A characterization theorem for the evolution semigroup generated by the sum of two unbounded operators

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Abstract

We consider a class of abstract evolution problems characterized by the sum of two unbounded linear operators A and B, where A is assumed to generate a positive semigroup of contractions on an L^1-space and B is positive. We study the relations between the semigroup generator G and the operator $A+B$. A characterization theorem for $G = \overline{A+B}$ is stated. The results are based on the spectral analysis of $B(\lambda - A)^{-1}$. The main point is to study the conditions under which the value 1 belongs to the resolvent set, the continuous spectrum, or the residual spectrum of $B(\lambda - A)^{-1}$.

Applications to the runaway problem in the kinetic theory of particle swarms and to the fragmentation problem describing polymer degradation are discussed in the light of the previous theory.

Key-words: Semigroup theory, abstract Cauchy problem, perturbation theory, fragmentation model

AMS subject classification: 47D06, 45K05, 34G10

1 Introduction

In this paper we consider the initial value problem in the general abstract form

$$\begin{cases} \dfrac{\partial f}{\partial t} &= Af + Bf\,, \quad t > 0 \\ \\ f(0) &= f_0, \end{cases} \tag{1.1}$$

where A and B are possibly unbounded operators acting on some L^1-space and the unknown vector function f belongs to this space.

In many applications the different nature of the two operators A and B, often both of them unbounded, makes the problem of the generation of the evolution semigroup corresponding to (1.1) unsolvable within the framework of classical perturbation theory. Examples of such a situation are easily found in kinetic theory, where the loss term is described by the operator A and the gain term is taken into account by the operator B. The evolution problem is essentially a balance equation and the Banach space L^1 is a natural choice as the norm of a non-negative element gives the number of particles. Other, not necessarily equivalent norms, may be important if we want to measure the number of collisions or other physical quantities.

The strategy which can be adapted to such evolution problems involves semigroup theory and is based on the fundamental work by Kato [Kat54] on Kolmogoroff equations. In general, the generator G of the evolution semigroup solving the Cauchy problem is a closed (but not necessarily the minimal closed) extension of $A+B$, but a full characterization of the generator requires additional assumptions.

A useful criterion that an extension of $A+B$ is a generator G of a substochastic semigroup was given by Voigt in [Voi87]. According to this criterion, under appropriate assumptions, a perturbation of the generator of a substochastic [1] semigroup by an unbounded positive operator has an extension that also generates a substochastic semigroup. Generalizations of the approach of Voigt were proved and successfully used by Arlotti [Arl87, Arl91] and Banasiak [Ban00, Ban01].

The perturbation problem with an unbounded operator in kinetic theory can be approached also within the framework of the general theory for the existence and uniqueness of solutions in an L^p-space setting, as developed by Beals and Protopopescu in [BP87], and also in Chapter XI of [GvdMP]. This theory was modified and extended in [vdM00] in order to deal with

[1] A semigroup $\{S(t)\}_{t\geq 0}$ on an L^1-space is called *stochastic* if it is positive and $\|S(t)f\|_1 = \|f\|_1$ for all $t \geq 0$ and $f \geq 0$ in the L^1-space. Positive contraction semigroups on an L^1-space are called *substochastic*.

the existence issues in the unbounded case. However, for phase spaces, force terms, collision frequencies, collision loss operators and boundary reflection operators no depending on time, in the L^1-setting and for a positive loss term not exceeding or even balancing the gain term, the kinetic equations studied in [BP87, GvdMP, vdM00] turn out to be applications of the abstract theory of this article.

In the general case, one can prove that the generator of the evolution semigroup $\{S(t)\}_{t\geq 0}$ solving the Cauchy problem (1.1) is an extension of the operator T defined by the right-hand side of (1.1). In fact, three mutually exclusive situations occur: *i)* T actually is the generator, *ii)* the generator is the closure of T, which implies that $\{S(t)\}_{t\geq 0}$ is stochastic, and *iii)* the generator is a nonminimal closed extension of T. As it will turn out, a necessary and sufficient condition for S to be stochastic, i.e., to satisfy

$$\|S(t)f\| = \|f\|, \quad \forall t \geq 0,\ \forall f \geq 0\,,$$

is that the generator coincides with the closure of T. In the basic application to kinetic theory, the total number of particles is preserved in time if and only if $\{S(t)\}_{t\geq 0}$ is stochastic. So only in this case we can claim that the obtained semigroup has physical relevance [Ban01], *unless* we can argue for physical processes leading to particles leaking from the system in spite of the apparent existence of detailed balance.

In this paper we study a class of abstract evolution problems determined by the sum of two possibly unbounded linear operators A and B, where A is assumed to generate a positive semigroup of contractions on an L^1-space and B is a positive operator. The main purpose of the study is to investigate the relations between the generator G and the operator $A + B$. In Section 3 a characterization theorem for all of three main situations discussed above, i.e., $G = A + B$, $G = \overline{A + B}$ and G is a nonminimal extension of $\overline{A + B}$, is proved. The results are based on the spectral analysis of $B(\lambda - A)^{-1}$ and it will in fact turn out that the three situations are characterized by whether 1 is in the resolvent set, the continuous spectrum or the residual spectrum of $B(\lambda - A)^{-1}$, respectively. In Section 4 we revise the existing treatment of the runaway problem in the kinetic theory of particle swarms [FvdMPF89, FvdM89] and characterize the generator. In the last section we apply the general theory to the fragmentation problem describing polymer degradation [ZMcG85], perform a spectral analysis of the problem and completely characterize the generated semigroup.

2 Statement of the abstract problem

Let (Σ, μ) and (Σ, ν) be two measure spaces consisting of the same σ-algebra of subsets of Σ but equipped with two positive Borel measures μ and ν, where μ is σ-finite and ν is absolutely continuous with respect to μ. Let X and Y denote the Banach lattices $X = L^1(\Sigma, d\mu)$ and $Y = L^1(\Sigma, d\nu)$, respectively, endowed with the standard norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. Then X and Y both consist of (equivalence classes of) measurable complex-valued functions on Σ and their intersection $X \cap Y$ can be identified with $L^1(\Sigma, d\mu + d\nu)$, where we have added the measures μ and ν to produce the positive Borel measure $\mu + \nu$ on Σ.

In the abstract setting, we make the following general assumptions:

i) A is the generator of a positive contraction semigroup (a so-called substochastic semigroup) $\{S_0(t)\}_{t\geq 0}$ on X.

