrangement* of u, u^*, is defined by

$$u^*(s) = \inf\{t \geq 0 : \mu(t) \leq s\} \tag{2.2}$$

for every nonnegative s. The *symmetric rearrangement* of u, $u^\star$, is defined by

$$u^\star(x,y) = u^*\left(\frac{\pi}{y}\left(x^2 + (y-1)^2\right)\right) \tag{2.3}$$

for every (x,y) from $\mathbb{H}^2$.

Clearly, μ is *nonnegative*, *decreasing* and *right-continuous*, and u^* coincides with the distribution function of μ. It is easy to show that

$$\{s \geq 0 : u^*(s) > t\} = [0, \mu(t)[\tag{2.4}$$

for every nonnegative t — i.e. *the set where u^* exceeds t is an interval on the real line whose end-points are* 0 *and the value of μ at t.*

Recall that the *geodesic disk* in $\mathbb{H}^2$ with center at (a,b) and radius r coincides with the Euclidean disk whose center is $(a, b\cosh r)$ and whose radius is $b\sinh r$; and that the Riemannian area and the Riemannian perimeter of a geodesic disk in $\mathbb{H}^2$ with radius r are $4\pi[\sinh(r/2)]^2$ and $2\pi\sinh r$, respectively. In other words, if a, b and s are real parameters, and b and s are positive, then the inequality $(x-a)^2+(y-b)^2 < (s/\pi)by$ defines the geodesic disk in $\mathbb{H}^2$ such that: center$= (a,b)$, measure$=$ s, perimeter$=$ $\sqrt{s^2+4\pi s}$, and radius$=$ $\log\left(1 + \frac{1}{2\pi}s + \frac{1}{2\pi}\sqrt{s^2+4\pi s}\right)$. Thus, equation (2.3) implies that the value of $u^\star$ at any point (x,y) depends only upon the Riemannian distance between (x,y) and $(0,1)$, and decreases as such a distance increases. Equations (2.3) and (2.4) give precisely that

$$\left\{(x,y) \in \mathbb{H}^2 : |u^\star(x,y)| > t\right\} = \textit{the open geodesic disk whose center is } (0,1) \textit{ and whose Riemannian measure equals } \mu(t) \tag{2.5}$$

for every nonnegative t.

Equations (2.4) and (2.5) tell us that $u, u^*, u^\star$ are *equidistributed*, i.e. both the distribution function of u^* and the distribution function of $u^\star$ coincide with μ, the distribution function of u. It follows that

$$\operatorname{ess\,sup}|u| = u^*(0) = u^\star(0,0), \tag{2.6a}$$

and

$$\int_{\mathbb{H}^2} \Phi\left(|u(x,y)|\right) d\mathcal{M} = \int_0^\infty \Phi\left(u^*(s)\right) ds = \int_{\mathbb{H}^2} \Phi\left(|u^\bigstar(x,y)|\right) d\mathcal{M} \tag{2.6b}$$

if Φ is a nonnegative increasing function defined in $[0,\infty[$ such that $\Phi(0) = 0$.

As shown in Section 4, a proof of Theorem 1 rests upon equations (2.6a) and (2.6b), and the following theorem.

Theorem 2. *Let u be a real-valued function defined on hyperbolic space $\mathbb{H}^2$. Assume u is Lipschitz-continuous and the Riemannian measure of $\{(x,y) \in \mathbb{H}^2 : |u(x,y)| > t\}$ is finite for every positive t. Let Φ be any Young function — i.e. assume Φ maps $[0,\infty[$ into $[0,\infty[$, $\Phi(0) = 0$, Φ is increasing and convex. Then*

$$\int_{\mathbb{H}^2} \Phi\left(|\nabla u|\right) d\mathcal{M} \geq \int_0^\infty \Phi\left(-\sqrt{4\pi s + s^2}\, \frac{du^*}{ds}(s)\right) ds, \tag{2.7a}$$

and

$$\textit{the right-hand side of (2.7a)} = \int_{\mathbb{H}^2} \Phi\left(|\nabla u^\bigstar|\right) d\mathcal{M}. \tag{2.7b}$$

Theorem 2 amounts to saying that Dirichlet-type integrals decrease under the symmetric rearrangement. (A counterpart of this, where $\mathbb{H}^2$ is replaced by an Euclidean space, is a popular tool, exhaustively discussed in [BrZi], for example.) Interestingly, Theorem 2 can be derived from rearrangement inequalities that have their roots in *merely combinatorial* arguments. A proof along these lines is due to A. Baernstein and W. Beckner, and is outlined in [Bae, Sections 3 and 4]. Here we insist on a more geometric approach, and provide a proof that is based on the isoperimetric theorem for $\mathbb{H}^2$ and may help to understand how the negative curvature of $\mathbb{H}^2$ comes into play.

