Identification of a time-dependent UV-photon source in an interstellar cloud

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Abstract

Consider an interstellar cloud that occupies the region $V \subset \mathbb{R}^3$, bounded by the known surface ∂V and assume that the scattering cross section σ_s and the total cross section σ are also known. Then, we prove that it is possible to identify the source $q = q(\mathbf{x}, t)$ that produces UV-photons inside the cloud, provided that the UV-photon distribution function arriving at a location $\widehat{\mathbf{x}}$, far from the cloud, is measured at times $\widehat{t}_0$, $\widehat{t}_1 = \widehat{t}_0 + \tau$, ..., $\widehat{t}_J = \widehat{t}_0 + J\tau$.

Keywords: photon transport, semigroups and linear evolution equations, inverse problems.

1 Introduction

In this paper, we shall consider the following *time dependent inverse problem* in photon transport.

Assume that the boundary surface ∂V of the region $V \subset \mathbb{R}^3$ occupied by an interstellar cloud [1], the scattering cross section σ_s and the total cross section σ are known. If the one-particle distribution function of UV-photons arriving at a location $\widehat{\mathbf{x}}$, "far" from the cloud, is measured at times $\widehat{t}_0$, $\widehat{t}_1 = \widehat{t}_0 + \tau$, ..., $\widehat{t}_J = \widehat{t}_0 + J\tau$, (by using some suitable instrument located within a satellite), then we show that it is possible to identify the *space* and *time* behaviour of the source that produces UV-photons inside the cloud.

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The knowledge of the UV-photon source characteristics is important because, together with the cross sections and the shape of ∂V, determines the form of the photon distribution function. In turn, interaction between UV-photons and the particles of the cloud (mainly hydrogen molecules and dust grains) play a crucial role in the chemistry of the cloud.

Note that the literature on time *independent* inverse problems in particle transport is rather abundant, seee the references listed in [2]. Only a few papers deal with *time dependent* inverse problem, see for instance [3, 4, 5, 6, 7]

2 The mathematical model

Let $N(\mathbf{x}, \mathbf{u}, t)$ be the one-particle distribution function of UV-photons which, at time t, are at $\mathbf{x}$ and have velocity $\mathbf{v} = c\,\mathbf{u}$ (where c is the speed of light). Then, the transport equation, the boundary condition and the initial condition have the form

$$\frac{\partial}{\partial t}N(\mathbf{x},\mathbf{u},t) = -\,c\,\mathbf{u}\cdot\nabla N(\mathbf{x},\mathbf{u},t) - c\sigma N(\mathbf{x},\mathbf{u},t) + \\ + \frac{c\sigma_s}{4\pi}\int_S N(\mathbf{x},\mathbf{u}',t)\,d\mathbf{u}' + q(\mathbf{x},t), \quad \mathbf{x}\in V_i,\ \mathbf{u}\in S,\ t>0 \tag{1a}$$

$$N(\mathbf{y},\mathbf{u},t) = 0 \qquad \text{if } \mathbf{y}\in\partial V \quad \text{and} \quad \mathbf{u}\cdot\boldsymbol{\nu}(\mathbf{y}) < 0 \tag{1b}$$

$$N(\mathbf{x},\mathbf{u},0) = N_0(\mathbf{x},\mathbf{u}), \qquad \mathbf{x}\in V, \quad \mathbf{u}\in S \tag{1c}$$

In (1), $V \in \mathbb{R}^3$ is the bounded and convex region occupied by the cloud, and V_i is the interior of V. Hence $V = V_i \cup \partial V$ where ∂V is the boundary surface, which is assumed to be closed and "regular" (in a sense that will be explained later on). Further, S is the surface of the unit sphere, $\mathbf{u} \in S$ is a unit vector, and $\boldsymbol{\nu}(\mathbf{y})$ is the outward directed normal to ∂V at $\mathbf{y}$. The scattering cross section σ_s and the total cross section σ (with $\sigma > \sigma_s$) are, for simplicity, assumed to be given positive constants within V (and zero outside).

Finally, $q(\mathbf{x}, t)$ represents the UV-photon source at any $\mathbf{x} \in V$ and $t > 0$ (and $q(\mathbf{x}, t) \equiv 0$ if $\mathbf{x} \notin V$).

In order to write the abstract version of the evolution problem (1) we introduce the Banach space $X = L^1(V \times S)$, with norm $\|f\| = \int_V d\mathbf{x} \int_S |f(\mathbf{x}, \mathbf{u})| \, d\mathbf{u}$.

