Existence results for set-valued variational inequalities via topological methods☆

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Abstract

In this paper we find existence results for nonlinear variational inequalities involving a multivalued map. Both cases of a lower semicontinuous multimap and an upper semicontinuous one are considered. We solve the problem using a linearization argument and a suitable continuation principle.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary, we use topological methods to establish existence results for a class of nonlinear variational inequalities on convex closed sets. The inequalities considered involve a quasi-linear operator of class S_+ and the nonlinear part is given by the sum of a Carathéodory map and a multivalued map (multimap). We take into account both the cases of elliptic variational inequalities and parabolic variational inequalities. We look for solutions in $W_0^{m,p}(\Omega) = W_0^{m,p}(\Omega, \mathbb{R})$ $(1 < p < \infty)$ and in $L^p([0,d], W_0^{m,p}(\Omega))$, $2 \leq p < \infty$, in the elliptic and parabolic case respectively. Problems of this kind have been studied by many authors and appear in many applications, such as the obstacle and bi-obstacle problem, or the elasto-plastic torsion problem, in which the set K is given by gradient conditions. We mention

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the works of Hu-Papageorgiou [8], Aizicovici-Papageorgiou-Staicu [1], Väth [16], Lan [12], the monograph of Carl-Lee-Motreanu [5], and the references therein. It is also worth to mention the work of Mordukhovich for the link between differential inclusions and variational inequalities (see [13, 14]) and of Kučera for the relation with partial differential equations (see [7]). Observe that if the set on which the variational inequality is valid coincides with the whole space $W_0^{m,p}(\Omega)$ or $L^p([0,d], W_0^{m,p}(\Omega))$, the solutions of the variational inequalities are weak solutions of elliptic and parabolic partial differential inclusions involving a second order differential operator in divergence form. Different methods have been applied to solve these problems, the more used ones being the method of upper and lower solutions (as e.g. in [5]) and the degree theory approach. The latter was first used for semilinear variational inequalities by Szulkin [15] and Miesermann [11]. In [12] the author proves existence results for variational inequalities involving a demicontinuous S-contractive, map A, i.e., $I - A$ is of S_+-type; he finds, as an application, weak solutions for semilinear second-order elliptic inequalities. Concerning the multivalued case, in [16] a fixed point index is constructed for the studied partial differential inclusion. In [8] and in [1] degree theory methods based on the degree map for multivalued perturbation of a (S_+) operator are applied: in [8] the authors prove existence results for a class of partial differential inclusions with an upper semicontinuous multivalued nonlinearity; in [1] multiplicity results are proved both for partial differential inclusions and variational inequalities with, as multimap involved, the generalized subdifferential of a locally Lipschitz function. On the other hand, we consider both the cases of an upper semicontinuous and a lower semicontinuous general kind of multivalued nonlinearity. To solve the problem we use a linearization argument and a continuation principle. More precisely, we define a suitable multivalued operator (multioperator) depending on a parameter λ, whose fixed points at level 1 are the solutions of the variational inequalities considered. We do not assume any regularity in terms of compactness, neither on the quasilinear operator nor on the nonlinearity part to apply the topological degree theory for completely continuous multimap (see [10]), to obtain the existence of at least a fixed point. Moreover, with this approach we do not require any restriction on the set K, as done in [5], see example 5.1. Given a domain $\mathcal{D} \subset \mathbb{R}^k$ we denote in the whole paper with $\|u\|_p$, $\|u\|_{m,p}$, $\|u\|_0$ the usual norm for $L^p(\mathcal{D}) = L^p(\mathcal{D}, \mathbb{R})$, $W^{m,p}(\mathcal{D}) = W^{m,p}(\mathcal{D}, \mathbb{R})$ and $W_0^{m,p}(\mathcal{D}) = W_0^{m,p}(\mathcal{D}, \mathbb{R})$ respectively.

2. Preliminaries

A multimap $G : \mathbb{R} \multimap \mathbb{R}$ is said to be:

(i) *upper semicontinuous (u.s.c.)*, if $G^{-1}(V) = \{x \in X : G(x) \subset V\}$ is an open subset of $\mathbb{R}$ for every open $V \subseteq \mathbb{R}$;

(ii) *lower semicontinuous (l.s.c.)*, if $G^{-1}(Q) = \{x \in X : G(x) \subset Q\}$ is a closed subset of $\mathbb{R}$ for every closed set $Q \subset \mathbb{R}$

(ii) *closed*, if its graph $G_F = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y \in G(x)\}$ is a closed subset of $\mathbb{R} \times \mathbb{R}$;

(iv) *completely continuous*, if it is u.s.c. and maps bounded sets into compact ones.

For u.s.c. multimaps the following relations hold.

Theorem 2.1 (see [10], Theorem 1.1.4.)**.** *An u.s.c. multimap $G : \mathbb{R} \multimap \mathbb{R}$ with closed values is a closed multimap.*

Theorem 2.2 (see [10], Theorem 1.1.5.)**.** *A closed multimap $G : \mathbb{R} \multimap \mathbb{R}$ with compact values, such that maps bounded sets into compact ones is u.s.c.*

A map $g : \mathbb{R}^k \to \mathbb{R}$ is said to be a *Carathéodory map* if it is measurable with respect to the first variable and continuous with respect to the other $k-1$ variables.
Let E be a Banach space and E^* its dual space, an operator $A : E \to E^*$ is said *to satisfy the S_+ condition* if and only if the weak convergence of a sequence $\{u_n\} \subset E$ to $u \in E$ and the condition $\limsup\limits_{n\to\infty}\langle Au_n, u_n - u\rangle \leq 0$ imply the strong convergence of $\{u_n\}$ to u in E.
Let $K \subset E$ be a convex closed set, let $V \subset E$ be a bounded open set, such that $V_K = V \cap K \neq \emptyset$, and let $G : \overline{V} \multimap K$ be a completely continuous multimap with compact and convex values such that $x \notin G(x)$ for any $x \in \partial V_K$, where $\overline{V}_K$ and ∂V_K denote the relative closure and the relative boundary of the set V_K in K. In such a setting the relative topological degree

$$\deg_K(i - G, \overline{V}_K)$$

of the corresponding multivalued vector field (multifiled) $i - G$ is well defined and satisfies the standard properties (see e.g. [10]). In particular, it satisfies the homotopy invariance property and the existence condition, i.e.

$$\deg_K(i - G, \overline{V}_K) \neq 0$$

implies that the fixed point set $\operatorname{Fix} G = \{x : x \in G(x)\}$ is a nonempty compact subset of V_K (see [10]).

Definition 2.1. *Two completely continuous multimaps $G_0, G_1 : V_K \multimap K$ with compact and convex values and the corresponding multifields are said to be homotopic if there exists a completely continuous family $\mathcal{G} : [0,1] \times V_K \multimap K$ with compact and convex values such that*

$$Fix\mathcal{G}(\lambda, \cdot) \cap \partial V_K = \emptyset, \quad \forall\, \lambda \in [0,1],$$

and $\mathcal{G}(0,\cdot) = G_0,\ \mathcal{G}(1,\cdot) = G_1$.