Let us denote by L_λ the resolvent operator $(\lambda - A)^{-1}$ of A such that

$$L_\lambda f = \int_0^\infty e^{-\lambda t} S_0(t)\, f\, dt, \quad f \in X, \quad \mathrm{Re}\,\lambda > 0\,. \tag{2.1}$$

Then we assume that the following two conditions are satisfied:

ii) for all $\lambda > 0$ we have $L_\lambda[X] \subset X \cap Y$, and

iii) for all $\lambda > 0$ we have

$$\lambda \|L_\lambda f\|_X + \|L_\lambda f\|_Y \leq \|f\|_X, \qquad f \geq 0 \text{ in } X. \tag{2.2}$$

Now let $B : X \cap Y \to X$ be a positive linear operator.

iv) We assume that $B : X \cap Y \to X$ is such that

$$\|Bf\|_X \leq \|f\|_Y, \qquad f \geq 0 \text{ in } X \cap Y. \tag{2.3}$$

We remark that under this assumption the operator B can be unbounded as an operator acting on X.

Example 2.1 *In kinetic theory, A is commonly given by the sum $T_0 + T_F + T_A$, where T_0 is the free streaming operator, T_F is the external force term, and T_A is the absorption (loss) term. The absorption operator is defined by $T_A = -\nu I$, where I is the identity operator and $\nu = \nu(\mu)$ is nonnegative and belongs to $L^{1,loc}(\Sigma, d\mu)$, i.e., ν is μ-integrable on every bounded subset of Σ.*

For simplicity we restrict ourselves to the one dimensional setting with constant acceleration so that $T = T_F + T_A = -a\dfrac{\partial}{\partial v} - \nu(v)$, *with* $a > 0$ *constant and* $v \in \mathbb{R}$. *Then the operator* L_λ *takes the form*

$$(L_\lambda f)(v) = \frac{1}{a}\int_{-\infty}^{v} \exp\left(-\frac{1}{a}\int_{v'}^{v} [\nu(v'') + \lambda]\, dv''\right) f(v')dv'. \tag{2.4}$$

In most applications the operator B *is the gain collision operator, which maps* $\{f \in L^1(\mathbb{R}) : f \geq 0, \nu f \in L^1(\mathbb{R})\}$ *into* $\{f \in L^1(\mathbb{R}) : f \geq 0\}$ *and satisfies*

$$\|Bf\|_{L^1(\mathbb{R})} \leq \|\nu f\|_{L^1(\mathbb{R})}, \qquad f \geq 0 \ \ \textit{for } f \in L^1(\mathbb{R}) \textit{ and } \nu f \in L^1(\mathbb{R}).$$

■

Now define for $n \in \mathbb{N}$ and $\lambda > 0$

$$T_\lambda^{(n)} f = \sum_{j=0}^{n} L_\lambda (BL_\lambda)^j f, \qquad f \geq 0 \text{ in } X, \tag{2.5}$$

where $\{L_\lambda\}_{\mathrm{Re}\,\lambda > 0}$ and B are assumed to satisfy hypotheses ii)-iv). Then for $f \geq 0$ in X we have

$$\begin{aligned}
\lambda\|T_\lambda^{(n)} f\|_X &= \lambda \sum_{j=0}^{n} \|L_\lambda(BL_\lambda)^j f\|_X \\
&\leq \lambda\|L_\lambda f\|_X + \sum_{j=1}^{n} \left\{\|(BL_\lambda)^j f\|_X - \|L_\lambda(BL_\lambda)^j f\|_Y\right\} \\
&\leq \lambda\|L_\lambda f\|_X + \sum_{j=1}^{n} \left\{\|L_\lambda(BL_\lambda)^{j-1} f\|_Y - \|L_\lambda(BL_\lambda)^j f\|_Y\right\} \\
&= \lambda\|L_\lambda f\|_X + \|L_\lambda f\|_Y - \|L_\lambda(BL_\lambda)^n f\|_Y \\
&\leq \|f\|_X - \|(BL_\lambda)^{n+1} f\|_X.
\end{aligned}$$

Now letting $n \to \infty$ in (2.5), we define

$$T_\lambda f = \lim_{n\to\infty} T_\lambda^{(n)} f \ , \tag{2.6}$$

which satisfies

$$\lambda\|T_\lambda f\|_X \leq \|f\|_X - \beta_\lambda(f), \qquad f \geq 0 \text{ in } X, \tag{2.7}$$

where

$$\beta_\lambda(f) = \lim_{n\to\infty} \|(BL_\lambda)^n f\|_X, \qquad f \geq 0 \text{ in } X,$$

extends to a bounded linear functional on X which is necessarily positive. Thus there exists $\varphi_\lambda \geq 0$ in $X^* = L^\infty(\Sigma, d\mu)$ — note that μ is σ-finite — such that

$$\beta_\lambda(f) = \langle f, \varphi_\lambda \rangle, \qquad f \geq 0 \text{ in } X.$$

We now introduce more restrictive assumptions. Let us consider the positive contraction semigroup $\{S_0(t)\}_{t\geq 0}$ on X whose resolvent $\{L_\lambda\}_{\mathrm{Re}\,\lambda>0}$ satisfies assumption ii) as well as the following equality:

iii′) for all $\lambda > 0$ we have

$$\lambda \|L_\lambda f\|_X + \|L_\lambda f\|_Y = \|f\|_X, \qquad f \geq 0 \text{ in } X. \tag{2.2'}$$

We will also consider positive operators $B : X \cap Y \to X$ satisfying the following equality:

iv′)

$$\|Bf\|_X = \|f\|_Y, \qquad f \geq 0 \text{ in } X \cap Y. \tag{2.3'}$$

If $\{L_\lambda\}_{\mathrm{Re}\,\lambda>0}$ and B are assumed to satisfy assumptions ii), iii′) and iv′), then all inequality signs in the previous two paragraphs turn into equality signs and (2.7) turns into the equality

$$\lambda \|T_\lambda f\|_X = \|f\|_X - \beta_\lambda(f), \qquad f \geq 0 \text{ in } X. \tag{2.7'}$$

Proposition 2.1 $\{T_\lambda\}_{\lambda>0}$ *is the resolvent of a positive contraction semigroup on X. Moreover, this semigroup is stochastic if and only if, for some (and hence all) $\lambda > 0$, conditions* ii), iii′) *and* iv′) *are satisfied and the linear functional $\beta_\lambda \equiv 0$.*

Proof. Let $\{B_m\}_{m\in\mathbb{N}}$ be an increasing sequence of bounded positive operators on X such that

$$\lim_{m\to\infty} \|Bf - B_m f\|_X = 0, \qquad f \geq 0 \text{ in } X \cap Y.$$