The *isoperimetric theorem* for hyperbolic space $\mathbb{H}^2$ is as follows. Let E be a sufficiently smooth subset of $\mathbb{H}^2$, and let

$$P = \int_{\partial E} \frac{1}{y}\sqrt{(dx)^2 + (dy)^2} \quad \text{and} \quad A = \int_E \frac{dxdy}{y^2}$$

— the Riemannian perimeter and the Riemannian area of E, respectively. If A is finite, then

$$P \geq \sqrt{4\pi A + A^2}; \tag{2.8}$$

moreover, P equals $\sqrt{4\pi A + A^2}$ if, and only if, E is a disk. One early proof of this theorem is due to E. Schmidt [Sch], another can be found in [BuZa, Section 10].

The curvature is a clue to the isoperimetric theorem for $\mathbb{H}^2$. Indeed, as observed in [Oss, Section 4], the isoperimetric inequality on a 2-dimensional manifold *of constant curvature* — either a sphere, or the Euclidean plane, or the hyperbolic plane — reads

$$(\text{perimeter}) \geq \left[4\pi \times (\text{area}) - (\text{curvature}) \times (\text{area})^2\right]^{1/2}. \tag{2.9}$$

The proofs given below in this section and in the subsequent ones indicate that: (i) The isoperimetric theorem for $\mathbb{H}^2$ is precisely what brings the weight

$$\sqrt{C_1 s + C_2 s^2},$$

$$C_1 = 4\pi \quad \text{and} \quad C_2 = -(\text{curvature}),$$

into the inequalities of Theorem 2. (ii) This same weight, which behaves like $\sqrt{s}$ when s is small and like s when s grows large, is precisely what causes the inequalities of Theorem 1 to hold in the form stated.

Proof of Theorem 2. Clearly there is no loss in generality if we assume $u \geq 0$. Federer's coarea formula implies that

$$\int_{u^{-1}(]u^*(s+h),u^*(s)])} y\sqrt{(u_x)^2+(u_y)^2}\,\frac{dxdy}{y^2} = \int_{u^*(s+h)}^{u^*(s)} dt \int_{u^{-1}(\{t\})} \frac{1}{y}\sqrt{(dx)^2+(dy)^2} \tag{2.10}$$

if $s \geq 0$ and $h \geq 0$. (An accessible version of Federer's coarea formula can be found in [Zi, Section 2.7], for example.) The isoperimetric theorem for hyperbolic space $\mathbb{H}^2$ implies that

$$\int_{u^{-1}(\{t\})} \frac{1}{y}\sqrt{(dx)^2+(dy)^2} \geq \sqrt{4\pi\mu(t) + [\mu(t)]^2} \tag{2.11}$$

for every nonnegative t. Therefore, if $s \geq 0$ and $h \geq 0$ we have

$$\int_{u^{-1}(]u^*(s+h),u^*(s)])} |\nabla u| d\mathcal{M} \geq \sqrt{4\pi s + s^2}\,[u^*(s) - u^*(s+h)]\,. \tag{2.12}$$

Inequality (2.12) is crucial: the whole Theorem 2 will be derived from it.