Theorem 2.3 (see [10], Theorem 3.1.4.)**.** *Let* $G : V_K \multimap K$ *be a completely continuous multimap with compact and convex values such that* $G(x) \cap \overline{V}_K \neq \emptyset$*,* $\forall\, x \in \partial V_K$ *and* $\operatorname{Fix} G \cap \partial V_K = \emptyset$*, then* $\deg_K(i - G, \partial V_K) = 1$.

3. Elliptic variational inequalities

We consider the following variational inequalities:

$$\begin{cases} \displaystyle\int_\Omega \sum_{|\alpha|\le m} A_\alpha(x,\eta(u(x)))(D^\alpha v(x) - D^\alpha u(x))\,dx \ge \\ \qquad \displaystyle\ge \int_\Omega (g(x,\eta(u(x))) + f(x,u(x)) + h)(v(x) - u(x))\,dx, \quad \forall\, v \in K \\ f(x,u(x)) \in F(x,u(x)) \ \text{ a.e. } x \in \Omega \end{cases} \tag{3.1}$$

$$\begin{cases} \displaystyle\int_\Omega \sum_{|\alpha|\le m} A_\alpha(x,\eta(u(x)))(D^\alpha v(x) - D^\alpha u(x))\,dx \ge \\ \qquad \displaystyle\ge \int_\Omega (g(x,\eta(u(x))) + f(x) + h)(v(x) - u(x))\,dx, \quad \forall\, v \in K \\ f(x) \in F(x,u(x)) \ \text{ a.e. } x \in \Omega \end{cases} \tag{3.2}$$

Where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, K a closed convex subset of $W_0^{m,p}(\Omega)$ $(1 < p < \infty)$ with $0 \in K$, α a multiindex, $\eta(u) = \{D^\alpha u : |\alpha| \le m\}$, the function A_α maps $\Omega \times \mathbb{R}^{N_m}$ into $\mathbb{R}$ (with $N_m = \frac{(N+m)!}{N!m!}$), $g : \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ and $F : \Omega \times \mathbb{R} \multimap \mathbb{R}$ are a given map and multimap respectively, finally $h \in (W_0^{m,p}(\Omega))^*$. Let q be such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, we assume the following hypotheses on the function $A_\alpha : \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$:

(A1) $x \to A_\alpha(x,\eta)$ is measurable in Ω for any $\eta \in \mathbb{R}^{N_m}$;
$\eta \to A_\alpha(x,\eta)$ is continuous for almost all (a.a.) $x \in \Omega$;
there exist a function $k_0 \in L^q(\Omega)$ and a constant ν such that

$$|A_\alpha(x,\eta)| \le k_0(x) + \nu(\|\eta\|^{p-1}), \text{ a.e. in } \Omega, \text{ for any } \eta \in \mathbb{R}^{N_m};$$

(A2) $\forall\, x \in \Omega$, and η, η', $\eta \neq \eta'$,

$$\sum_{|\alpha|\le m} (A_\alpha(x,\eta) - A_\alpha(x,\eta'))(\eta_\alpha - \eta'_\alpha) > 0;$$

(A3) there exist a function $k_1 \in L^1(\Omega)$ and a constant μ such that

$$\sum_{|\alpha| \leq m} A_\alpha(x, \eta)\eta_\alpha \geq \mu \|\eta\|^p - k_1(x), \text{ a.e. in } \Omega \text{ and } \forall\, \eta \in \mathbb{R}^{N_m}.$$

As a consequence the function A_α generates an operator A from $W_0^{m,p}(\Omega)$ into its dual $(W_0^{m,p}(\Omega))^*$ defined by

$$\langle Au, \varphi\rangle = \int_\Omega \sum_{|\alpha| \leq m} A_\alpha(x, \eta(u(x)))D^\alpha \varphi(x)\, dx.$$

A typical example that satisfies (A1)-(A3) is the p-Laplacian. Moreover, under previous hypotheses the operator $A : W_0^{m,p}(\Omega) \to (W_0^{m,p}(\Omega))^*$ is continuous, bounded, monotone, and satisfies the S_+ condition (see e.g. [5]).

Remark 3.1. *Observe that if the set K coincides with the whole space $W_0^{m,p}(\Omega)$, the solutions of the variational inequalities* (3.1) *and* (3.2) *are weak solutions of the following partial differential inclusion:*

$$\begin{cases} -h \in \displaystyle\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \eta(u(x))) + g(x, \eta(u(x))) + F(x, u(x)) \; in \; \Omega \\ u(x) = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad on \; \partial\Omega \end{cases} \tag{3.3}$$

Given $q \in W_0^{m,p}(\Omega)$ and $\lambda \in [0, 1]$, consider the linearized variational inequality:

$$\langle A(u), v - u\rangle \geq \lambda \int_\Omega (g(x, \eta(q(x))) + f(x, q(x)) + h)(v(x) - u(x))\, dx, \;\; \forall\, v \in K \tag{3.4}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory selection of the multimap F (i.e. $f(x, q(x)) \in F(x, q(x))$ a.e. in Ω).

Theorem 3.1. *Let $A_\alpha : \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ satisfy hypotheses (A1)-(A3) and $g : \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ be a Carathéodory map such that*

$$|g(x, \eta)| \leq k_2(x) + c_1(\|\eta\|^\sigma) \;\; a.e. \;\; in\; \Omega, \forall\, \eta \in \mathbb{R}^{N_m}$$

with $k_2 \in L^q(\Omega)$, $c_1 > 0$ and $1 \leq \sigma < p - 1$.
Let $F : \Omega \times \mathbb{R} \multimap \mathbb{R}$ be a measurable multimap with closed convex values such that

(i) $F(x, \cdot) : \mathbb{R} \multimap \mathbb{R}$ is l.s.c. $\forall\, x \in \Omega$

(ii) $\|F(x, u)\| \leq a(x) + b|u|^\sigma$ a.e. in Ω, $\forall\, u \in \mathbb{R}$ with $a \in L^q(\Omega)$, $b > 0$ and $1 \leq \sigma < p - 1$.

Then the problem (3.1) *has at least a solution.*

Proof. Hypotheses (i)-(ii) on the multimap F imply the existence of a Carathéodory selection and hence the variational inequality (3.4) is well defined. We can assume without loss of generality (w.l.o.g.) that $0 \in K$: if this is not the case, we can consider an element $u_0 \in K$ and solve the analogous problem:

$$\begin{cases} \langle \bar{A}(w), v' - w \rangle \geq \displaystyle\int_\Omega (\bar{g}(x, \eta(w(x))) + \bar{f}(x, w(x)) + h)(v'(x) - w(x))\, dx, \ \forall\, v' \in K_1 \\ \bar{f}(x, w(x)) \in \bar{F}(x, w(x)) \ \text{ a.e. } x \in \Omega \end{cases}$$

where $K_1 = K - u_0$, $w = u - u_0$, $\bar{A} : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$, $\bar{g} : \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ and $\bar{F} : \Omega \times \mathbb{R} \multimap \mathbb{R}$ are defined by $\bar{A}(w) = A(w + u_0)$, $\bar{g}(x, \eta(w(x))) = g(x, \eta(w(x) + u_0(x)))$, $\bar{F}(x, w(x)) = F(x, w(x) + u_0(x))$, respectively.
We split the proof in several steps and for sake of simplicity we assume $m = 1$. Let U_f the solution set of (3.4). Denote with T the multioperator

$$\begin{array}{rccl} T: & Q \times [0,1] & \multimap & K \\ & (q, \lambda) & \mapsto & \{U_f, f(x, q(x)) \in F(x, q(x))\} \end{array}$$

where $Q \subset W_0^{1,p}(\Omega)$ is a suitable closed convex set.