In fact, choosing an increasing sequence $\{\Sigma_m\}_{m=1}^\infty$ of μ-measurable sets with union Σ on which the Radon-Nikodym derivative of ν with respect to μ is bounded, we may choose $B_m f = B(f\chi_{\Sigma_m})$,[2] where χ_E denotes the characteristic function of E. Then standard semigroup theory (i.e. adding the bounded linear operator B_m to the generator A of $\{S_0(t)\}_{t\geq 0}$) implies that

$$T_{\lambda,m} f = \sum_{j=0}^{\infty} L_\lambda (B_m L_\lambda)^j f, \qquad f \geq 0 \text{ in } X, \tag{2.8}$$

is the resolvent of the positive semigroup $\{S_{(m)}\}_{t\geq 0}$ generated by $A + B_m$. Applying the theorem of dominated convergence to (2.8), we see that for all $t \geq 0$

$$\lim_{m\to\infty} \|T_\lambda f - T_{\lambda,m} f\|_X = 0, \qquad f \geq 0 \text{ in } X \cap Y. \tag{2.9}$$

Now note that the semigroups $\{S_{(m)}\}_{t\geq 0}$ form an increasing sequence of substochastic semigroups. Thus there exists a positive contraction semigroup $\{S(t)\}_{t\geq 0}$ on X such that

$$\lim_{m\to\infty} \|S(t)f - S_{(m)}(t)f\|_X = 0, \qquad f \geq 0 \text{ in } X \cap Y. \tag{2.10}$$

Then the first part of Proposition 2.1 follows immediately from (2.9) and (2.10).

To prove the second part, it is clear that the stochasticity of $\{S(t)\}_{t\geq 0}$ implies that for all $\lambda > 0$ the equality signs hold in the derivation leading from (2.5) to (2.7) and that $\beta_\lambda \equiv 0$. But this is only possibly if iii′) and iv′) as well as (2.7′) hold for all $\lambda > 0$. Conversely, if assumptions iii′) and iv′) hold and $\beta_\lambda \equiv 0$ for some $\lambda > 0$, then, in view of (2.7′), we have for $f \geq 0$ in X

$$\int_0^\infty e^{-\lambda t} \left(\|f\|_X - \|S(t)f\|_X\right) dt = \frac{1}{\lambda}\|f\|_X - \frac{1}{\lambda}\|f\|_X = 0,$$

while $\|f\|_X - \|S(t)f\|_X \geq 0$ for every $t \geq 0$. Hence, the integrand must vanish and therefore $\{S(t)\}_{t\geq 0}$ is stochastic. ■

3 Characterizing the semigroup generator G

We now characterize the generator of the semigroup $S(t)$ in terms of properties of the operator BL_λ. We begin by observing that the kernel of $I - BL_\lambda$ is trivial, i.e., that 1 cannot be an eigenvalue of BL_λ.

Indeed, using (2.5) we immediately have

$$T_\lambda^{(n)}(I - BL_\lambda) = L_\lambda - L_\lambda (BL_\lambda)^{n+1}.$$

[2] Here we use that μ is σ-finite and ν is absolutely continuous with respect to μ.

Taking the strong limit as $n \to \infty$ we immediately get for $\lambda > 0$

$$T_\lambda(I - BL_\lambda) = L_\lambda. \tag{3.1}$$

Now let us denote the generator of $\{S(t)\}_{t\geq 0}$ by G. Then

$$\mathcal{D}(G) = T_\lambda[X] \supset L_\lambda[X] = \mathcal{D}(A),$$

and $\mathrm{Ker}\,(I - BL_\lambda) \subset \mathrm{Ker}\, L_\lambda = \{0\}$, so that $1 \notin \sigma_p(BL_\lambda)$ (i.e., 1 is not an eigenvalue of BL_λ).

In [FMvdM02] we have established the following result.

Theorem 3.1 *Let G denote the generator of the semigroup $S(t)$ whose resolvent is T_λ defined by* (2.6)*. Then the following statements are equivalent:*

1) $\mathcal{D}(G) = \mathcal{D}(A)$ and $G = A + B$;

2) $I - BL_\lambda$ is invertible on X for some $\lambda > 0$;

3) $I - BL_\lambda$ is invertible on X for all $\mathrm{Re}\,\lambda > 0$.

Corollary 3.1 *The equivalent conditions of Theorem* 3.1 *are fulfilled if at least one the following sufficient conditions is satisfied:*

a) B is bounded on X;

b) BL_λ is weakly compact on X;

c) the norm of BL_λ is strictly less than one.

Proof. Each of the sufficient conditions implies that $I - BL_\lambda$ is boundedly invertible on X. ∎

We now link the property of 1 being a point of the resolvent set $\rho(BL_\lambda)$, the continuous spectrum $\sigma_c(BL_\lambda)$, or the residual spectrum $\sigma_r(BL_\lambda)$ of BL_λ to three distinct characterizations of the generator G. For convenience we include Theorem 3.1 as Part 1) of Theorem 3.2.

Theorem 3.2 *We have the following characterizations:*

1) $1 \in \rho(BL_\lambda)$ *for some (and hence all)* $\lambda > 0$ *if and only if* $\mathcal{D}(G) = \mathcal{D}(A)$;

2) $1 \in \sigma_c(BL_\lambda)$ *for some (and hence all)* $\lambda > 0$ *if and only if* $\mathcal{D}(G) \supsetneqq \mathcal{D}(A)$ *and* $G = \overline{A + B}$;

3) $1 \in \sigma_r(BL_\lambda)$ *for some (and hence all)* $\lambda > 0$ *if and only if* $G \supsetneqq \overline{A + B}$.

Proof. If $1 \in \rho(BL_\lambda)$, then $I - BL_\lambda$ is boundedly invertible on X and therefore, in view of (3.1) and $\operatorname{Ker} T_\lambda = \operatorname{Ker} L_\lambda = \{0\}$, we have $\mathcal{D}(G) = \operatorname{Im} T_\lambda = \operatorname{Im} L_\lambda = \mathcal{D}(A)$, and conversely. Since clearly $\operatorname{Im} T_\lambda = \operatorname{Im} L_\lambda$ does not depend on λ, we obtain that $1 \in \rho(BL_\lambda)$ for all $\lambda > 0$ whenever this is true for some $\lambda > 0$.