Let us commence by showing that u^* is locally Lipschitz-continuous — a property implicit in the statement of Theorem 2. Basic properties of both distribution function and decreasing rearrangement — equations (2.1), (2.2) and (2.4) — imply that

$$\int_{u^{-1}(]u^*(s+h),u^*(s)[)} d\mathcal{M} \leq h \tag{2.13}$$

if $s \geq 0$ and $h \geq 0$. Moreover,

$$\int_{u^{-1}(\{t\})} |\nabla u| d\mathcal{M} = 0 \tag{2.14}$$

for every nonnegative t, because either $u^{-1}(\{t\})$ has Riemannian measure zero or $|\nabla u|$ vanishes almost everywhere on $u^{-1}(\{t\})$. Inequalities (2.12) and (2.13), and equation (2.14) imply that

$$\sqrt{4\pi s + s^2}\,[u^*(s) - u^*(s+h)] \leq h \cdot \operatorname{ess\,sup} |\nabla u| \tag{2.15}$$

if $s \geq 0$ and $h \geq 0$. The above mentioned property follows. Inequality (2.12) gives immediately that

$$\frac{d}{ds}\int_{\{(x,y)\in\mathbb{H}^2 : u(x,y)>u^*(s)\}} |\nabla u| d\mathcal{M} \geq \sqrt{4\pi s + s^2}\left[-\frac{du^*}{ds}(s)\right] \tag{2.16}$$

for almost every positive s.

Now we prove that

$$\frac{d}{ds}\int_{\{(x,y)\in\mathbb{H}^2 : u(x,y)>u^*(s)\}} \Phi\left(|\nabla u|\right) d\mathcal{M} \geq \Phi\left(\sqrt{4\pi s + s^2}\left[-\frac{du^*}{ds}(s)\right]\right) \tag{2.17}$$

for almost every positive s. There are exactly three alternatives: (i) s belongs to some exceptional set having one-dimensional Lebesgue measure zero; (ii) du^*/ds vanishes at s; (iii) a neighborhood of s exists where u^* decreases strictly.

If either (i) or (ii) holds, there is nothing to prove. If (iii) is in force, a simple argument shows that

$$\int_{u^{-1}(]u^*(s+h),u^*(s)])} d\mathcal{M} = h \tag{2.18}$$

if h is positive and small enough. Then Jensen's inequality for convex functions gives

$$\frac{1}{h}\int_{u^{-1}(]u^*(s+h),u^*(s)])} \Phi\left(|\nabla u|\right) d\mathcal{M} \geq \Phi\left(\frac{1}{h}\int_{u^{-1}(]u^*(s+h),u^*(s)])} |\nabla u| d\mathcal{M}\right), \tag{2.19}$$

consequently we have

$$\frac{d}{ds}\int_{\{(x,y)\in\mathbb{H}^2 : u(x,y)>u^*(s)\}} \Phi(|\nabla u|) d\mathcal{M} \geq \Phi\left(\frac{d}{ds}\int_{\{(x,y)\in\mathbb{H}^2 : u(x,y)>u^*(s)\}} |\nabla u| d\mathcal{M}\right). \tag{2.20}$$

Inequalities (2.16) and (2.20) yield (2.17).

The proof of inequality (2.7a) is now at hand. Indeed, (2.7a) follows from inequality (2.17) and the obvious inequality

$$\int_{\mathbb{H}^2} \Phi\left(|\nabla u|\right) d\mathcal{M} \geq \int_0^\infty ds \frac{d}{ds}\int_{\{(x,y)\in\mathbb{H}^2 : u(x,y)>u^*(s)\}} \Phi\left(|\nabla u|\right) d\mathcal{M}. \tag{2.21}$$

The remaining part of the proof runs this way. Define s by

$$s(x,y) = \frac{\pi}{y}\left[x^2 + (y-1)^2\right], \tag{2.22}$$

and observe that: (i) $s(x,y) \geq 0$ if $y > 0$; (ii) s obeys the partial differential equation

$$y^2\left(s_x^2 + s_y^2\right) = 4\pi s + s^2; \tag{2.23}$$

(iii) for every positive t, the level line where $s(x,y) = t$ is a geodesic circle of $\mathbb{H}^2$ whose Riemannian length equals $\sqrt{4\pi t + t^2}$. Derive from (i), (ii) and (iii) and Federer's coarea formula that

$$\int_{\mathbb{H}^2} f(s(x,y))d\mathcal{M} = \int_0^\infty f(t)dt \tag{2.24}$$

if f is defined in $[0,\infty[$ and decays fast enough near 0 and ∞. Deduce from the definition of $u^\bigstar$ — equation (2.3) — and from equation (2.23) that

$$|\nabla u^\bigstar| = \sqrt{4\pi s + s^2}\left[-\frac{du^*}{ds}(s)\right]. \tag{2.25}$$

Conclude the proof by observing that equations (2.24) and (2.25) give (2.7b).