Note that $\|N\|$ is the total number of UV-photons within V at time t. We also define the following operators

$$
\begin{aligned}
(Bf)(\mathbf{x}, \mathbf{u}) &= -c\,\mathbf{u} \cdot \nabla f(\mathbf{x}, \mathbf{u}) - c\sigma f(\mathbf{x}, \mathbf{u}), \qquad R(B) \subset X, \\
& \quad D(B) = \{f \colon f \in X,\ \mathbf{u} \cdot \nabla f \in X, f \text{ satisfies} \\
& \qquad \text{the boundary condition (1b)}\}, \qquad (2)
\end{aligned}
$$

$$
(Kf)(\mathbf{x}) = \frac{c\,\sigma}{4\pi} \int_S f(\mathbf{x}, \mathbf{u}') \, d\mathbf{u}', \qquad D(K) = X, \quad R(K) \subset X. \qquad (3)
$$

In Lemma 2.1, we state the properties of B and K which will be used later on.

Lemma 2.1

i) $K \in \mathcal{B}(X)$, *i.e.* K *is a bounded operator, with* $\|K\| \leq c\,\sigma_s$*;*

ii) $B \in \mathcal{G}(1, -c\,\sigma; X)$, *i.e.* B *is the generator of the strongly continuous semigroup* $\{\exp(tB), t \geq 0\}$*, such that* $\|\exp(tB)\| \leq \exp(-c\,\sigma t)$, $\forall\, t \geq 0$.

Proof. i) immediately follows from definition (3) whereas *ii)* is a standard result in particle transport theory [8, 9] □

Here, we only recall that the resolvent operator $(I - \tau B)^{-1}$ has the form

$$(Gg)(\mathbf{x},\mathbf{u}) = \big((I-\tau B)^{-1}g)\big)(\mathbf{x},\mathbf{u}) =$$
$$= \frac{1}{\tau c}\int_0^{R(\mathbf{x},\mathbf{u})} \exp\left(-\frac{1+\tau c\sigma}{\tau c} r\right) g(\mathbf{x}-r\mathbf{u},\mathbf{u})\,dr$$
$$\forall g \in X, \quad \tau > -1/c\sigma \tag{4a}$$

with

$$\|(I-\tau B)^{-1}\| \leq \frac{1}{1+\tau c\sigma}. \tag{4b}$$

In (4a), $R(\mathbf{x},\mathbf{u})$ is such that $\mathbf{y} = \mathbf{x} - R(\mathbf{x},\mathbf{u})\mathbf{u} \in \partial V$, for each given $\mathbf{x} \in V_i$ and $\mathbf{u} \in S$. In other words, for each given $\mathbf{x} \in V_i$, $\mathbf{y} = \mathbf{x} - R(\mathbf{x},\mathbf{u})\mathbf{u}$ $\forall \mathbf{u} \in S$ is the equation of the boundary surface ∂V. Such a surface is assumed to be such that $R(\mathbf{x},\mathbf{u})$ is a continuous function of $(\mathbf{x},\mathbf{u}) \in V \times S$, with $R(\mathbf{x},\mathbf{u}) = 0$ if $\mathbf{x} \in \partial V$ and $\mathbf{u}$ is directed towards V_i.

Relation (4a) implies that, $\forall \tau > -1/c\sigma$, the resolvent operator $(I - \tau B)^{-1}$ has the following properties

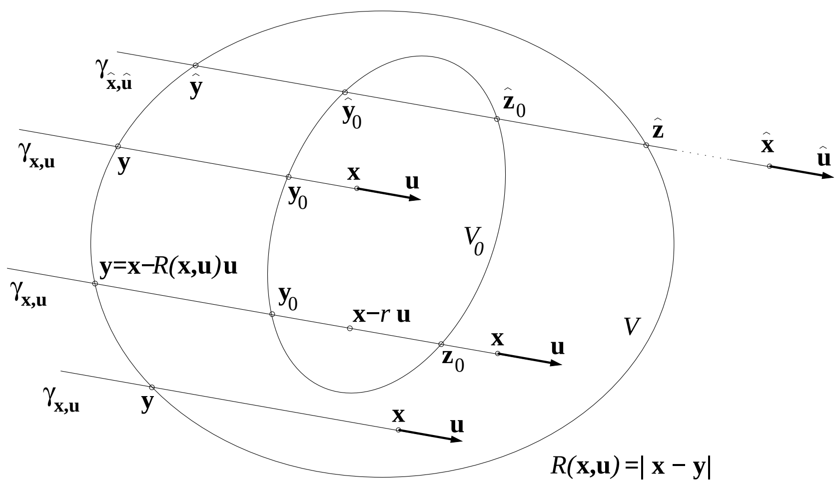

Figura 2.1 *The convex regions $V = V_i \cup \partial V$ and $V_0 = V_{0i} \cup \partial V_0$, with $V_0 \subset V_i$; the location $\widehat{\mathbf{x}}$ "far" from the cloud, with $\gamma_{\widehat{\mathbf{x}},\widehat{\mathbf{u}}} \cap V_{0i} \neq \emptyset$*