Let us now prove parts 2) and 3), with the exception of the implication leading from some to all $\lambda > 0$. A vector $w \in X$ belongs to $\mathcal{D}(\overline{A+B})$ if and only if there exist $f \in X$ and a sequence $\{z_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}(A+B)$ such that $\|z_n - w\|_X \to 0$ and $\|(\lambda - (A+B))z_n - f\|_X \to 0$ (in which case $(\lambda - \overline{A+B})w = f$). Since $\mathcal{D}(A) = \operatorname{Im} L_\lambda$, we write $z_n = L_\lambda g_n$ for some $g_n \in X$ and rewrite the second limit in the form

$$\|(I - BL_\lambda)g_n - f\|_X = \|(\lambda - (A+B))z_n - f\|_X \to 0,$$

so that f belongs to the closure of $\operatorname{Im}(I - BL_\lambda)$ in X. Moreover,

$$\begin{aligned}\|z_n - T_\lambda f\|_X &= \|L_\lambda g_n - T_\lambda f\|_X = \|T_\lambda (I - BL_\lambda)g_n - T_\lambda f\|_X \\ &= \|T_\lambda[(I - BL_\lambda)g_n - f]\|_X \le \frac{1}{\lambda}\|(I - BL_\lambda)g_n - f\|_X \to 0,\end{aligned}$$

so that $w = T_\lambda f$. In other words,

$$\mathcal{D}(\overline{A+B}) = T_\lambda \left[\overline{\operatorname{Im}(I - BL_\lambda)}\right],$$

which implies that $\mathcal{D}(\overline{A+B}) = \mathcal{D}(G)$ (which in turn equals $T_\lambda[X]$) if and only if $I - BL_\lambda$ has a dense range in X. It is easily seen that, in general, $G \supset \overline{A+B}$.

Since $G = \overline{A+B}$ and $G \supsetneq \overline{A+B}$ are both statements that do not depend on λ, 2) and 3) are each true for all $\lambda > 0$ if they are each true for some $\lambda > 0$. ∎

We now derive a necessary and sufficient condition for the stochasticity of the semigroup $\{S(t)\}_{t\ge0}$ generated by G.

Theorem 3.3 *Let the conditions* ii), iii′) *and* iv′) *be satisfied. Then the semigroup* $\{S(t)\}_{t\ge0}$ *is stochastic if and only if* $G = \overline{A+B}$, *i.e., if and only if either condition* 1) *or condition* 2) *of Theorem* 3.2 *is fulfilled.*

Proof. In view of (2.7′), stochasticity of $\{S(t)\}_{t\ge0}$ is obviously equivalent to requiring that $\beta_\lambda(f) = 0$ for every $f \in X$. The latter amounts to requiring that $\varphi_\lambda \equiv 0$. Because for all $f \ge 0$ in X

$$\beta_\lambda(BL_\lambda f) = \lim_{n\to\infty} \|(BL_\lambda)^{n+1} f\|_X = \beta_\lambda(f),$$

we have $(I - (BL_\lambda)^*)\varphi_\lambda \equiv 0$ in $X^* = L^\infty(\Sigma, d\mu)$. Thus stochasticity of $\{S(t)\}_{t\geq 0}$ is equivalent to requiring that $1 \notin \sigma_p((BL_\lambda)^*)$. However, since in any case $1 \notin \sigma_p(BL_\lambda)$, we see that stochasticity of $\{S(t)\}_{t\geq 0}$ is equivalent to requiring that $1 \notin \sigma_r(BL_\lambda)$. Theorem 3.3 follows now immediately from part c) of Theorem 3.2. ∎

Constructing T_λ as the strong limit of $\{T_\lambda^{(n)}\}_{n\in\mathbb{N}}$ or constructing T_λ as the strong monotonically increasing limit of $\{T_{\lambda,m}\}_{m\in\mathbb{N}}$ if $B_m \uparrow B$ strongly, always leads to a so-called minimal evolution semigroup. By this we mean a substochastic semigroup $\{S(t)\}_{t\geq 0}$ on X such that its resolvent $\{T_\lambda\}_{\mathrm{Re}\,\lambda>0}$ satisfies the identity (3.1) *and can be obtained by iterating* (3.1). Clearly, (3.1) is equivalent to the Dyson-Phillips integral formula

$$S(t)f - \int_0^\infty S(t-\tau)BS_0(\tau)f\,d\tau = S_0(t)f, \qquad f \geq 0 \text{ in } X\,, \tag{3.2}$$

where $t \geq 0$, plus the rule that $\{S(t)f\}_{t\geq 0}$ can be obtained by iterating (3.2).

The question is if there exist more than one substochastic semigroup $\{\tilde{S}(t)\}_{t\geq 0}$ on X (with corresponding resolvent $\{\tilde{T}_\lambda\}_{\mathrm{Re}\,\lambda>0}$) which satisfies (3.2) [or equivalently: for which the resolvent satisfies (3.1)]. Then it is easily seen by applying infinitely many iteration steps that any such semigroup (resp., any such resolvent) satisfies $\tilde{S}(t)f \geq S(t)f$ (resp., $\tilde{T}_\lambda f \geq T_\lambda f$) for every $f \geq 0$ in X. Therefore it is justified to call $\{S(t)\}_{t\geq 0}$ (resp., $\{T_\lambda\}_{\mathrm{Re}\,\lambda>0}$) the *minimal evolution semigroup* (resp., the *minimal resolvent*).

When $1 \in \rho(BL_\lambda) \cup \sigma_c(BL_\lambda)$ for some (and hence all) $\lambda > 0$ and therefore the semigroup $\{S(t)\}_{t\geq 0}$ is stochastic, the minimal semigroup as discussed in the preceding paragraph is the only semigroup satisfying (3.1), for the simple reason that $I - BL_\lambda$ has a dense range in X for every $\lambda > 0$. The situation might (or might not) change drastically if $1 \in \sigma_r(BL_\lambda)$ for some (and hence all) $\lambda > 0$ and the semigroup $\{S(t)\}_{t\geq 0}$ is substochastic but not stochastic. To understand what might happen, we consider an alternative substochastic semigroup $\{\tilde{S}(t)\}_{t\geq 0}$ whose resolvent $\{\tilde{T}_\lambda\}_{\mathrm{Re}\,\lambda>0}$ satisfies the identity

$$\tilde{T}_\lambda(I - BL_\lambda) = L_\lambda, \qquad \mathrm{Re}\,\lambda > 0. \tag{3.3}$$

Then (3.3) is equivalent to the Dyson-Phillips integral equation

$$\tilde{S}(t)f - \int_0^\infty \tilde{S}(t-\tau)BS_0(\tau)f\,d\tau = S_0(t)f, \qquad f \geq 0 \text{ in } X. \tag{3.4}$$

The important point is that $\tilde{T}_\lambda$ and $\tilde{S}(t)$ may no longer be obtained by iterating (3.3) and (3.4), respectively.