3. Lemmas.

Lemma 3.1. *(i) The following inequality*

$$\frac{\int_0^\infty s|du(s)|}{\int_0^\infty |u(s)|ds} \geq 1 \tag{3.1}$$

holds for every non-zero real-valued function u such that: u has bounded variation, the integral $\int_0^\infty s|du(s)|$ is finite, and $u(\infty) = 0$. The right-hand side of (3.1) is the minimum value of the left-hand side, and any positive decreasing function is a minimizer.

(ii) If $1 < p < \infty$, then

$$\frac{\int_0^\infty |su'(s)|^p ds}{\int_0^\infty |u(s)|^p ds} > p^{-p} \tag{3.2}$$

for every non-zero real-valued function u such that: u is absolutely continuous, the integral $\int_0^\infty |su'(s)|^p ds$ is finite, and $u(\infty) = 0$. The right-hand side of (3.2) is the greatest lower bound — not attained — of the left-hand side; a minimizing sequence is given by

$$u_k(s) = s^{-1/p+1/k}e^{-s} \qquad (k = 1, 2, 3, \ldots).$$

Lemma 3.2. *(i) The following inequality*

$$\frac{\left\{\int_0^\infty \sqrt{s}|du(s)|\right\}^2}{\int_0^\infty [u(s)]^2 ds} \geq 1 \tag{3.3}$$

holds for every non-zero real-valued function u such that: u has bounded variation, the integral $\int_0^\infty \sqrt{s}|du(s)|$ is finite, and $u(\infty) = 0$. The right-hand side of (3.3) is the minimum value of the left-hand side; the characteristic function of the interval $[0,1]$, and any rescaled version of it, are minimizers.

(ii) Let $1 < p < 2$ and $q = 2p/(2-p)$. Then

$$\frac{\left\{\int_0^\infty s^{p/2}|u'(s)|^p ds\right\}^{2/p}}{\left\{\int_0^\infty |u(s)|^q ds\right\}^{2/q}} \geq \frac{4\pi(q/2-1)^{2/q}}{q^2 sin(2\pi/q)} \tag{3.4}$$

for every non-zero real-valued function u such that: u is absolutely continuous, the integral $\int_0^\infty s^{p/2}|u'(s)|^p ds$ is finite, and $u(\infty) = 0$. The right-hand side is the minimum value of the left-hand side; a minimizer is given by

$$u(s) = \left[1 + s^{q/(q-2)}\right]^{-2/q},$$

and any other minimizer is a rescaled version of this.

Lemma 3.3. *(i) Suppose A and B are nonnegative constants. The following inequality*

$$\frac{\left\{\int_0^\infty \sqrt{s^2+4\pi s}\,|du(s)|\right\}^2}{A\left\{\int_0^\infty |u(s)|\,ds\right\}^2 + B\int_0^\infty |u(s)|^2\,ds} \geq 1 \tag{3.5a}$$

holds for every non-zero u if, and only if,

$$A \leq 1 \qquad \text{and} \qquad B \leq 4\pi. \tag{3.5b}$$

Here u is any real-valued function such that: u has bounded variation, the integrals $\int_0^\infty s|du(s)|$ and $\int_0^\infty \sqrt{s}\,|du(s)|$ are finite, and $u(\infty) = 0$.

(ii) Let $1 \leq p < 2$ and $q = 2p/(2-p)$, and suppose A and B are nonnegative constants. The following inequality

$$\frac{\left\{\int_0^\infty \left(s^2+4\pi s\right)^{p/2}|u'(s)|^p\,ds\right\}^{2/p}}{A\left\{\int_0^\infty |u(s)|^p\,ds\right\}^{2/p} + B\left\{\int_0^\infty |u(s)|^q\,ds\right\}^{2/q}} \geq 1 \tag{3.6a}$$

holds for every non-zero u if, and only if,

$$A \leq p^{-2} \qquad \text{and} \qquad B \leq \frac{(4\pi/q)^2(q/2-1)^{2/q}}{sin(2\pi/q)}. \tag{3.6b}$$

Here u is any real-valued function such that: u is absolutely continuous, the integrals $\int_0^\infty |su'(s)|^p ds$ and $\int_0^\infty s^{p/2}|u'(s)|^p ds$ are finite, and $u(\infty) = 0$.