Step 1. The multioperator T has nonempty closed convex values.
Indeed, consider the functional $G : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined as:

$$G(u) = \lambda \int_\Omega (g(x, q(x), Dq(x)) + f(x, q(x)) + h) u(x)\, dx$$

We have:

$$\begin{aligned} |G(u)| &= \left| \lambda \int_\Omega (g(x, q(x), Dq(x)) + f(x, q(x)) + h) u(x)\, dx \right| \leq \\ &\leq \int_\Omega (|g(x, q(x), Dq(x))| + |f(x, q(x))| + |h|) |u(x)|\, dx \leq \\ &\leq \int_\Omega (k_2(x) + c_1(|q(x)|^{p-1} + \|Dq(x)\|^{p-1}) + a(x) + b|q(x)|^{p-1} + |h|)|u(x)|\, dx \leq \\ &\leq (\|k_2\|_q + \|a\|_q + \|h\|_q)\|u\|_p + |\Omega|^{1 - \frac{\sigma+1}{p}} (c_1(\|q\|_p^\sigma + \|Dq\|_p^\sigma) + b\|q\|_p^\sigma)\|u\|_p \leq \\ &\leq C\|u\|_0 \end{aligned}$$

hence G is a linear and continuous operator, i.e. $G \in (W_0^{1,p}(\Omega))^*$. We notice that for $\lambda = 0$, $G \equiv 0$. Let $\chi(u)$ be the indicator function of K

$$\chi(u) = \begin{cases} 0 & u \in K \\ +\infty & u \in W_0^{1,p}(\Omega) \setminus K \end{cases}$$

The problem (3.4) can be rewritten in the following equivalent form (see [17])

$$G \in \partial\chi(u) + A(u), u \in K$$

where

$$\partial\chi(u) = \begin{cases} u^* \in (W_0^{1,p}(\Omega))^* \ : \ \langle u^*, u - v\rangle \geq 0 \ \ \forall\, v \in K & u \in K \\ \emptyset & u \in W_0^{1,p}(\Omega) \setminus K \end{cases}$$

The mapping $\partial\chi : W_0^{1,p}(\Omega) \multimap (W_0^{1,p}(\Omega))^*$ is maximal monotone, then, for the regularity properties of the operator A it follows that for any $b \in (W_0^{1,p}(\Omega))^*$ the inclusion

$$b \in \partial\chi(u) + A(u)$$

has at least a solution $u \in K$ (see [4]). In particular there exist solutions when $b = G$. Moreover, from the monotonicity and the continuity of the operator A we have that (3.4) is equivalent to the problem

$$\begin{aligned} &\langle A(v), v - u\rangle \geq \\ &\quad \geq \lambda \int_\Omega (g(x, q(x), Dq(x)) + f(x, q(x)) + h)(v(x) - u(x))\, dx \quad \forall\, v \in K. \end{aligned} \tag{3.5}$$

and, since F has convex values, the multioperator T has closed and convex values.

Step 2. The multioperator T is a closed operator.
Let $q_n \to q_0$ in $W_0^{1,p}(\Omega)$, $\lambda_n \to \lambda_0$, $u_n \to u_0$ in $W_0^{1,p}(\Omega)$ where $u_n \in T(q_n, \lambda_n)$, then, $\forall\, v \in K$,

$$\begin{aligned} &\langle A(u_n), v - u_n\rangle \geq \\ &\quad \geq \lambda_n \int_\Omega \big(g(x, q_n(x), Dq_n(x)) + f(x, q_n(x)) + h\big)(v(x) - u_n(x))\, dx \end{aligned} \tag{3.6}$$

From the convergence of q_n in $W^{1,p}(\Omega)$, we can extract a subsequence $\{q_{n_k}\} \subset \{q_n\}$ such that:

$$\lim_{k\to\infty} \lambda_{n_k} \int_\Omega (g(x, q_{n_k}(x), Dq_{n_k}(x)) + f(x, q_{n_k}(x)) + h)(v(x) - u_{n_k}(x))\, dx =$$

$$= \liminf_{n\to\infty} \lambda_n \int_\Omega (g(x, q_n(x), Dq_n(x)) + f(x, q_n(x)) + h)(v(x) - u_n(x))\, dx$$

and $q_{n_k} \to q_0$, $Dq_{n_k} \to Dq_0$ a.e. in Ω. From the continuity of g with respect to the second and the third argument, the continuity of f with respect to the second argument, the Lebesgue convergence Theorem and Hölder inequality we have

$$\lim_{k\to\infty} \lambda_{n_k} \int_\Omega (g(x, q_{n_k}(x), Dq_{n_k}(x)) + f(x, q_{n_k}(x)) + h)(v(x) - u_{n_k}(x))\, dx =$$

$$= \lambda_0 \int_\Omega (g(x, q_0(x), Dq_0(x)) + f(x, q_0(x)) + h)(v(x) - u_0(x))\, dx.$$

Moreover K is closed, hence $u_0 \in K$ and from the continuity of A we have,

$$\lim_{n\to\infty} \langle A(u_n), v - u_n\rangle = \langle A(u_0), v - u_0\rangle.$$

Then

$$\begin{aligned}\langle A(u_0),&v - u_0\rangle = \lim_{n\to\infty} \langle A(u_n), v - u_n\rangle \geq \\ &\geq \liminf_{n\to\infty} \lambda_n \int_\Omega (g(x, q_n(x), Dq_n(x)) + f(x, q_n(x)) + h)(v(x) - u_n(x))\, dx = \\ &=\lambda_0 \int_\Omega (g(x, q_0(x), Dq_0(x)) + f(x, q_0(x)) + h)(v(x) - u_0(x))\, dx\end{aligned}$$

then $u_0 \in T(\lambda_0, q_0)$ and T is closed.