Having found a solution $\tilde{T}_\lambda$ of (3.3) that is a positive operator on X of norm $\leq (1/\lambda)$ (where $\lambda > 0$), we denote by 1 the function identically equal to 1 on Σ, as element of $X^* = L^\infty(\Sigma, d\mu)$, and define

$$\tilde{\varphi}_\lambda = 1 - \lambda \tilde{T}_\lambda^* 1 \in X^* = L^\infty(\Sigma, d\mu), \tag{3.5}$$

where $\langle \tilde{T}_\lambda f, \varphi \rangle = \langle f, \tilde{T}_\lambda^* \varphi \rangle$ for all $f \in X$ and $\varphi \in X^*$. Then we easily compute for $f \geq 0$ in X

$$\begin{aligned}
\langle f, (BL_\lambda)^* \tilde{\varphi}_\lambda \rangle &= \langle BL_\lambda f, \tilde{\varphi}_\lambda \rangle \\
&= \langle BL_\lambda f, 1 \rangle - \lambda \langle BL_\lambda f, \tilde{T}_\lambda^* 1 \rangle \\
&= \langle BL_\lambda f, 1 \rangle - \lambda \langle \tilde{T}_\lambda BL_\lambda f, 1 \rangle \\
&= \langle BL_\lambda f, 1 \rangle - \lambda \langle [\tilde{T}_\lambda - L_\lambda] f, 1 \rangle \\
&= \|BL_\lambda f\|_X - \lambda \langle \tilde{T}_\lambda f, 1 \rangle + \lambda \|L_\lambda f\|_X \\
&= \|f\|_X - \lambda \langle \tilde{T}_\lambda f, 1 \rangle = \langle f, 1 - \lambda \tilde{T}_\lambda^* 1 \rangle = \langle f, \tilde{\varphi}_\lambda \rangle,
\end{aligned}$$

which implies that $(BL_\lambda)^* \tilde{\varphi}_\lambda = \tilde{\varphi}_\lambda$. Moreover, $\tilde{\varphi}_\lambda \not\equiv 0$, because otherwise we would have for all $f \geq 0$ in X

$$0 = \langle f, \tilde{\varphi}_\lambda \rangle = \langle f, 1 \rangle - \lambda \langle f, \tilde{T}_\lambda^* 1 \rangle = \|f\|_X - \lambda \|\tilde{T}_\lambda f\|_X,$$

which would turn $\{\tilde{T}_\lambda\}_{\lambda > 0}$ into the resolvent of a stochastic semigroup. In other words, every substochastic semigroup $\{\tilde{S}(t)\}_{t \geq 0}$ with resolvent $\{\tilde{T}_\lambda\}_{\lambda > 0}$ satisfying (3.3) leads to a positive eigenvector $\tilde{\varphi}_\lambda$ of $(BL_\lambda)^*$ at the eigenvalue 1. This eigenvector must necessarily satisfy

$$0 \leq \tilde{\varphi}_\lambda \leq \varphi_\lambda \leq 1,$$

because the substochastic semigroup obtained by iterating the Dyson-Phillips integral equation (3.4) is the minimal substochastic semigroup whose resolvent satisfies (3.3).

4 Application: The runaway problem

In this section we revise some mathematical aspects of the behaviour of charged particles moving within a host medium under the influence of a constant electric field. When the collision process is not sufficient to slow down the most energetic particles and to force the system towards relaxation, a travelling wave in velocity space is generated. In this case the particle swarm exhibits a phenomenon called *runaway*: electrons are continuously accelerated without limit (for the physical problem see [CPF, Kum84] and the references quoted therein.)

Necessary conditions and sufficient conditions for the existence, uniqueness and attractivity of a steady-state solution and the occurrence of travelling waves have been studied in [FvdMPF89, FvdM89].

Let $a > 0$ be the constant electrostatic acceleration and let ν be the collision frequency between an electron and the host medium, where we assume that $\nu \in L^{1,loc}(\mathbb{R}; dv)$ is nonnegative. Introduce the Banach spaces $X = L^1(\mathbb{R}; dv)$ and $X_\nu = L^1(\mathbb{R}; \nu(v)dv)$ with their usual norms.[3] Define $T_F = -a\frac{\partial}{\partial v}$, $A = T_F + T_A = -a\frac{\partial}{\partial v} - \nu(v)$ on suitable domains contained in $X = L^1(\mathbb{R}; dv)$, and let

$$(Bf)(v) = \int_{-\infty}^{\infty} k(v, v')\nu(v')f(v')\,dv', \tag{4.1}$$

where k is positive and $\int_{-\infty}^{\infty} k(v, v')\,dv = 1$ for every v'.

Using the preceding definitions we can write the linear Boltzmann equation for electron swarms in the concise form

$$\frac{\partial f}{\partial t} = T_F f + T_A f + Bf\,, \quad t > 0$$

equipped by the initial data $f(0) = f_0 \in X$.

In the following we will explain how the dependence of the collision frequency $\nu(v)$ upon the speed v for large values of v is crucial in finding the solution of the problem and characterizing the generator G.

Defining L_λ as in (2.4) by

$$(L_\lambda f)(v) = \frac{1}{a}\int_{-\infty}^{v} \exp\left\{-\frac{\lambda}{a}(v - v') - \frac{1}{a}\int_{v'}^{v} \nu(\hat{v})d\hat{v}\right\} f(v')\,dv', \tag{4.3}$$

we have for $f \geq 0$ in X

$$\int_{-\infty}^{\infty} [\lambda + \nu(v)](L_\lambda f)(v)\,dv =$$

$$= \begin{cases} \displaystyle\int_{-\infty}^{\infty} f(v')\,dv' & \text{if Re } \lambda > 0 \\ \displaystyle\int_{-\infty}^{\infty} \left[1 - \exp\left\{-\frac{1}{a}\int_{v'}^{\infty} \nu(\hat{v})d\hat{v}\right\}\right] f(v')dv' & \text{if } \lambda = 0, \end{cases}$$

[3] In fact, X_ν is a Banach space if ν is positive a.e.

and

$$\int_{-\infty}^{\infty} \nu(v)(L_0 f)(v)\,dv = \int_{-\infty}^{\infty} f(v')\,dv' \qquad \text{if } \int_{-\infty}^{\infty} \nu(v)dv = \infty$$
$$\int_{-\infty}^{\infty} \nu(v)(L_0 f)(v)\,dv \leq \left[1 - \exp\left\{-\frac{\|\nu\|_X}{a}\right\}\right] \|f\|_X \quad \text{if } \nu \in L^1(\mathbb{R})\,.$$

As a result,

$$\|BL_\lambda\| \leq \begin{cases} 1 \text{ if } \nu \notin L^1(\mathbb{R}) \\ 1 - \ \exp\left\{-\frac{\|\nu\|_X}{a}\right\} \text{ if } \nu \in L^1(\mathbb{R})\,. \end{cases}$$

Therefore, the norm of BL_λ is less than 1 whenever $\nu \in L^1(\mathbb{R})$.

Thus, according to Proposition 3.1, we have $G = A + B$ if at least one of the three conditions (1) BL_λ is weakly compact on X for some (and hence) all $\lambda > 0$, (2) ν is bounded, and (3) $\nu \in L^1(\mathbb{R})$, is satisfied. We do not have any example in which $G \supsetneq \overline{A + B}$ and hence the semigroup $\{S(t) : t \geq 0\}$ is not stochastic.