Lemma 2.2

i) $(I-\tau B)^{-1}g \in X_+ \ \forall g \in X_+$, *where* $X_+ = \{g\colon g \in X, g(\mathbf{x},\mathbf{u}) \geq 0$ *at a.a.* $(\mathbf{x},\mathbf{u}) \in V \times S\}$ *is the closed positive cone of* X*;*

ii) *if* $g \in X_+$ *and* $g > 0$ *along a finite portion of the half straight line* $\gamma_{\mathbf{x},\mathbf{u}} = \{\mathbf{y}\colon \mathbf{y} = \mathbf{x} - r\mathbf{u}, r \geq 0\}$, *see Figure 2.1, then* $\big((\alpha I - B)^{-1}g(\mathbf{x},\mathbf{u})\big) > 0 \ \forall (\mathbf{x},\mathbf{u})$.

Consider now the abstract version of system (1), [9]:

$$\begin{cases} \dfrac{d}{dt}N(t) = (B+K)N(t) + q(t), \quad t > 0 \\ N(0) = N_0 \end{cases} \tag{5}$$

where $N(t) = N(\cdot,\cdot,t)$ and $q(t) = q(\cdot,t)$ map $[0,\infty)$ into the Banach space X and N_0 is a given element of $D(B+K) = D(B)$.

The unique strict solution of the initial value problem (5) can be written as follows

$$N(t) = \exp\big(t(B+K)\big)N_0 + \int_0^t \exp\big((t-s)(B+K)\big)q(s)\,ds, \quad t \geq 0, \tag{6}$$

where $\{\exp\big(t(B+K)\big),\ t \geq 0\}$ is the semigroup generated by $(B+K)$.

Remark 2.1

i) *By using some standard results of perturbation theory, [8], we have from Lemma 2.1 that* $(B+K) \in \mathcal{G}(1, -c(\sigma - \sigma_s); X)$, *i.e.* $(B+K)$ *is the generator of the strongly continuous semigroup* $\{\exp\big(t(B+K)\big)\}$, *such that* $\|\exp\big(t(B+K)\big)\| \leq \exp\big(-c(\sigma-\sigma_s)t\big) \ \forall t \geq 0\big)$.

ii) *Relation (6) holds provided that* $N_0 \in D(B+K) = D(B)$ *and* $q(t)$ *is a continuously differentiable map from* $[0,\infty)$ *into* X. *If* $q(t)$ *is only continuous, then (6) follows from (5) but the converse is not necessary true, [8].*

□

3 The time-discretization procedure

Assume that the source term $q(\mathbf{x}, t)$ in (1a) is strictly positive if $\mathbf{x} \in V_{0i}$, where V_{0i} is the interior of a convex region $V_0 \subset V_i$, bounded by the "regular" surface ∂V_0, see Fig. 2.1.

Remark 3.1 *The region V_0 is where the stars, emitting the UV-photons, are contained.* □

Suppose also, see the Introduction, that the values $\widehat{N}_j = N(\widehat{\mathbf{x}}, \widehat{\mathbf{u}}, \widehat{t}_j)$ of the UV-photon distribution functions are measured at a location $\widehat{\mathbf{x}}$ far from the cloud (far-field measurements), with $\widehat{\mathbf{u}}$ such that $\gamma_{\widehat{\mathbf{x}},\widehat{\mathbf{u}}} \cap V_{0i} \neq 0$, see Fig. 2.1, and with $\widehat{t}_j = \widehat{t}_0 + j\tau$, $j = 0, 1, \ldots, J$. Then, we have that $\widehat{N}_j = N(\widehat{\mathbf{x}}, \widehat{\mathbf{u}}, \widehat{t}_j) = N(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}, t_j)$ where $\widehat{\mathbf{z}}$ is the "first" intersection of $\gamma_{\widehat{\mathbf{x}},\widehat{\mathbf{u}}}$ with ∂V and $t_j = \widehat{t}_j - \widehat{t}$ with $\widehat{t} = |\widehat{\mathbf{x}} - \widehat{\mathbf{z}}|/c$. In what follows, we shall choose $\widehat{t}_0 = \widehat{t}$, i.e. $t_0 = 0$ and $t_j = (\widehat{t}_0 + j\tau) - \widehat{t} = j\tau$.