STEP 3. The multioperator T is a compact operator with compact and convex values.
To prove this, let $q_n \in W_0^{1,p}(\Omega)$ be such that $\|q_n\|_0 < N$, $\forall\, n$, with N a positive constant, and let $u_n \in T(\lambda_n, q_n)$. Since, by hypothesis, $0 \in K$ we may consider (3.4) with $v \equiv 0$, obtaining

$$\begin{aligned}\mu\|u_n\|_0^p - k_1(x) \leq \langle A(u_n), u_n\rangle &\leq \\ &\leq \lambda_n \int_\Omega (g(x, q_n(x), Dq_n(x)) + f(x, q_n(x)) + h)u_n(x)\, dx \leq \\ &\leq \lambda_n \int_\Omega (|g(x, q_n(x), Dq_n(x))| + |f(x, q_n(x))| + |h|)|u_n(x)|\, dx \leq \\ &\leq \big(\|k_2\|_q + \|a\|_q + c_1|\Omega|^{1-\frac{\sigma+1}{p}}\|Dq_n\|_p^\sigma\big)\|u_n\|_p + \\ &+ \big((c_1 + b)|\Omega|^{1-\frac{\sigma+1}{p}}\|q_n\|_p^\sigma + \|h\|_q\big)\|u_n\|_p \leq \\ &\leq C(\|u_n\|_0)\end{aligned}$$

Since $p > 1$, by the Young inequality u_n is uniformly bounded, i.e., there exists a subsequence, that weakly converges in $W_0^{1,p}(\Omega)$ to $u_0 \in W_0^{1,p}(\Omega)$. Moreover from the convexity and the closure of K we have $u_0 \in K$. It follows

$$0 \leq \lim_{n\to\infty} \left| \int_\Omega (g(x, q_n(x), Dq_n(x)) + f(x, q_n(x)) + h)(u_0(x) - u_n(x))\, dx \right| \leq$$

$$\leq \lim_{n\to\infty} \Big(\|k_2\|_q + \|a\|_q + c_1 |\Omega|^{1-\frac{\sigma+1}{p}} \|Dq_n\|_p^\sigma +$$

$$+ (c_1 + b)|\Omega|^{1-\frac{\sigma+1}{p}} \|q_n\|_p^\sigma + \|h\|_q \Big) \|u_0 - u_n\|_p = 0$$

Substituting $v = u_0$ in (3.1), we have

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u_0 \rangle \leq$$

$$\leq \lim_{n\to\infty} \lambda_n \int_\Omega (g(x, q_n(x), Dq_n(x)) + f(x, q_n(x)) + h)(u_n(x) - u_0(x)) = 0$$

Since A satisfies the S^+ condition, $u_n \to u_0$ in $W^{1,p}(\Omega)$, that is T is a compact operator. Finally by Step 2, T has closed values, hence has compact values.

Step 4. The fixed point set $\{u \in T(u, \lambda), \lambda \in [0, 1]\}$ is a-priori bounded. In fact, let $u \in T(u, \lambda)$, as before we have:

$$\mu \|u\|_0^p - k_1(x) \leq$$

$$\leq \left(\|k_2\|_q + \|a\|_q + c_1 |\Omega|^{1-\frac{\sigma+1}{p}} \|Du\|_p^\sigma + (c_1 + b)|\Omega|^{1-\frac{\sigma+1}{p}} \|u\|_p^\sigma + \|h\|_q \right) \|u\|_p \leq$$

$$\leq C(\|u\|_0^{\sigma+1} + \|u\|_0)$$

Again, since $\sigma < p-1$ and $p > 1$, by the Young inequality, u is uniformly bounded as u varies among all the solutions of the original variational inequality. Therefore, there exists a constant $M > 0$ such that $\|u\|_0 < M$ for any $u \in T(u, \lambda)$ and for any $\lambda \in [0, 1]$.

Let $Q = B_M(0) \cap K$, by Theorem 2.3, we have $\deg_K(i - T_0, Q) \neq 0$, then for the homotopy invariance property $\deg_K(i - T_1, Q) \neq 0$, hence there exists a fixed point $u \in T(u, 1)$, i.e. a solution of (3.1). ■

Now, to solve (3.2), given $q \in W_0^{m,p}(\Omega)$ and $\lambda \in [0, 1]$, consider the linearized variational inequality:

$$\langle A(u), v - u \rangle \geq \lambda \int_\Omega (g(x, \eta(q(x))) + f(x) + h)(v(x) - u(x))\, dx, \ \forall\, v \in K \quad (3.7)$$

where $f : \Omega \to \mathbb{R}$ is a measurable selection of the multimap F (i.e. $f(x) \in F(x, q(x))$ a.e. in Ω).

Theorem 3.2. *Let $A_\alpha : \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ satisfy hypotheses (A1)-(A3) and $g : \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ be a Carathéodory map such that*

$$|g(x,\eta)| \leq k_2(x) + c_1(\|\eta\|^\sigma) \ a.e. \ in\, \Omega, \, \forall\, \eta \in \mathbb{R}^{N_m}$$

with $k_2 \in L^q(\Omega)$, $c_1 > 0$ and $1 \leq \sigma < p-1$.
Let $F : \Omega \times \mathbb{R} \multimap \mathbb{R}$ be a multimap with closed and convex values such that

(i) $F(\cdot, u)$ is measurable for all $u \in \mathbb{R}$;

(ii) $F(x, \cdot) : \mathbb{R} \multimap \mathbb{R}$ is u.s.c., for all $x \in \Omega$;

(iii) $\|F(x,u)\| \leq a(x) + b|u|^\sigma$ a.e. in Ω, $\forall\, u \in \mathbb{R}$ with $a \in L^q$, $b > 0$ and $1 \leq \sigma < p-1$.

Then the problem (3.2) *has at least a solution.*

Proof. Under hypotheses (i)-(ii) the multimap F admits a measurable selection $f : \Omega \to \mathbb{R}$. So (3.7) is well defined. As before we assume $m = 1$ and it is possible to prove the existence of at least a solution of (3.7). The proof scheme is similar to Theorem 3.1 but we need to prove the closeness of the multimap T in a different way. Denoting with U_f the solution set of (3.7), we introduce the solution multioperator

$$\begin{array}{llll} T: & Q \subset W_0^{1,p}(\Omega) \times [0,1] & \multimap & W_0^{1,p}(\Omega) \\ & (q, \lambda) & \to & \{U_f, f(x) \in F(x, q(x))\} \end{array}$$

where Q is a suitable closed convex set. Let $q_n \to q_0$ in $W_0^{1,p}(\Omega)$, $\lambda_n \to \lambda_0$, $u_n \to u_0$ in $W_0^{1,p}(\Omega)$ where $u_n \in T(q_n, \lambda_n)$, we want to prove that $u_0 \in T(q_0, \lambda_0)$. From $u_n \in T(q_n, \lambda_n)$ we have

$$\begin{aligned} &\langle A(u_n), v - u_n \rangle \geq \\ &\quad \geq \lambda_n \int_\Omega (g(x, q_n(x), Dq_n(x)) + f_n(x) + h)(v(x) - u_n(x))\, dx \ \forall\, v \in K, \end{aligned} \tag{3.8}$$

where $f_n(x) \in F(x, q_n(x))$ for a.a. $x \in \Omega$. Since the sequence $\{q_n\}$ converges in $W^{1,p}(\Omega)$, there exists a subsequence $\{q_{n_k}\}$ in $L^p(\Omega)$ converging to q_0 a.e. in Ω. From the Egoroff's Theorem the sequence q_{n_k} converges almost uniformly to q_0, i.e. there exists a zero-measure set O such that $q_{n_k}(x)$ converges uniformly to $q_0(x)$ for all $x \in \Omega \setminus O$. Moreover from the hypothesis (iii) on F, $|f_{n_k}(x)| \leq \|F(x, q_{n_k}(x))\| \leq a(x) + b|q_{n_k}|^\sigma$. Hence there exists a constant $L > 0$ such that