The reader interested to such problem can compare this approach with the results obtained in [FvdMPF89].

5 Application: The Fragmentation Model

In this section we consider a simple model for the fragmentation process which arises in many fields such as erosion, polymer degradation and oxidation. Recently, mathematical modelling for molecule fragmentation has received much attention from mathematicians.

Let $\varphi(x,t)$ be the concentration of x-mers as a function of time. Let $F(x,y)$ denote the intrinsic rate at which an $(x+y)$-mer breaks up into an x-mer and a y-mer. Then, according to the model studied in [ZMcG85, McLLMcB97, AB79], φ obeys

$$\frac{\partial \varphi}{\partial t} = -\varphi(x)\int_0^x F(y, x-y)\,dy + 2\int_x^\infty F(x, y-x)\varphi(y)\,dy. \tag{5.1}$$

We consider only the special case in which $F(x,y)$ depends on the total size $(x+y)$ of the fragmenting object, and in particular the case where $F(x,y) = (x+y)^{\alpha-1}$ for some constant $\alpha \in \mathbb{R}$. For this, formally conservative, model equation (5.1) reads as follows:

$$\frac{\partial \varphi}{\partial t} = -\varphi(x)x^\alpha + 2\int_x^\infty y^{\alpha-1}\varphi(y)\,dy. \tag{5.2}$$

This model was investigated from the analytical point of view in [AB79, ZMcG85] and using semigroup theory in [McLLMcB97]. A characterization of the semigroup generator for equation (5.2) was obtained by Banasiak in [Ban01], by means of an extension of the Kato-Voigt perturbation theorem for substochastic semigroups.

The case $\alpha > 0$ corresponds to the stochastic (conservative) semigroup which is generated by the closure of the operator K_α appearing on the right-hand side of (5.2). For $\alpha > 0$ one can produce multiple solutions and offer an explanation of the non-uniqueness in such a model (see [Ban02a], and also [Ban02b]).

The case $\alpha < 0$ corresponds to a generator which is a proper extension of $\overline{K_\alpha}$ and the (generated) semigroup is not stochastic. In this case the solutions of (5.2) do not conserve mass, as observed for explicit solutions in [ZMcG86, McGZ87]. The very fast fragmentation of very small particles creates "zero" size particles with non-zero mass. This phenomenon which corresponds to an infinitely large fragmentation rate is called "shattering" transition.

The previous models are formally conservative. However, in many situations the process involves continuous mass loss, for instance, due to dissolution of the porous surface of molecules leading to fragmentation. In a recent paper [BL02], Banasiak and Lamb apply substochastic semigroup theory to analyze molecule fragmentation models with mass loss and to give a mathematical validation of the mass loss known in the chemical physics literature as "shattering" transition and the creation of zero-size particles called "fractal dust."

In this section we limit ourselves to studying equation (5.2).

Let us study the equation in the Banach space $X = L^1(\mathbb{R}^+; xdx)$, because the total mass for nonnegative distribustion φ is given by

$$\int_0^\infty \varphi(x)x\,dx.$$

Then, in terms of the abstract theory, A is the multiplication by $-x^\alpha$, which can be interpreted as a collision frequency, so that

$$(L_\lambda\varphi)(x) = ((\lambda - A)^{-1}\varphi)(x) = \frac{\varphi(x)}{\lambda + x^\alpha}, \quad \text{for } \lambda > 0,$$

Furthermore, $Y = L^1(\mathbb{R}^+; x^{1+\alpha}\,dx)$ and

$$(B\varphi)(x) = 2\int_x^\infty y^{\alpha-1}\varphi(y)\,dy,$$

so that

$$(BL_\lambda f)(x) = 2\int_x^\infty \frac{y^{\alpha-1}}{\lambda + y^\alpha} f(y)\, dy\,, \tag{5.3}$$

and

$$((BL_\lambda)^*\varphi)(x) = \frac{2x^{\alpha-2}}{\lambda + x^\alpha}\int_0^x y\,\varphi(y)\, dy. \tag{5.4}$$

We easily check the following conditions

$$\begin{aligned} \lambda\|L_\lambda\varphi\|_X + \|L_\lambda\varphi\|_Y = \|\varphi\|_X, &\qquad \varphi \geq 0 \text{ in } X; \\ \|B\varphi\|_X = \|\varphi\|_Y, &\qquad \varphi \geq 0 \text{ in } X \cap Y = L^1(\mathbb{R}^+; x(1+x^\alpha)dx), \end{aligned}$$

which correspond with the general assumptions iii ′) and iv ′).

We apply the previous abstract theory to the fragmentation model (5.2) to characterize the generator G.

Let $f \in X$ be nonnegative and let $\lambda > 0$. BL_λ is bounded on X as follows from the positivity and ii). From (5.3) and (5.4) we have

$$\begin{aligned} \|BL_\lambda f\|_X &= \int_0^\infty \left\{2\int_x^\infty \frac{y^{\alpha-1}}{\lambda + y^\alpha} f(y)\, dy\right\} xdx = \\ &= \int_0^\infty \frac{y^{\alpha-1} f(y)}{\lambda + y^\alpha}\int_0^y 2x\, dxdy = \int_0^\infty \frac{y^\alpha}{\lambda + y^\alpha} f(y)\, ydy. \end{aligned}$$

Hence,

$$\|BL_\lambda\|_{X\to X} = \sup_{y\geq 0}\left|\frac{y^\alpha}{\lambda + y^\alpha}\right| = \begin{cases} \dfrac{1}{1+\lambda} & \text{if } \alpha = 0 \\ 1 & \text{if } \alpha \neq 0 \end{cases}. \tag{5.5}$$

Let us now consider the equation $(I - \zeta(BL_\lambda)^*)\varphi = 0$, where $\zeta \in \mathbb{C}$. Putting $\psi(x) = \int_0^x y\,\varphi(y)\, dy$, we solve

$$\frac{\psi'(x)}{\psi(x)} = \frac{2\zeta x^{\alpha-1}}{\lambda + x^\alpha}, \qquad\qquad \varphi(x) = \frac{\psi'(x)}{x}.$$

We now distinguish three cases, depending on the sign of α.