Correspondingly, (6) gives

$$\widehat{N}_j = N(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}, t_j) = \Big(\exp\big(t_j(B+K)\big) N_0 \Big)(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) + \\ + \left(\int_0^{t_j} \exp\big((t_j - s)(B+K)\big) q(s) \Big) \, ds \right) (\widehat{\mathbf{z}}, \widehat{\mathbf{u}}). \tag{7}$$

However, it is not easy "to extract" some information on the space and time behaviour of the source $q(s) = q(\cdot, s)$ from (7), where the J left-hand sides $\widehat{N}_j$ ar assumed to be known, e.g. from experimental measurements.

In fact, it seems much more reasonable to discretize (5) (in a "semi-implicit" way), as follows,[10]

$$\begin{cases} \dfrac{n_{j+1} - n_j}{\tau} = B n_{j+1} + K N_j + q(t_j), & j = 0, 1, \ldots, J-1 \\ n_0 = N_0 \end{cases} \tag{8}$$

where $n_j = n_j(\mathbf{x},\mathbf{u})$ "approximates" $N(\mathbf{x},\mathbf{u},t_j) = N_j(\mathbf{x},\mathbf{u})$. We have from (8), $\forall(\mathbf{x},\mathbf{u}) \in V \times S$,

$$\begin{cases} n_{j+1}(\mathbf{x},\mathbf{u}) = (Gn_j)(\mathbf{x},\mathbf{u}) + \tau\big(G[Kn_j + q(t_j)]\big)(\mathbf{x},\mathbf{u}), & j = 0,1,\ldots,J-1 \\ n_0(\mathbf{x},\mathbf{u}) = N_0(\mathbf{x},\mathbf{u}) \end{cases} \tag{9}$$

Remark 3.2

i) As a result of the semi-implicit discretization (8), relation (9) is obtained, where the explicit form of $G = (I - \tau B)^{-1}$ *is known, see (4a).*

ii) It can be shown that $\|N_j - n_j\| \leq$ *(a positive constant)* $\cdot\, \tau\ \forall j$*, provided that* $N_0 \in D(B^2)$ *and* $q(t)$ *is regular enough, [10, 11]*

4 Identification of the source

Consider the location $\widehat{\mathbf{x}}$ "far" from the cloud (hence $\widehat{\mathbf{x}} \notin V$) and a unit vector $\widehat{\mathbf{u}}$ such that $\gamma_{\widehat{\mathbf{x}},\widehat{\mathbf{u}}} \cap V_{0i} \neq \emptyset$, see Figure 2.1. Assume that the photon distribution functions $N(\widehat{\mathbf{x}},\widehat{\mathbf{u}},\widehat{t}_0)$, $N(\widehat{\mathbf{x}},\widehat{\mathbf{u}},\widehat{t}_1)$, $\ldots$, $N(\widehat{\mathbf{x}},\widehat{\mathbf{u}},\widehat{t}_J)$ are measured. As a consequence, $N(\widehat{\mathbf{z}},\widehat{\mathbf{u}},t_0) = N(\widehat{\mathbf{x}},\widehat{\mathbf{u}},\widehat{t}_0)$, $N(\widehat{\mathbf{z}},\widehat{\mathbf{u}},t_1) = N(\widehat{\mathbf{x}},\widehat{\mathbf{u}},\widehat{t}_1)$, $\ldots$, $N(\widehat{\mathbf{z}},\widehat{\mathbf{u}},t_J) = N(\widehat{\mathbf{x}},\widehat{\mathbf{u}},\widehat{t}_J)$ are known quantities (where we recall that $t_j = \widehat{t}_j - \widehat{t} = j\tau$, with $\widehat{t} = |\widehat{\mathbf{x}} - \widehat{\mathbf{z}}|/c$ and $\widehat{t}_0 = \widehat{t}$).

This implies that $n_0(\mathbf{x},\mathbf{u})$ $(= N_0(\widehat{\mathbf{z}},\widehat{\mathbf{u}}) = N(\widehat{\mathbf{z}},\widehat{\mathbf{u}},0))$, $n_1(\widehat{\mathbf{z}},\widehat{\mathbf{u}})$, $\ldots$, $n_J(\widehat{\mathbf{z}},\widehat{\mathbf{u}})$ are also known.