$\|f_{n_k}(x)\|_q \leq L$, and there exists a subsequence $\{f_{n_k}\}$, denoted as the sequence, that weakly converges in $L^q(\Omega)$ to a function f_0. From Mazur's lemma a convex combination $\{\tilde{f}_{n_k}\}$ of $\{f_{n_k}\}$, converges to f_0 with respect to the norm of $L^1(\Omega)$. Passing to a subsequence we can assume that $\{\tilde{f}_{n_k}\}$ converges a.e. to f_0. We show that $f_0(x) \in F(x, q_0(x))$ for a.a. $x \in \Omega$. From the upper semicontinuity of the multimap F there exists an index k_0 such that

$$F(x, q_{n_k}(x)) \subset W_\varepsilon(F(x, q_0(x)))$$

$\forall\, x \in \Omega \setminus O$ and $k \geq k_0$. Then

$$f_{n_k}(x) \in W_\varepsilon(F(x, q_0(x)))$$

for a.a. $x \in \Omega$. From the convexity of the values of F

$$\tilde{f}_{n_k}(x) \in W_\varepsilon(F(x, q_0(x))) \;\; k \geq k_0$$

for a.a. $x \in \Omega$. It follows

$$f_0(x) \in F(x, q_0(x))$$

for a.a. $x \in \Omega$. Since $W_0^{1,p}(\Omega)$ is closed we have $u_0 \in W_0^{1,p}(\Omega)$; moreover, by the continuity of the operator A,

$$\lim_{n\to\infty} \langle A(u_n), v - u_n \rangle = \langle A(u_0), v - u_0 \rangle$$

and hence

$$\begin{aligned}\langle A(u_0), &v - u_0 \rangle = \lim_{n\to\infty} \langle A(u_n), v - u_n \rangle \geq \\ &\geq \liminf_{n\to\infty} \lambda_n \int_\Omega (g(x, q_n(x), Dq_n(x)) + f_n(x) + h)(v(x) - u_n(x))\, dx = \\ &= \lambda_0 \int_\Omega (g(x, q_0(x), Dq_0(x)) + f_0(x) + h)(v(x) - u_0(x))\, dx\end{aligned}$$

The last equality follows from the continuity of the functions q_n, the weak convergence up to subsequence of f_n and the strong convergence of the sequence u_n. Finally K is closed, $u_0 \in K$ and hence $u_0 \in T(\lambda_0, q_0)$. ■

Remark 3.2. *Problems as* (3.3) *appear in many applications, e.g. the obstacle problem. In this case, given* $\psi \in L^p(\Omega)$, *the set* K *is*

$$K = \{u \in W_0^{1,p}(\Omega) \; : \; u(x) \geq \psi(x), \textit{ for a.e.}\, x \in \Omega\}.$$

4. Evolution variational inequalities

We consider now the parabolic case. To this aim let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, $J = [0, d]$, and K a closed convex subset of $X_0 = L^p(J, W_0^{m,p}(\Omega))$ $(2 \leq p < \infty)$ we look for functions $u \in Y_0 \cap K$ $u(0, \cdot) = \bar{u}(\cdot)$, $Y_0 = \{u \in X_0,\ u_t \in X_0^*\}$, solutions of the following variational inequalities:

$$\begin{cases} \displaystyle\int_J\int_\Omega \frac{\partial u}{\partial t}(v(t,x) - u(t,x))\,dx\,dt + \\ \displaystyle+\int_J\int_\Omega \sum_{|\alpha|\leq m} A_\alpha(t,x,\eta(u(t,x)))(D^\alpha v(t,x) - D^\alpha u(t,x))\,dx\,dt \geq \\ \displaystyle\geq \int_J\int_\Omega (g(t,x,\eta(u(t,x))) + f(t,x,u(t,x)) + h)(v(t,x) - u(t,x))\,dx\,dt, \\ \hfill \forall\, v \in K \\ f(t,x,u(t,x)) \in F(t,x,u(t,x)) \ \text{ a.e. in } \ J \times \Omega \end{cases} \tag{4.1}$$

$$\begin{cases} \displaystyle\int_J\int_\Omega \frac{\partial u}{\partial t}(v(t,x) - u(t,x))\,dx\,dt + \\ \displaystyle\quad+\int_J\int_\Omega \sum_{|\alpha|\leq m} A_\alpha(t,x,\eta(u(t,x)))(D^\alpha v(t,x) - D^\alpha u(t,x))\,dx\,dt \geq \\ \displaystyle\quad\geq \int_J\int_\Omega (g(t,x,\eta(u(t,x))) + f(t,x) + h)(v(t,x) - u(t,x))\,dx\,dt, \\ \hfill \forall\, v \in K \\ f(t,x) \in F(t,x,u(t,x)) \ \text{ a.e. in } \ J \times \Omega \end{cases} \tag{4.2}$$

where, as before, α is a multiindex, $\eta(u) = \{D^\alpha u \ : \ |\alpha| \leq m\}$, the function A_α maps $J \times \Omega \times \mathbb{R}^{N_m}$ into $\mathbb{R}$ (with $N_m = \frac{(N+m)!}{N!m!}$), and where $g : J \times \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ and $F : J \times \Omega \times \mathbb{R} \multimap \mathbb{R}$ are given map and multimap respectively, finally $h \in X_0^*$. Let q be such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, we assume the following hypotheses on the function $A_\alpha : J \times \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$:

(A4) $(t,x) \to A_\alpha(t,x,\eta)$ is measurable in $J \times \Omega$ for any $\eta \in \mathbb{R}^{N_m}$;
$\eta \to A_\alpha(t,x,\eta)$ is continuous for almost all $x \in \Omega$
there exist a function $k_0 \in L^q(J \times \Omega)$ and a constant ν such that

$$|A_\alpha(t,x,\eta)| \leq k_0(t,x) + \nu(\|\eta\|^{p-1}), \text{ a.e. in } J \times \Omega, \text{ for any } \eta \in \mathbb{R}^{N_m};$$

(A5) $\forall\,(t,x) \in J \times \Omega$, and η, η', $\eta \neq \eta'$,

$$\sum_{|\alpha|\leq m} (A_\alpha(t,x,\eta) - A_\alpha(t,x,\eta'))(\eta_\alpha - \eta'_\alpha) > 0;$$

(A6) there exist a function $k_1 \in L^1(J \times \Omega)$ and a constant $\mu > 0$ such that

$$\sum_{|\alpha|\leq m} A_\alpha(t,x,\eta)\eta_\alpha \geq \mu\|\eta\|^p - k_1(t,x), \text{ a.e. in } J \times \Omega \text{ and } \forall\,\eta \in \mathbb{R}^{N_m}.$$

Remark 4.1. *Observe that if the set K coincides with the whole space X_0, the solutions of the variational inequalities* (4.1) *and* (4.2) *are weak solutions of the following parabolic partial differential inclusion:*

$$\begin{cases} -h \in \dfrac{\partial u}{\partial t} + \displaystyle\sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \eta(u(t,x))) + \\ \qquad + g(t,x,\eta(u(t,x))) + F(t,x,u(t,x)) \qquad\qquad \text{in } J \times \Omega \\ u(t,x) = 0 \qquad \text{on } \ J \times \partial\Omega \\ u(0,x) = \overline{u}(x) \quad \text{a.e. in } \ \Omega \end{cases} \tag{4.3}$$