Case 1: $\alpha > \mathbf{0}$: We get

$$\psi(x) = c(\lambda + x^\alpha)^{2\zeta/\alpha}, \qquad\qquad \varphi(x) = 2c\zeta x^{\alpha-2}(\lambda + x^\alpha)^{2\frac{\zeta}{\alpha}-1}.$$

Then

$$\zeta((BL_\lambda)^*\varphi)(x) = \frac{2\zeta x^{\alpha-2}}{\lambda + x^\alpha} \int_0^x y\varphi(y)\,dy =$$
$$= \frac{2\zeta x^{\alpha-2}}{\lambda + x^\alpha}\,[\psi(x) - \psi(0)] =$$
$$= \varphi(x) - 2c\zeta\lambda^{2\zeta/\alpha}\frac{x^{\alpha-2}}{\lambda + x^\alpha}.$$

Thus, for $\zeta \in \mathbb{C}$, the only solution of $(I - \zeta(BL_\lambda)^*)\varphi = 0$ occurs if $c = 0$ and this is the trivial solution. Hence, for all $\zeta \in \mathbb{C}$ we have $\ker(I - \zeta(BL_\lambda)^*) = \{0\}$, so that $\mathrm{Im}\,(I - \zeta BL_\lambda)$ is dense in X for all $\zeta \in \mathbb{C}$. In other words, $\sigma_r(BL_\lambda) \setminus \{0\} = \emptyset$.

Further, it is easily seen that for $\mathrm{Re}\ s > \frac{2}{\alpha} > 0$ the functions $f_s(x) = (\lambda + x^\alpha)^{-s}$ belong to X and satisfy $BL_\lambda f_s = \frac{2}{\alpha s} f_s$. Thus

$$\left\{z \in \mathbb{C}\colon \left|z - \frac{1}{2}\right| < \frac{1}{2}\right\} \subset \sigma_p(BL_\lambda),$$

i.e., this open disk consists of eigenvalues of BL_λ only. By the above, the boundary of this disk is contained in $\sigma_c(BL_\lambda)$. As a result, we have $1 \in \sigma_c(BL_\lambda)$.

Case 2: $\boldsymbol{\alpha < 0}$: In this case the functions ψ and φ take the form

$$\psi(x) = \frac{cx^{2\zeta}}{(1 + \lambda x^{|\alpha|})^{2\zeta/|\alpha|}}, \qquad \varphi(x) = \frac{2c\zeta}{x^{2(1-\zeta)}} \cdot \frac{1}{(1 + \lambda x^{|\alpha|})^{1+2\zeta/|\alpha|}},$$

and therefore

$$\zeta((BL_\lambda)^*\varphi)(x) = \frac{2\zeta}{x^2(1 + \lambda x^{|\alpha|})} \int_0^x \frac{2c\zeta}{y^{1-2\zeta}} \cdot \frac{dy}{(1 + \lambda y^{|\alpha|})^{1+2\zeta/|\alpha|}} =$$
$$= \frac{2\zeta}{x^2(1 + \lambda x^{|\alpha|})} \cdot \left[\frac{cy^{2\zeta}}{(1 + \lambda y^{|\alpha|})^{2\zeta/|\alpha|}}\right]_{y=0}^{y=x}.$$

Thus φ is a solution of the equation $\zeta(BL_\lambda)^*\varphi = \varphi$ if and only if $\varphi \in L^\infty(\mathbb{R}^+, xdx)$, which happens for $c \neq 0$ if and only if $\mathrm{Re}\,\zeta \geq 1$. Hence,

$$\ker(I - \zeta(BL_\lambda)^*) = \begin{cases} \mathrm{span}\left\{\dfrac{2\zeta}{x^{2(1-\zeta)}(1 + \lambda x^{|\alpha|})^{1+2\zeta/|\alpha|}}\right\} & \text{if } \mathrm{Re}\,\zeta \geq 1 \\ \{0\} & \text{otherwise.} \end{cases}$$

Consequently,

$$\left\{z \in \mathbb{C}\colon \left|z - \frac{1}{2}\right| > \frac{1}{2}\right\} \subset \sigma_c(BL_\lambda) \cup \rho(BL_\lambda) \tag{5.6}$$

$$\left\{z \in \mathbb{C}\colon \left|z - \frac{1}{2}\right| \leq \frac{1}{2}, z \neq 0\right\} \subset \sigma_r(BL_\lambda), \tag{5.7}$$

In particular, $1 \in \sigma_r(BL_\lambda)$.

Case 3: $\alpha = \mathbf{0}$: We get

$$\psi(x) = cx^{2\zeta/(\lambda+1)}, \qquad \varphi(x) = \frac{2c\zeta}{\lambda+1}\, x^{2(\zeta-\lambda-1)/(\lambda+1)},$$

and hence, for $c \neq 0$, $\varphi \in L^\infty(\mathbb{R}^+; x\,dx)$ if and only if $\operatorname{Re}\zeta = \lambda + 1$. Therefore,

$$\ker(I - \zeta(BL_\lambda)^*) = \begin{cases} \text{span } \left\{x^{2(\zeta-\lambda-1)/(\lambda+1)}\right\} & \text{if } \operatorname{Re}\zeta = \lambda+1 \\ \{0\} & \text{otherwise.} \end{cases}$$

Hence,

$$\left\{z \in \mathbb{C}\colon \left|z - \frac{1}{2(\lambda+1)}\right| = \frac{1}{2(\lambda+1)}\right\} \setminus \{0\} \subset \sigma_r(BL_\lambda).$$

Moreover, $1 \in \rho(BL_\lambda)$, as a result of (5.5).

The following theorem specifies the semigroup generator G for the three cases involved in the fragmentation model.

Theorem 5.1 *The generator G of the evolution semigroup is given by the following three cases:*

1) *For $\alpha = 0$ we have $G = A + B$;*

2) *For $\alpha > 0$ we have $G = \overline{A+B} \supsetneq A + B$;*

3) *For $\alpha < 0$ we have $G \supsetneq \overline{A+B}$.*

Proof. If $\alpha = 0$, $\|BL_\lambda\| < 1$; condition c) in Corollary 3.1 is satisfied and hence from Theorem 3.3 we conclude that $G = A+B$. Moreover if $\alpha = 0$, we have $1 \in \rho(BL_\lambda)$; thus from Theorem 3.1 we get again $G = A + B$. If $\alpha > 0$ we have $1 \in \sigma_c(BL_\lambda)$, thus Theorem 3.2 we have $G = \overline{A+B}$. Finally, if $\alpha < 0$, we have $1 \in \sigma_r(BL_\lambda)$; from Theorem 3.2 we deduce that $G \supsetneq \overline{A+B}$. ■

We conclude this section by giving an explicit expression for T_λ. For $\alpha > 0$ we first solve the equation $(I - BL_\lambda)f = g$ for $g \in X \cap C^1(0,\infty)$ with $x^2 g(x) \to 0$ as $x \to \infty$. Then $f \in X \cap C^1(0,\infty)$ with $x^2 f(x) \to 0$ as $x \to \infty$ and

$$f'(x) + \frac{2x^{\alpha-1}}{\lambda + x^\alpha} f(x) = g'(x),$$

which can be written in the equivalent form

$$\frac{d}{dx}\left\{(\lambda + x^\alpha)^{2/\alpha} f(x)\right\} = \frac{d}{dx}\left\{(\lambda + x^\alpha)^{2/\alpha} g(x)\right\} - \frac{2x^{\alpha-1}}{(\lambda + x^\alpha)^{1-\frac{2}{\alpha}}} g(x).$$