We have from (4a) and (9),

$$n_{j+1}(\mathbf{x},\mathbf{u}) = (Gn_j)(\mathbf{x},\mathbf{u}) + \tau(GKn_j)(\mathbf{x},\mathbf{u})$$

$$\text{if } \mathbf{x} \in V - V_0 \text{ and } \gamma_{\mathbf{x},\mathbf{u}} \cap V_{0i} = \emptyset \tag{10a}$$

$$n_{j+1}(\mathbf{x},\mathbf{u}) = (Gn_j)(\mathbf{x},\mathbf{u}) + \tau(GKn_j)(\mathbf{x},\mathbf{u}) + (Hq(t_j))(\mathbf{x},\mathbf{u})$$

$$\text{if } \mathbf{x} \in V - V_0 \text{ and } \gamma_{\mathbf{x},\mathbf{u}} \cap V_{0i} \neq \emptyset, \text{or if } \mathbf{x} \in V_{0i}, \tag{10b}$$

where

$$(Hq(t_j))(\mathbf{x},\mathbf{u}) = \frac{1}{\tau c}\int_{|\mathbf{x}-\mathbf{z}_0|}^{|\mathbf{x}-\mathbf{y}_0|} dr \exp\left(-\frac{1+\tau c\sigma}{\tau c} r\right) q(\mathbf{x} - r\mathbf{u}, t_j)$$

$$\text{if } \mathbf{x} \in V - V_0 \text{ and } \gamma_{\mathbf{x},\mathbf{u}} \cap V_{0i} \neq \emptyset \tag{10c}$$

$$(Hq(t_j))(\mathbf{x},\mathbf{u}) = \frac{1}{\tau c}\int_{0}^{|\mathbf{x}-\mathbf{y}_0|} dr \exp\left(-\frac{1+\tau c\sigma}{\tau c} r\right) q(\mathbf{x} - r\mathbf{u}, t_j) \quad \text{if } \mathbf{x} \in V_{0i} \tag{10d}$$

see Figure 2.1.

In particular, if $\mathbf{x} = \widehat{\mathbf{z}}$, $\mathbf{u} = \widehat{\mathbf{u}}$ and $\gamma_{\widehat{\mathbf{z}},\widehat{\mathbf{u}}} \cap V_{0i} = \gamma_{\widehat{\mathbf{x}},\widehat{\mathbf{u}}} \cap V_{0i} \neq \emptyset$, see Figure 2.1, (10b) becomes

$$n_{j+1}(\widehat{\mathbf{z}},\widehat{\mathbf{u}}) = (Gn_j)(\widehat{\mathbf{z}},\widehat{\mathbf{u}}) + \tau(GKn_j)(\widehat{\mathbf{z}},\widehat{\mathbf{u}}) + (Hq(t_j))(\widehat{\mathbf{z}},\widehat{\mathbf{u}}). \tag{11}$$

Assume now that $q(\mathbf{x}, t_{j-1})$ is konwn $\forall \mathbf{x} \in V_0$; then (10a) + (10b), with $j-1$ instead of j, give $n_j((\mathbf{x},\mathbf{u}))$ at any $((\mathbf{x},\mathbf{u})) \in V \times S$. As a consequence, the first and the second term on the r.h.s. of (11) are known, whereas $n_{j+1}(\widehat{\mathbf{z}},\widehat{\mathbf{u}})$ is measured (hence, it is also known). Thus, (11) should determine the source term $q(t_j) = q(\cdot, t_j)$, see later on.

In order to understand how (11) identifies the source $q(\mathbf{x}, t_j)$, we re-write (11) as follows

$$n_{j+1}(\widehat{\mathbf{z}},\widehat{\mathbf{u}}) = \nu_j(\widehat{\mathbf{z}},\widehat{\mathbf{u}}) + Hq(t_j) \tag{12}$$

where $\nu_j(\mathbf{x}, \mathbf{u})$ is the sum of the first two terms on the r.h.s. of (10a), (10b):

$$\nu_j(\mathbf{x}, \mathbf{u}) = (Gn_j)(\mathbf{x}, \mathbf{u}) + \tau(GKn_j)(\mathbf{x}, \mathbf{u}) \tag{13a}$$

and $\widehat{H}$ is defined by

$$\widehat{H}g = (Hg)(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) = \frac{1}{\tau c} \int_{|\widehat{\mathbf{z}}-\widehat{\mathbf{z}_0}|}^{|\widehat{\mathbf{z}}-\widehat{\mathbf{y}_0}|} dr \, \exp\left(-\frac{1+\tau c\sigma}{\tau c} r\right) g(\widehat{\mathbf{z}} - r\widehat{\mathbf{u}}), \quad \forall g \in L^\infty(V_0). \tag{13b}$$