Lemma 3.4. *Let $2 < p < \infty$, and denote $p/(p-1)$ by p'. Then*

$$\frac{\sup|u|}{\left\{\int_0^\infty \left(s^2+4\pi s\right)^{p/2}|u'(s)|^p\,ds\right\}^{2/p}} \leq (4\pi)^{-1/p}\left\{\frac{\Gamma(1-p'/2)\Gamma(p'-1)}{\Gamma(p'/2)}\right\}^{1/p'} \tag{3.7}$$

for every non-zero real-valued function u such that: u is absolutely continuous, the integrals $\int_0^\infty |su'(s)|^p\,ds$ and $\int_0^\infty s^{p/2}\,|u'(s)|^p\,ds$ are finite, and $u(\infty) = 0$. The right-hand side of (3.7) is exactly the maximum value of the left-hand side.

Proof of Lemma 3.1. A proof of statement (i) can be easily figured out and is omitted here. Statement (ii) follows from [HLP, Theorem 328] — a variant of Hardy's inequality.

Proof of Lemma 3.2. (i) Replacing u by $]0,\infty[\ni s \mapsto \int_s^\infty |du(t)|$ leaves the set of competing functions invariant and decreases the left-hand side of (3.3). Thus, there is no loss of generality if the extra assumption is used that u is nonnegative-valued and decreasing.

We have

$$\int_0^s \frac{u(t)}{\sqrt{t}}dt \geq u(s)\int_0^s \frac{1}{\sqrt{t}}dt$$

because of the monotonicity of u, and consequently

$$u(s) \leq \frac{1}{2\sqrt{s}}\int_0^s \frac{u(t)}{\sqrt{t}}dt$$

and

$$[u(s)]^2 \leq \frac{d}{ds}\left\{\int_0^s \frac{u(t)}{2\sqrt{t}}dt\right\}^2$$

for every positive s. Therefore

$$\int_0^\infty [u(s)]^2 ds \leq \left\{\int_0^\infty \frac{u(t)}{2\sqrt{t}}dt\right\}^2$$

On the other hand, the equation $u(s) = \int_s^\infty [-du(t)]$ gives

$$\int_0^\infty \frac{u(t)}{2\sqrt{t}}dt = \int_0^\infty \sqrt{s}[-du(s)].$$

We have shown

$$\int_0^\infty [u(s)]^2 ds \leq \left\{\int_0^\infty \sqrt{s}[-du(s)]\right\}^2.$$

Inequality (3.3) is demonstrated. The remaining part of statement (i) follows from a straightforward inspection.

(ii) A theorem by Bliss [Bs] says that if $1 < p < q$ then

$$\frac{\left\{\int_0^\infty |w'(t)|^p\, dt\right\}^{1/p}}{\left\{\int_0^\infty |w(t)|^q\, t^{-1-q(1-1/p)}dt\right\}^{1/q}} \geq \left(q - \frac{q}{p}\right)^{1/q} \left\{\frac{\Gamma\left(\frac{pq}{q-p}\right)}{\Gamma\left(\frac{q}{q-p}\right)\Gamma\left(p\frac{q-1}{q-p}\right)}\right\}^{1/p-1/q}$$

for every non-zero real-valued function w such that w is absolutely continuous, the integral $\int_0^\infty |w'(t)|^p dt$ is finite, and $w(0) = 0$. The right-hand side of the last inequality is the *minimum value* of the left-hand side; a *minimizer* is given by

$$w(t) = t\left(t^{q/p-1} + 1\right)^{-p/(q-p)},$$

and any other minimizer is a rescaled version of this. (Bliss' theorem has proved instrumental in investigating Sobolev-type inequalities; it relies upon typical methods of the classical calculus of variations, and on the circumstance that appropriate solutions to the relevant Euler equation — a differential equation of the Emden-Fowler type — are available in a close form. Incidentally, statement 270 in [HLP] is flawed by a misprint.)

Letting $q = 2p/(2-p)$ and applying Bliss' theorem to test functions of this form

$$]0,\infty[\ni t \mapsto u\left(t^{1-q/2}\right)$$

results in statement (ii).