By integration we obtain

$$f(x) = g(x) + \frac{2}{(\lambda + x^\alpha)^{2/\alpha}} \int_x^\infty \frac{y^{\alpha-1}}{(\lambda + y^\alpha)^{1-\frac{2}{\alpha}}} g(y)\, dy, \tag{5.8}$$

where the integration constant vanishes since $x^2 f(x) \to 0$ and $x^2 g(x) \to 0$ as $x \to \infty$. As a result,

$$(T_\lambda g)(x) = \frac{g(x)}{\lambda + x^\alpha} + \frac{2}{(\lambda + x^\alpha)^{1+\frac{2}{\alpha}}} \int_x^\infty \frac{y^{\alpha-1}}{(\lambda + y^\alpha)^{1-\frac{2}{\alpha}}} g(y)\, dy. \tag{5.9}$$

For $g \geq 0$ in $X \cap C^1(0,\infty)$ with $x^2 g(x) \to 0$ as $x \to \infty$ we easily get

$$\|T_\lambda g\|_X = \int_0^\infty \frac{y}{\lambda + y^\alpha} g(y)\, dy + \frac{1}{\lambda} \int_0^\infty \frac{y^{\alpha+1}}{\lambda + y^\alpha} g(y)\, dy = \frac{1}{\lambda}\|g\|_X,$$

as to be expected. Hence, (5.9) holds for every $g \in X$. For $\alpha = 0$ we get in analogy with (5.8)

$$f(x) = g(x) + \frac{2}{\lambda + 1} x^{-\frac{2}{\lambda+1}} \int_x^\infty y^{\frac{1-\lambda}{1+\lambda}} g(y)\, dy, \tag{5.10}$$

where $g \in X \cap C^1(0,\infty)$ with $x^2 g(x) \to 0$ as $x \to \infty$. Thus

$$(T_\lambda g)(x) = \frac{1}{\lambda + 1}\left[g(x) + \frac{2}{\lambda + 1} x^{-\frac{2}{\lambda+1}} \int_x^\infty y^{\frac{1-\lambda}{1+\lambda}} g(y)\, dy\right], \tag{5.11}$$

where $g \in X$ after suitable extension from a dense domain.

For $\alpha < 0$ the above derivation is somewhat more complicated. Solving $(I - BL_\lambda)f = g$ for $g \in X \cap C^1(0,\infty)$ with $g(+\infty) = 0$ (so that $f \in C^1(0,\infty)$ with $f(+\infty) = 0$, *but not necessarily* $f \in X$), we first obtain

$$\frac{d}{dx}\left\{\frac{x^2}{(1 + \lambda x^{|\alpha|})} f(x)\right\} = \frac{d}{dx}\left\{\frac{x^2}{(1 + \lambda x^{|\alpha|})} g(x)\right\} - \frac{2x}{(1 + \lambda x^{|\alpha|})^{1+\frac{2}{|\alpha|}}} g(x),$$

and subsequently

$$f(x) = g(x) + \frac{(1+\lambda x^{|\alpha|})^{2/|\alpha|}}{x^2} \int_x^\infty \frac{2y}{(1+\lambda y^{|\alpha|})^{1+\frac{2}{|\alpha|}}} g(y)\, dy, \tag{5.12}$$

where the integration constant vanishes. Multiplying (5.12) by x and integrating with respect to $x \in \mathbb{R}^+$ it is easily seen that one cannot choose g in a dense linear subset of g and have $f \in X$. Thus $\mathrm{Im}\,(I - BL_\lambda)$ is not dense in X, as proved above by different means. Let us now compute T_λ and φ_λ for $\alpha < 0$. By induction we obtain

$$((BL_\lambda)^{m+1} f)(x) = 2 \int_x^\infty \frac{f(y)}{y(1+\lambda y^{|\alpha|})} \frac{F_\alpha(x,y)^m}{m!}\, dy,$$

where

$$F_\alpha(x,y) = 2 \int_x^\infty \frac{dz}{z(1+\lambda z^{|\alpha|})} = \log \left\{ \left(\frac{y}{x}\right)^2 \left(\frac{1+\lambda x^{|\alpha|}}{1+\lambda y^{|\alpha|}}\right)^{\frac{2}{|\alpha|}} \right\}.$$

Then

$$\begin{aligned}(T_\lambda f)(x) &= (L_\lambda f)(x) + \sum_{m=0}^{\infty} (L_\lambda (BL_\lambda)^{m+1} f)(x) = \\ &= \frac{x^{|\alpha|}}{1+\lambda x^{|\alpha|}} \left\{ f(x) + 2 \int_x^\infty \frac{f(y)}{y(1+\lambda y^{|\alpha|})} \left(\frac{y}{x}\right)^2 \left(\frac{1+\lambda x^{|\alpha|}}{1+\lambda y^{|\alpha|}}\right)^{\frac{2}{|\alpha|}} dy \right\},\end{aligned}$$

where for $f \geq 0$ in X

$$\begin{aligned}\langle f, \varphi_\lambda \rangle &= \|f\|_X - \lambda \|T_\lambda f\|_X = \\ &= \int_0^\infty \frac{x\, f(x)}{1+\lambda x^{|\alpha|}}\, dx - 2\lambda \int_0^\infty \frac{y\, f(y)}{(1+\lambda y^{|\alpha|})^{1+\frac{2}{|\alpha|}}} \int_0^y \frac{x^{|\alpha|-1}\, dx}{(1+\lambda x^{|\alpha|})^{1-\frac{2}{|\alpha|}}}\, dy = \\ &= \int_0^\infty \frac{x\, f(x)}{1+\lambda x^{|\alpha|}}\, dx - \int_0^\infty \frac{y\, f(y)}{(1+\lambda y^{|\alpha|})^{1+\frac{2}{|\alpha|}}} \left[(1+\lambda x^{|\alpha|})^{\frac{2}{|\alpha|}} \right]_{x=0}^{y} dy = \\ &= \int_0^\infty \frac{y f(y)}{(1+\lambda y^{|\alpha|})^{1+\frac{2}{|\alpha|}}}\, dy,\end{aligned}$$

which implies that

$$\varphi_\lambda(x) = \frac{1}{(1+\lambda x^{|\alpha|})^{1+\frac{2}{|\alpha|}}}\,. \tag{5.13}$$

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