Further, we introduce a "suitable" family Φ of source functions $\varphi(\mathbf{x})$, such that

(α) $\varphi(x) > 0$, $\forall \mathbf{x} \in V_{0i}$, $\varphi(\mathbf{x}) \equiv 0$, $\forall \mathbf{x} \notin V_{0i}$;

(β) $\varphi \in L^\infty(V_0)$;

(γ) if $\varphi, \varphi_1 \in \Phi$, then either $\varphi(\mathbf{x}) > \varphi_1(\mathbf{x})$ or $\varphi(\mathbf{x}) < \varphi_1(\mathbf{x})$ $\forall \mathbf{x} \in V_{0i}$ (correspondingly, we shall write $\varphi > \varphi_1$ or $\varphi < \varphi_1$);

(δ) if φ, $\varphi_1 \in \Phi$, then $\varphi_2 = (\varphi + \varphi_1)/2 \in \Phi$;

(ε) Φ is a closed subset of the Banach space $L^\infty(V_0)$.

The family Φ, whose structure might have been suggested by astrophysicists, will be used to find approximate expressions of the J source terms $q(t_j) = q(\cdot, t_j)$, $j = 0, 1, \ldots, J-1$.

Remark 4.1 *Perhaps, the simplest way to construct Φ is the following. Choose the phisically reasonable "minimal" and "maximal" sources φ_m and φ_M, satisfing (α), (β) and such that $\varphi_m(\mathbf{x}) < \varphi_M(\mathbf{x})$ $\forall \mathbf{x} \in V_{0i}$.*
Then, $\Phi = \Phi_{[0,1]} = \{\varphi_h \colon \varphi_h = (1-h)\varphi_m + h\varphi_M,\ h \in [0,1]\}$. □

Remark 4.2 *If $\varphi_h \in \Phi_{[0,1]}$, we obtain from (13b) that $\widehat{H}\phi_h = (1-h)\widehat{H}\varphi_m + h\widehat{H}\varphi_M$. Then, (12) leads to the value $\widehat{h} \in [0,1]$ such that $n_{j+1}(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) = \nu_j(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) + (1-\widehat{h})\widehat{H}\varphi_m + \widehat{h}\widehat{H}\varphi_M$, where n_{j+1} is measured and ν_j is known. Correspondingly, the approximated value of the same is given by $q(t_j) = q(\cdot, t_j) = (1-\widehat{h})\varphi_m(\cdot) + \widehat{h}\varphi_M(\cdot)$.* □

Going back to definition (13b) and considering a "general" family Φ, it immediately follows that

$$\widehat{H}\varphi < \widehat{H}\varphi_1, \qquad \forall \varphi, \varphi_1 \text{ with } \varphi < \varphi_1. \tag{14}$$

We remark that (14) and the procedure that follows may still hold also if $\widehat{H}$ is *nonlinear* (and, of course, it satisfies suitable assumptions).

As a first step, consider (12) with $j = 0$:

$$n_1(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) = \nu_0(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) + \widehat{H}q(t_0). \tag{15}$$

Since the value $n_1(\widehat{\mathbf{z}}, \widehat{\mathbf{u}})$ of the photon distribution function at time t_1 is known as a result of some experimental procedure and $\nu_0(\widehat{\mathbf{z}}, \widehat{\mathbf{u}})$ is defined by (13a) with $j = 0$ and with n_0 given, assume that $\varphi_{1-} \in \Phi$ and $\varphi_{1+} \in \Phi$ exist, such that

$$n_1(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) > n_{1-} = \nu_0(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) + \widehat{H}\varphi_{1-}, \tag{16a}$$

$$n_1(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) < n_{1+} = \nu_0(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) + \widehat{H}\varphi_{1+}. \tag{16b}$$

Note that, if the family Φ is suitably chosen, it should be possible to find φ_{1-} and φ_{1+} such that (16a) and (16b) are satisfied. (Otherwise, if for instance $\Phi = \Phi_{[0,1]}$ of Remark 4.1, one might choose a "smaller" φ_m and a "larger" φ_M.)