Let A from $J \times W_0^{m,p}(\Omega)$ into the dual $(W_0^{m,p}(\Omega))^*$ be defined by

$$\langle A(t,u), \varphi\rangle = \int_\Omega \sum_{|\alpha|\leq m} A_\alpha(t,x,\eta(u(x)))D^\alpha\varphi(x)\,dx.$$

Defining $\widetilde{A} : Y_0 \to Y_0^*$ as $\widetilde{A}(u)(t) = A(t,u(t))$, by hypotheses (A4)-(A6), we have that the operator $\widetilde{A}$ is continuous, bounded and satisfies the S_+ condition (see [8]). Moreover, we define the operator $L : Y_0 \subseteq X_0 \to X_0^*$ as $L(u) = u_t$. It is known that the operator $L : Y_0 \subseteq X_0 \to X_0^*$ is a closed maximal monotone operator (see [8]). Given $q \in X_0$ and $\lambda \in [0,1]$, we can consider the linearized variational inequality:

$$\begin{aligned} &\langle L(u) + \widetilde{A}(u), v - u\rangle \geq \\ &\geq \lambda \int_J \int_\Omega (g(t,x,\eta(q(t,x))) + f(t,x,q(t,x)) + h)(v(t,x) - u(t,x))\,dx\,dt, \end{aligned} \tag{4.4}$$

for all $v \in K$, where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory selection of the multimap F (i.e. $f(t,x,q(t,x)) \in F(t,x,q(t,x))$ a.e. in Ω).
We have three possible cases for the set K. It has non empty interior, denoted

by $\operatorname{int}(K)$; eather it has non empty relatively interior, in this case we solve the problem on X_0', the smallest subspace of X_0 containing K or it is reduced to a single point $K = \{0\}$. We solve the problem in the case $0 \in \operatorname{int}(K)$ and the other two cases follow easily.

Theorem 4.1. *Let $Y_0 \cap \operatorname{int}(K) \neq \emptyset$, $A_\alpha : J \times \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ satisfy hypotheses (A4)-(A6), and $g : J \times \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ be a Carathéodory map such that*

$$|g(t,x,\eta)| \leq k_2(t,x) + c_1(\|\eta\|^\sigma) \text{ a.e. in } J \times \Omega,\ \forall\, \eta \in \mathbb{R}^{N_m}$$

with $k_2 \in L^q(J \times \Omega)$, $c_1 > 0$ and $1 \leq \sigma < p-1$.
Let $F : J \times \Omega \times \mathbb{R} \multimap \mathbb{R}$ be a measurable multimap with closed convex values such that

(i) $F(t,x,\cdot) : \mathbb{R} \multimap \mathbb{R}$ *is l.s.c.* $\forall\,(t,x) \in J \times \Omega$

(ii) $\|F(t,x,u)\| \leq a(t,x) + b|u|^\sigma$ *a.e. in* $J \times \Omega$, $\forall\, u \in \mathbb{R}$ *with* $a \in L^q(J \times \Omega)$, $b > 0$ *and* $1 \leq \sigma < p-1$.

Then the problem (4.1) *has at least a solution.*

Proof. Hypotheses (i)-(ii) on the multimap F imply the existence of a Carathéodory selection and hence the variational inequality (3.4) is well defined. As before, for sake of simplicity we assume $m = 1$ and, denoting with U_f the solution set of (4.4), we define the multioperator T as

$$\begin{array}{rccl} T: & Q \times [0,1] & \multimap & K \\ & (q,\lambda) & \mapsto & \{U_f, f(t,x,q(x)) \in F(t,x,q(x))\} \end{array}$$

where $Q \subset X_0$ is a suitable closed convex set.
As for the elliptic variational inequalities we can assume w.l.o.g. that $0 \in K$.
The multioperator T is a compact multioperator with nonempty closed convex values.
Indeed, as before we consider the linear and continuous functional $G : X_0 \to \mathbb{R}$ defined as:

$$G(u) = \lambda \int_J \int_\Omega (g(t,x,q(t,x),Dq(t,x)) + f(t,x,q(t,x)) + h)u(t,x)\,dx\,dt$$

So, denoting with $\chi(u)$ the indicator function of K, problem (4.4) can be rewritten in the following equivalent form (see [17])

$$G \in \partial\chi + L(u) + \widetilde{A}(u), u \in K$$

The operator $\partial\chi(u) + L(u)$ is a maximal monotone operator as sum of two maximal monotone operators, then, for the regularity properties of the operator $\widetilde{A}$ it follows that for any $b \in X_0^*$ the inclusion

$$b \in \partial\chi(u) + L(u) + \widetilde{A}(u)$$

has at least a solution $u \in K$ (see [4]). In particular there exist solutions when $b = G$. Moreover, from the monotonicity and continuity of the operator $\widetilde{A}$ and from the monotonicity and linearity of the operator L we have that (4.4) is equivalent to the problem

$$\begin{aligned}\langle L(v) + \widetilde{A}(v), v - u\rangle \geq \\ \geq \lambda \int_J \int_\Omega (g(t,x,q(t,x),Dq(t,x)) + f(t,x,q(t,x)) + h)(v(t,x) - u(t,x))\,dx\,dt\end{aligned} \tag{4.5}$$

for all $v \in K$. Therefore, recalling that F has convex values, the multioperator T has closed and convex values.
T is a closed operator.
To this aim let $\{q_n\} \subset Q$, $q_n \to q_0$ in X_0, $\lambda_n \to \lambda_0$ in $[0,1]$ and $u_n \to u_0$ in X_0 with $u_n \in T(q_n, \lambda_n)$, we claim that $u_0 \in T(q_0, \lambda_0)$.
We find an estimate for $\|Lu_n\|$. Notice that since K is closed convex and $0 \in \operatorname{int}(K)$ we have that for any $v \in X_0$ there exists $\gamma \in \mathbb{R}$ and a $v_k \in K$ such that $v = \gamma v_k$. So

$$\|Lu_n\| = \sup_{v \in X_0; \|v\|_{X_0} \leq 1} |\langle Lu_n, v\rangle| = \sup_{\|\gamma v_k\| \leq 1} |\langle Lu_n, \gamma v_k\rangle|$$

Since $v_k \in K$ we obtain

$$\begin{aligned}|\langle Lu_n, \gamma v_k\rangle| = |\gamma|\,|\langle Lu_n, v_k\rangle| = |\gamma|\,|\langle Lu_n, -v_k\rangle| \leq \\ \leq |\gamma|\,|\langle Lu_n, u_n - v_k\rangle| + |\gamma|\,|\langle Lu_n, u_n\rangle|\,.\end{aligned}$$

Now

$$\begin{aligned}\Big|\langle Lu_n, u_n - v_k\rangle\Big| \leq \Big|\lambda_n \int_J \int_\Omega (g(t,x,q_n(t,x),Dq_n(t,x)) + \\ + f(t,x,q_n(t,x)) + h)(u_n(t,x) - v_k(t,x))\,dx\,dt\Big| + \Big|\langle \widetilde{A}u_n, u_n - v_k\rangle\Big|\end{aligned}$$