Proof of Lemma 3.3. Let us focus on statement (ii) — the proof of (i) is similar.

Suppose $1 < p < 2$ and $q = 2p/(2-p)$, and that A and B obey (3.6b). If u is any competing function, we have

$$\left\{\int_0^\infty \left(s^2+4\pi s\right)^{p/2}|u'(s)|^p\,ds\right\}^{p/2} \geq \\ t^{1-p/2}\int_0^\infty |su'(s)|^p\,ds + (1-t)^{1-p/2}(4\pi)^{p/2}\int_0^\infty s^{p/2}|u'(s)|^p\,ds$$

for every t such that $0 \leq t \leq 1$, consequently

$$\left\{\int_0^\infty \left(s^2+4\pi s\right)^{p/2}|u'(s)|^p\,ds\right\}^{2/p} \geq \\ \left\{\int_0^\infty |su'(s)|^p\,ds\right\}^{2/p} + 4\pi\left\{\int_0^\infty s^{p/2}|u'(s)|^p\,ds\right\}^{2/p}.$$

We used the formula

$$(a+b)^k = \max\left\{t^{1-k}a^k + (1-t)^{1-k}b^k : 0 \leq t \leq 1\right\},$$

where $0 < k < 1$ and a and b are positive. Therefore Lemmas 3.1 and 3.2 yield

$$\left\{\int_0^\infty \left(s^2+4\pi s\right)^{p/2}|u'(s)|^p\,ds\right\}^{2/p} \geq \\ p^{-2}\left\{\int_0^\infty |u(s)|^p\,ds\right\}^{2/p} + \frac{(4\pi/q)^2(q/2-1)^{2/q}}{\sin(2\pi/q)}\left\{\int_0^\infty |u(s)|^q\,ds\right\}^{2/q}.$$

Inequality (3.6a) follows.

Suppose A and B are positive constants and that inequality (3.6a) holds for every test function u. If λ is any positive constant and u is rescaled — i.e. replaced by $]0,\infty[\ni s \mapsto u(\lambda s)$ — inequality (3.6a) becomes

$$\frac{\left\{\int_0^\infty \left(s^2+4\pi\lambda s\right)^{p/2}|u'(s)|^p\,ds\right\}^{2/p}}{A\left\{\int_0^\infty |u(s)|^p\,ds\right\}^{2/p} + B\lambda\left\{\int_0^\infty |u(s)|^q\,ds\right\}^{2/q}} \geq 1.$$

Letting $\lambda \longrightarrow 0$ gives

$$\frac{\int_0^\infty |su'(s)|^p ds}{\int_0^\infty |u(s)|^p ds} \geq A,$$

letting $\lambda \longrightarrow \infty$ gives

$$\frac{\left\{\int_0^\infty s^{p/2}|u'(s)|^p ds\right\}^{2/p}}{\left\{\int_0^\infty |u(s)|^q ds\right\}^{2/q}} \geq B.$$

Inequalities (3.6b) follow, via Lemmas 3.1 and 3.2. The proof is complete.

Proof of Lemma 3.4. Clearly

$$\sup |u| \leq \int_0^\infty |u'(s)|\, ds,$$

and

$$\int_0^\infty |u'(s)|\, ds \leq \left\{\int_0^\infty \left(s^2 + 4\pi s\right)^{p/2} |u'(s)|^p\, ds\right\}^{1/p} \left\{\int_0^\infty \left(s^2 + 4\pi s\right)^{-p'/2} ds\right\}^{1/p'}$$

by Hölder inequality. Hence

$$\frac{\sup |u|}{\left\{\int_0^\infty (s^2 + 4\pi s)^{p/2}\, |u'(s)|^p\, ds\right\}^{1/p}} \leq \left\{\int_0^\infty \left(s^2 + 4\pi s\right)^{-p'/2} ds\right\}^{1/p'}.$$

Equality holds in these inequalities if u obeys

$$u(\infty) = 0, \qquad u'(s) = -\left(s^2 + 4\pi s\right)^{-p'/2}.$$

The Lemma follows.