Further, consider $\dfrac{\varphi_{1-} + \varphi_{1+}}{2} \in \Phi$ and assume, for instance, that

$$n_1(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) < \nu_0(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) + \widehat{H}\left(\frac{\varphi_{1-} + \varphi_{1+}}{2}\right), \quad \text{i.e.} \quad n_1(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) < \frac{n_{1-} + n_{1+}}{2}$$

because of the linearity of $\widehat{H}$. Then, if we put $\varphi_{2-} = \varphi_{1-}$, $\varphi_{2+} = \dfrac{\varphi_{1-} + \varphi_{1+}}{2}$, we have

$$n_1(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) > n_{2-} = \nu_0(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) + \widehat{H}\varphi_{2-} = n_{1-}$$

$$n_1(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) < n_{2+} = \nu_0(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) + \widehat{H}\varphi_{2+} = \frac{n_{1-} + n_{1+}}{2}$$

and also

$$\varphi_{1-} = \varphi_{2-} < \varphi_{2+} < \varphi_{1+}, \qquad n_{1-} = n_{2-} < n_1(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) < n_{2+} < n_{1+}$$

By iterating the above procedure (for which *only* $\widehat{H}\varphi_{1-}$ and $\widehat{H}\varphi_{1+}$ need to be evaluated), we find the four monotone sequences

$$\{\varphi_{j-}\} \subset \Phi \subset L^\infty(V_0), \qquad \{\varphi_{j+}\} \subset \Phi \subset L^\infty(V_0),$$

$$\{n_{j-}\} \subset \mathbb{R}_+, \qquad \{n_{j+}\} \subset \mathbb{R}_+.$$

It is not difficult to show, see for instance [12], that φ_{j-} and φ_{j+} are Cauchy sequences in $L^\infty(V_0)$ whereas n_{j-} and n_{j+} are Cauchy sequences in $\mathbb{R}$. Correspondingly, we have

$$\lim_{j\to\infty} \varphi_{j-} = \lim_{j\to\infty} \varphi_{j+} = \varphi_{\infty,1} \in \Phi \quad \text{(because } \Phi \text{ is a closed subset of } L^\infty(V_0)\text{)},$$

$$\lim_{j\to\infty} n_{j-} = \lim_{j\to\infty} n_{j+} = n_1(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}),$$

$$n_1(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) = \nu_0(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) + \widehat{H}\varphi_{\infty,1} \tag{17}$$

Remark 4.3 *According to (17), $\varphi_{\infty,1} \in \Phi$ is the "best approximation" within the family Φ to the "phisical" source $q(t_0) = q(\cdot, t_0)$ appearing in (15). Then, going back to (9) with $j = 0$ and putting*

$$\tilde{n}_1(\mathbf{x}, \mathbf{u}) = (Gn_0)(\mathbf{x}, \mathbf{u}) + \tau(GKn_0)(\mathbf{x}, \mathbf{u}) + \tau(G\varphi_{\infty,1})(\mathbf{x}, \mathbf{u}), \quad (\mathbf{x}, \mathbf{u}) \in V \times S \tag{18}$$

we conclude that $\tilde{n}_1(\mathbf{x},\mathbf{u})$ *should be a reasonable approximation to* $n_1(\mathbf{x},\mathbf{u})$ *at any* $(\mathbf{x},\mathbf{u}) \in V \times S$ *(and, of course,* $\tilde{n}_1(\widehat{\mathbf{z}},\widehat{\mathbf{u}}) = n_1(\widehat{\mathbf{z}},\widehat{\mathbf{u}})$ *because of (17)).* □

As a second step, we consider (12) with $j = 1$:

$$n_2(\widehat{\mathbf{z}},\widehat{\mathbf{u}}) = \nu_1(\widehat{\mathbf{z}},\widehat{\mathbf{u}}) + \widehat{H}q(t_1), \tag{19}$$

where $n_2(\widehat{\mathbf{z}},\widehat{\mathbf{u}})$ is known because it is the value of the photon distribution function, measured at time t_2. However, since $\nu_1(\widehat{\mathbf{z}},\widehat{\mathbf{u}})$ is given by (13a) with $j = 1$, such a quantity can be evaluated if we know $n_1(\mathbf{x},\mathbf{u})$ and not only the single value $n_1(\widehat{\mathbf{z}},\widehat{\mathbf{u}})$. On the other hand, (18) gives $\tilde{n}_1(\mathbf{x},\mathbf{u})$ which approximates $n_1(\mathbf{x},\mathbf{u})$. As a consequence, we can evaluate $\tilde{\nu}_1(\mathbf{x},\mathbf{u})$, defined by (13a) with $j = 1$, $\mathbf{x} = \widehat{\mathbf{z}}$, $\mathbf{u} = \widehat{\mathbf{u}}$, and with $\tilde{n}_1(\mathbf{x},\mathbf{u})$ instead of $n_1(\mathbf{x},\mathbf{u})$. Then, the procedure to identify the source $q(t_1)$ leads to the element $\tilde{\varphi}_{\infty,2} \in \Phi$ such that

$$n_2(\widehat{\mathbf{z}},\widehat{\mathbf{u}}) = \tilde{\nu}_1(\widehat{\mathbf{z}},\widehat{\mathbf{u}}) + \widehat{H}\tilde{\varphi}_{\infty,2} \tag{20}$$

rather than to the element $\varphi_{\infty,2}$ such that

$$n_2(\widehat{\mathbf{z}},\widehat{\mathbf{u}}) = \nu_1(\widehat{\mathbf{z}},\widehat{\mathbf{u}}) + \widehat{H}\varphi_{\infty,2}. \tag{21}$$