From growth conditions on maps g and F and on the operator A we have:

$$\begin{aligned}\Big|\langle Lu_n, u_n - v_k\rangle\Big| &\le (\|a\|_q + \|k_2\|_q + \|k_0\|_q + \|h\|_{X_0^*})(\|u_n\|_{X_0} + \|v_k\|_{X_0}) + \\ &+ c_1(d|\Omega|)^{(1-\frac{\sigma+1}{p})}\|Dq_n\|_{X_0}^{\sigma}(\|u_n\|_{X_0} + \|v_k\|_{X_0}) + \\ &+ (c_1 + b)(d|\Omega|)^{(1-\frac{\sigma+1}{p})}\|q_n\|_{X_0}^{\sigma}(\|u_n\|_{X_0} + \|v_k\|_{X_0}) + \\ &+ \nu\left(\|Du_n\|_{X_0}^{p-1} + \|u_n\|_{X_0}^{p-1}\right)(\|u_n\|_{X_0} + \|v_k\|_{X_0})\end{aligned}$$

Since, from the convergence of the sequences $\{q_n\}$ and $\{u_n\}$ we have the existence of two constants $M_1 > 0$ and $M_2 > 0$ such that $\|q_n\|_{X_0} \le M_1$ and $\|u_n\|_{X_0} \le M_2$, we obtain the existence of a constant N_1 such that

$$|\langle Lu_n, u_n - v_k\rangle| \le N_1$$

Choosing $v \equiv 0$ in (4.4), as before we have the existence of a constant N_2 such that

$$|\langle Lu_n, u_n\rangle| \le N_2$$

that is the norm of Lu_n is uniformly bounded. Then, up to subsequence, there exists $v_0 \in X_0^*$ such that $Lu_n \rightharpoonup v_0$. By the definition of the operator L we have that $v_0 = Lu_0$, i.e., $u_n \rightharpoonup u_0$ in Y_0 up to subsequence. From the compact embedding $Y_0 \subset L^p(J, L^p(\Omega))$ and the continuous embedding $Y_0 \subset C(J, L^p(\Omega))$ follows $u_n(0) \to u_0(0)$ and $u_n(d) \to u_0(d)$ and

$$\begin{aligned}&\lim_{n\to\infty}\langle Lu_n, u_n - u_0\rangle = \\ &\quad = \lim_{n\to\infty}\left(\frac{1}{2}\|u_n(d) - u_0(d)\|_2^2 - \frac{1}{2}\|u_n(d) - u_0(d)\|_2^2 - \langle Lu_0, u_n - u_0\rangle\right) = 0.\end{aligned}$$

Hence, recalling the convergence $u_n \to u_0$ in X_0 and that K is closed ($u_0 \in K$) we have

$$\begin{aligned}&\limsup_{n\to\infty}\ \left(\langle \widetilde{A}(u_n), u_n - u_0\rangle + \langle Lu_n, u_n - u_0\rangle\right) \le \\ &\quad \le \limsup_{n\to\infty}\lambda_n \int_J\int_\Omega (g(t,x,q(t,x),Dq(t,x)) + f(t,x,q(t,x)) + h)\cdot \qquad (4.6) \\ &\qquad\qquad \cdot (u_n(t,x) - u_0(t,x))\,dx\,dt = 0\end{aligned}$$

The operator $\widetilde{A}$ satisfies the S_+ condition; from the previous inequality we have that $u_n \to u_0$ in Y_0, in particular $Lu_n \to Lu_0$ in X_0^*. Finally

$$\langle Lu_0, v-u_0\rangle + \langle \widetilde{A}(u_0), v-u_0\rangle = \lim_{k\to\infty} \langle Lu_{n_k}, v-u_{n_k}\rangle + \langle \widetilde{A}(u_{n_k}), v-u_{n_k}\rangle \geq$$

$$= \int_J\int_\Omega (g(t,x,q_0(t,x), Dq_0(t,x)) + f(t,x,q_0(t,x)) + h)(v(t,x)-u_0(t,x))\,dx\,dt$$

where u_{n_k} and q_{n_k} are the sequences that verify the inferior limit. Then $u_0 \in T(q_0,\lambda_0)$.
To prove the compactness, let $q_n \in X_0$ be such that $\|q_n\|_0 < N$, $\forall\, n$, with N a positive constant, and let $u_n \in T(\lambda_n, q_n)$. Observe that

$$\langle L(u), u\rangle = \frac{1}{2}\|u(d)\|_2^2 - \frac{1}{2}\|u(0)\|_2^2.$$

Moreover, from (A6) we have that

$$\begin{aligned}\langle \widetilde{A}(u), u\rangle &\geq \int_J\int_\Omega (\mu\|\eta(u(t,x))\|^p - k_1(t,x))\,dx\,dt = \\ &= \int_J \mu\|u(t)\|_0^p\,dt - \int_J\int_\Omega k_1(t,x)\,dx\,dt = \mu\|u\|_{X_0}^p - \widetilde{k}_1.\end{aligned}$$

By hypothesis $0 \in K$; we may consider (4.1) with $v \equiv 0$, obtaining

$$\begin{aligned}&\mu\|u_n\|_{X_0}^p - \widetilde{k}_1 - \frac{1}{2}\|u_n(0)\|_2^2 \leq \\ &\quad\leq \mu\|u_n\|_{X_0}^p - \widetilde{k}_1 - \frac{1}{2}\|u_n(0)\|_2^2 + \frac{1}{2}\|u_n(d)\|_2^2 \leq \langle L(u_n) + \widetilde{A}(u_n), u_n\rangle \leq \\ &\quad\leq \left|\lambda_n \int_J\int_\Omega (g(t,x,q_n(t,x), Dq_n(t,x)) + f(t,x,q_n(t,x)) + h)u_n(t,x)\,dx\,dt\right| + \\ &\quad+ \left|\langle \widetilde{A}u_n, u_n\rangle\right|.\end{aligned}$$

From the growth conditions on maps g and F and on the operator A, we obtain

$$\mu\|u_n\|_{X_0}^p - \widetilde{k}_1 - \frac{1}{2}\|u_n(0)\|_2^2 \leq C\|u_n\|_{X_0}.$$

Since $p \geq 2$, by the Young inequality u_n is uniformly bounded, i.e., there exists a subsequence, that weakly converges in X_0 to $u_0 \in X$. Moreover from the convexity and the closure of K we have $u_0 \in K$. As before it is possible to prove the uniform boundedness of $\|Lu_n\|$, hence to show that $u_n \rightharpoonup u_0$ in Y_0 up to subsequence. Since Y_0 is compactly embedded in $L^p(J, L^p(\Omega))$, then $u_n \to u_0$

in $L^p(J, L^p(\Omega))$. So inequality (4.6) holds and we have the strong convergence $u_n \to u_0$ in Y_0, i.e. the compactness of the operator T.
Again we have that the fixed point set $\{u \in T(u,\lambda), \lambda \in [0,1]\}$ is a-priori bounded. Therefore, there exists a constant $M > 0$ such that $\|u\|_{X_0} < M$ for any $u \in T(u,\lambda)$ and for any $\lambda \in [0,1]$, then by the properties of the relative topological degree as in Theorem 3.1, we obtain a solution of (4.1). ■
For parabolic variational inequalities the existence theorem for u.s.c. multimap F is still valid.