4. Proof of Theorem 1.

Let u be any real-valued Lipschitz-continuous compactly supported function defined in hyperbolic space $\mathbb{H}^2$. The theory outlined in Section 2 tells us that the rearrangement of u, called $u^\star$ there, is Lipschitz-continuous and supported by a geodesic disk of finite radius. Moreover, $u^\star$ obeys

$$\int_{\mathbb{H}^2} |\nabla u^\star|^p\, d\mathcal{M} \leq \int_{\mathbb{H}^2} |\nabla u|^p\, d\mathcal{M} \tag{4.1}$$

for every p larger than (or equal to) 1, and

$$\int_{\mathbb{H}^2} (u^\star)^q\, d\mathcal{M} = \int_{\mathbb{H}^2} |u|^q\, d\mathcal{M}. \tag{4.2}$$

Hence the set of competing functions and the left-hand sides of inequalities (1.1)-(1.4) are *invariant* under the mapping $u \mapsto u^\star$, and the right-hand sides of the same inequalities *decrease* under this mapping. In other words, the competing functions that really count in the present context are *circular waves* u obeying

$$u(x, y) = u^*\left(\frac{\pi}{y}\left(x^2 + (y-1)^2\right)\right), \tag{4.3}$$

where u^* is defined in $[0,\infty[$, is decreasing and locally Lipschitz-continuous, and vanishes in a neighborhood of infinity.

Equation (4.3) gives

$$\int_{\mathbb{R}^2_+} y^p \left(u_x^2 + u_y^2\right)^{p/2} \frac{dxdy}{y^2} = \int_0^\infty \left(s^2 + 4\pi s\right)^{p/2} \left[-\frac{du^*}{ds}(s)\right]^p ds \tag{4.4}$$

and

$$\int_{\mathbb{R}^2_+} |u|^q \frac{dxdy}{y^2} = \int_0^\infty [u^*(s)]^q ds. \tag{4.5}$$

Equations (4.4) and (4.5) turn inequalities (1.1)-(1.4) into the following set:

$$(i) \quad \left\{\int_0^\infty u^*(s)ds\right\}^2 + 4\pi \int_0^\infty [u^*(s)]^2 ds \le \left\{\int_0^\infty \sqrt{s^2+4\pi s}\left[-\frac{du^*}{ds}(s)\right] ds\right\}^2; \tag{4.6}$$

$$(ii) \quad p^{-2}\left\{\int_0^\infty [u^*(s)]^p ds\right\}^{2/p} + \frac{(4\pi/q)^2(q/2-1)^{2/q}}{\sin(2\pi/q)}\left\{\int_0^\infty [u^*(s)]^q ds\right\}^{2/q} \le \left\{\int_0^\infty \left(s^2+4\pi s\right)^{p/2}\left[-\frac{du^*}{ds}(s)\right]^p ds\right\}^{2/p}, \tag{4.7}$$

where $1 < p < 2$ and $q = 2p/(2-p)$;

$$(iii) \quad u^*(0) \le (4\pi)^{-1/p}\left\{\frac{\Gamma\left(\frac{p-2}{2(p-1)}\right)\Gamma\left(\frac{1}{p-1}\right)}{\Gamma\left(\frac{p}{2(p-1)}\right)}\right\}^{1-1/p} \times \left\{\int_0^\infty \left(s^2+4\pi s\right)^{p/2}\left[-\frac{du^*}{ds}(s)\right]^p ds\right\}^{2/p}, \tag{4.8}$$

where $p > 2$;

$$(iv) \quad \int_0^\infty [u^*(s)]^p ds \le p^p \int_0^\infty \left(s^2+4\pi s\right)^{p/2}\left[-\frac{du^*}{ds}(s)\right]^p ds, \tag{4.9}$$

where $1 \le p < \infty$. Inequalities (4.6), (4.7) and (4.9) follow from Lemma 3.3; inequality (4.8) follows from Lemma 3.4. The same lemmas — together with appropriate density arguments — show that the inequalities in question are sharp.

The proof of Theorem 1 is complete.

References

[Ad] R.A. Adams, *Sobolev Spaces*, Academic Press, 1975.

[Au1] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, J. Diff. Geometry **11** (1976), 573–598.

[Au2] T. Aubin, *Espaces de Sobolev sur les variétés Riemanniennes*, Bull. Sci. Math. **100** (1976), 149–173.

[Au3] T. Aubin, *Nonlinear Analysis on Manifolds: Monge-Ampère Equations*, Grundlehren der mathematischen Wissenshaften, vol.252, Springer-Verlag, 1982.

[Bae] A. Baernstein II, *A unified approach to symmetrization*, Pages 47–91 in Partial Differential Equations of Elliptic Type (A. Alvino & E. Fabes & G. Talenti, eds.), Symposia Mathematica 35, Cambridge Univ. Press, 1994.

[Ban] C. Bandle, *Isoperimetric Inequalities and Applications*, Monographs and Studies in Math. 7, Pitman, 1980.

[Bi] L. Bianchi, *Lezioni di Geometria Differenziale*, vol. 1, Zanichelli, 1927.

[Bl] W. Blaschke, *Vorlesungen über Differentialgeometrie vol. 1*, Grundleheren der mathematischen Wissenshaften, vol. 1, Springer-Verlag, 1930.

[Bs] G.A. Bliss, *An integral inequality*, J. London Math. Soc. **5** (1930), 40–46.

[BrZi] J. Brothers & W. Ziemer, *Minimal rearrangements of Sobolev functions*, J. Reine Angew. Math. **384** (1988), 153–179.

[BuZa] Yu.D. Burago & V.A. Zalgaller, *Geometric inequalities*, Springer-Verlag, 1988.

[Ci] A. Cianchi, *A sharp form of Poincaré type inequalities on balls and spheres*, J. Appl. Math. Phys. (ZAMP) **40** (1989), 558–569.

[Fr] L.E. Fraenkel, *On steady vortex rings with swirl and a Sobolev inequality*, Pages 13–26 in Progress in partial differential equations: calculus of variations, applications (C. Bandle & J. Bemelmans & M. Chipot & M. Grüter & J. Saint Jean Paulin, eds.), Pitman Research Notes in Math. vol.266, 1992.

[HLP] G.H. Hardy & J.E. Littlewood & G. Pólya, *Inequalities*, Cambridge Univ. Press (first edition 1934, second edition 1952).

[Kw] B. Kawohl, *Rearrangements and convexity of level sets in PDE*, Lecture Notes in Math. 1150, Springer-Verlag, 1985.

[Ma] V.G. Maz'ja, *Sobolev Spaces*, Springer-Verlag, 1985.

[Mk] H.P. McKean, *An upper bound to the spectrum of Δ on a manifold of negative curvature*, J. Differential Geometry **4** (1970), 359–366.

[Oss] R. Osserman, *The isoperimetric inequality*, Bull. Amer. Math. Soc. **84** (1978), 1182–1238.

[PS] G. Pólya & G. Szegö, *Isoperimetric inequalities in mathematical physics*, Princeton Univ. Press, 1951.

[SG] G. Sansone & J. Gerretsen, *Lectures on the theory of functions of a complex variable vol. 2*, Wotters-Noordhoff, 1970.

[Sch] E. Schmidt,, *Über die isoperimetrische Aufgabe im n-dimensionalen Raum konstanter negativer Krümmung I: Die isoperimetrischen Ungleichungen in hyperbolischen Ebene und für Rotationskörper im n-dimensionalen hyperbolischen Raum*, Math. Z. **46** (1940), 204–230.

[Sie] C.L. Siegel, *Topics in Complex Functions Theory, vol. 2: Automorphic Functions and Abelian Integrals*, Interscience Tracts in Pure Appl. Math. 25, Wiley-Interscience, 1971.

[Ta1] G. Talenti, *Some inequalities of Sobolev type on two-dimensional spheres*, Pages 401-408 in General Inequalities 5 (W. Walter, ed.), Int. Series Numer. Math. 80, Birkhäuser-Verlag, 1987.

[Ta2] G. Talenti, *Inequalities in rearrangement invariant function spaces*, Pages 177-231 in Nonlinear Analysis, Function Spaces and Applications 5 (M. Krbec & A. Kufner & B. Opic & J. Rakosnik, eds.), Prometheus Publ, Prague, 1994.

[Zi] W.P. Ziemer, *Weakly Differentiable Functions: Sobolev Spaces and Functions of Bounded Variation*, Graduate Texts in Math. 120, Springer-Verlag, 1989.

Dipartimento di Matematica dell'Università, viale Morgagni 67A, I-50134 Firenze, Italy