Theorem 4.2. *Let $Y_0 \cap \mathrm{int}(K) \neq \emptyset$, $A_\alpha : J \times \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ satisfy hypotheses (A4)-(A6), and $g : J \times \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ be a Carathéodory map such that*

$$|g(t,x,\eta)| \leq k_2(t,x) + c_1(\|\eta\|^\sigma) \text{ a.e. in } J \times \Omega,\ \forall\, \eta \in \mathbb{R}^{N_m}$$

with $k_2 \in L^q(J \times \Omega)$, $c_1 > 0$ and $1 \leq \sigma < p-1$.
Let $F : J \times \Omega \times \mathbb{R} \multimap \mathbb{R}$ be a multimap with closed and convex values such that

- *(i) $F(\cdot,\cdot,u)$ is measurable for all $u \in \mathbb{R}$;*
- *(ii) $F(t,x,\cdot) : \mathbb{R} \multimap \mathbb{R}$ is u.s.c., for all $(t,x) \in J \times \Omega$;*
- *(iii) $\|F(t,x,u)\| \leq a(t,x) + b|u|^\sigma$ a.e. in $J \times \Omega$, $\forall\, u \in \mathbb{R}$ with $a \in L^q(J \times \Omega)$, $b > 0$ and $1 \leq \sigma < p-1$.*

Then the problem (4.2) has at least a solution.

Proof. As before we consider the linearized problem

$$\begin{gathered}\langle L(u) + \widetilde{A}(u), v-u\rangle \geq \\ \geq \lambda \int_J \int_\Omega (g(t,x,\eta(q(t,x))) + f(t,x) + h)(v(t,x) - u(t,x))\, dx\, dt,\ \forall v \in K\end{gathered} \tag{4.7}$$

where $f : J \times \Omega \to \mathbb{R}$ is a measurable selection of the multimap F (i.e. $f(t,x) \in F(t,x,q(t,x))$ a.e. in $J \times \Omega$). Moreover, denoting with U_f the solution set of (4.7), we introduce the solution multioperator

$$\begin{array}{rlcl} T: & Q \subset X_0 \times [0,1] & \multimap & X_0 \\ & (q,\lambda) & \to & \{U_f, f(t,x) \in F(t,x,q(t,x))\} \end{array}$$

where Q is a suitable closed convex set. Given $q_n \to q_0$ and $u_n \to u_0$ in X_0, $\lambda_n \to \lambda_0$, with $u_n \in T(q_n,\lambda_n)$, as in Theorem 3.2 it is possible to find a sequence of selections $\{f_n\} \subset L^q(J, L^q(\Omega))$, $f_n(t,x) \in F(t,x,q_n(t,x))$ a.e. in $J \times \Omega$, such that $\{f_n\}$ weakly converges to f_0 with $f_0(t,x) \in F(t,x,q_0(t,x))$. Now, as in the proof of Theorem 4.1 we have

$$\lim_{n\to\infty} \langle L(u_n) + \widetilde{A}(u_n), v - u_n\rangle = \langle L(u_0) + \widetilde{A}(u_0), v - u_0\rangle.$$

Hence

$$\langle L(u_0) + \widetilde{A}(u_0), v - u_0\rangle = \lim_{n\to\infty} \langle L(u_n) + \widetilde{A}(u_n), v - u_n\rangle \geq$$

$$\geq \liminf_{n\to\infty} \lambda_n \int_J \int_\Omega (g(t,x,q_n(t,x), Dq_n(t,x)) + f_n(t,x) + h)\cdot$$

$$\cdot (v(t,x) - u_n(t,x))\, dx\, dt \ =$$

$$= \lambda_0 \int_J \int_\Omega (g(t,x,q_0(t,x), Dq_0(t,x)) + f_0(t,x) + h)(v(t,x) - u_0(t,x))\, dx\, dt$$

and we came to the conclusion. ■

5. Examples

Example 5.1. *We show an example of a minimization problem with respect to a set K that is not closed with respect to the maximum and the minimum, as is required in [5]. We consider the following problem:*

$$\min_{u\in K} \left\{ \int_C |Du|^2\, dx \quad u \in u_0 + W_0^{1,2}(C, \mathbb{R}) \right\} \tag{5.1}$$

with C the unit ball with center at zero, $u_0 = y^2 - 1$, and

$$K = \{f(\rho,\theta) = \rho\cos 2\theta + a\sin(2\pi\rho) + b\sin(4\pi\rho),\ a, b \in \mathbb{R}\}.$$

Finding minimizers of Problem (5.1) *is equivalent to solve the variational inequality:*

$$\langle Du, Dv - Du\rangle \geq 0 \quad \forall\, v \in K,$$

i.e. we obtain a variational inequality of type (3.1.) We can apply Theorem 3.1 obtaining the existence of a solution.

Example 5.2. *We consider a problem of heat dissipation in an isotropic homogeneous bounded body $B \subset \mathbb{R}^3$, which has to be maintained at a constant temperature u_0. The problem is expressed by the following system*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \Delta u - D(t,x) = \displaystyle\sum_{i=1}^{N} q_i(t,u) f_i(x) & t \in [0,T],\ x \in B \\ u(t,x) = u_0(t,x) & t \in [0,T],\ x \in \partial B \end{cases}$$

The function $u(t,x) \in W^{1,p}([0,T] \times B, \mathbb{R})$ *describes the change of temperature at point* x *and time* t *due to the dispersion* $D(t,x)$*. Heat is supplied by* N *sources* $f_i \in L^\infty(B_i, \mathbb{R}),\ i = 1,\dots,N$ *of bounded heating output* $q_i(t,u),\ i = 1,\dots,N$*, where* $B_i \subset B$*, in order to keep the body* B *at a constant temperature for any* $t > 0$*. The heating output is represented by* N *measurable functions* $q_i : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$*, continuous with respect to the second variable.*
Now given a constant $c \in \mathbb{R}$ *we consider all the possible amounts of heat subject to the constraint*

$$\sum_{i=1}^{N} q_i(t,u) = c, \tag{5.2}$$

for all $t \in [0,T]$ *and* $x \in B$*. Defining the multimap* $F : [0,T] \times \Omega \times \mathbb{R} \multimap \mathbb{R}$ *as*

$$F(t,x,u) = \left\{ \sum_{i=1}^{N} q_i(t,u) f_i(x), q_i \textit{ satisfying } (5.2) \right\}$$

we obtain an analogous problem as (4.3)*, i.e.*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \Delta u - D(t,x) \in F(t,x,u) & t \in [0,T],\ x \in B \\ u(t,x) = u_0(t,x) & t \in [0,T],\ x \in \partial B. \end{cases}$$

In this way we can obtain a solution that is optimal with respect to the controls $q_i(t,u)$ *satisfying* (5.2) *(see [3]).*

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