However, since $\tilde{\nu}_1(\widehat{\mathbf{z}},\widehat{\mathbf{u}})$ should be a good approximation to $\nu_1(\widehat{\mathbf{z}},\widehat{\mathbf{u}})$, $\tilde{\varphi}_{\infty,2}$ is likely to be "close" to $\varphi_{\infty,2}$, see Section 5.

Further, by using $\tilde{n}_1(\mathbf{x},\mathbf{u})$ and $\varphi_{\infty,2}(\mathbf{x})$, from (9) with $j = 1$ we have that

$$\tilde{n}_2(\mathbf{x},\mathbf{u}) = (G\tilde{n}_1)(\mathbf{x},\mathbf{u}) + \tau(GK\tilde{n}_1)(\mathbf{x},\mathbf{u}) + \tau(G\tilde{\varphi}_{\infty,2})(\mathbf{x},\mathbf{u}) \tag{22}$$

should be a reasonable approximation to $n_2(\mathbf{x},\mathbf{u})$ $\forall(\mathbf{x},\mathbf{u}) \in V \times S$. The final result of the above procedure is the set $\{\varphi_{\infty,1}(\mathbf{x}),\ \tilde{\varphi}_{\infty,2}(\mathbf{x}),\ \ldots,\ \tilde{\varphi}_{\infty,J}(\mathbf{x})\}$ that is in some sense, the best approximation within the family Φ to the set of the "physical" source terms $\{q(\mathbf{x},t_0),\ q(\mathbf{x},t_1),\ \ldots\ q(\mathbf{x},t_{J-1})\}$.

5 Concluding remarks

1. If the family Φ is particularly well chosen (or it is "large enough"), then the set $\{q(\mathbf{x}, t_0),\ \dots\ q(\mathbf{x}, t_{J-1})\}$ is contained in Φ. Corresponingly, $\varphi_{\infty,1}(\mathbf{x}) = q(\mathbf{x}, t_0)$, $\varphi_{\infty,2}(\mathbf{x}) = \tilde{\varphi}_{\infty,2}(\mathbf{x}) = q(\mathbf{x}, t_1), \dots\ , \varphi_{\infty,J}(\mathbf{x}) = \tilde{\varphi}_{\infty,J}(\mathbf{x}) = q(\mathbf{x}, t_{J-1})$, due to the uniqueness of our limit procedure within Φ.

   Thus, in such a lucky case, we are able to identidy *exactly* the source term.

   In particular, assume that q depends on t but not on $\mathbf{x} \in V_0$. Then, we can take $\Phi = \{\varphi \colon q_m \leq \varphi(\mathbf{x}) = \text{ a constant } \leq q_M\} \subset \mathbb{R}_+$ and, if q_m and q_M are suitably chosen, the set $\{q(t_0),\ q(t_1),\ \dots,\ q(t_{J-1})\}$ is contained in Φ.

2. Assume now that a family Ψ is also considered, with $\Psi \cap \Phi = \emptyset$ and such that (α)-(ε) of Section 4 are satisfied. Then, the procedures of Section 4 lead to the set $\{\psi_{\infty,1}(\mathbf{x}),\ \tilde{\psi}_{\infty,2}(\mathbf{x}),\ \dots,\ \tilde{\psi}_{\infty,J}(\mathbf{x})\}$ as the best approximation within ψ to the physical source terms $\{q(\mathbf{x}, t_0),\ \dots,\ q(\mathbf{x}, t_{J-1})\}$. This kind of non-uniqueness is obiouvsly due to the possibility of choosing among several different families Φ, Ψ, ... Of course, the most reasonable choice should be suggested by experimental evidence, e.g. by a partial knowledge of the position of the stars which emit UV-photons inside the cloud.

3. As far as the errors involved in the procedures of Section 4, assume that

   i) the experimental values $n_{j+1}(\widehat{\mathbf{z}}, \widehat{\mathbf{u}})$ are exact, i.e. they are measured with a very small experimental error, see (12);

   ii) the "true" photon distribution function $n_{j+1}(\mathbf{x}, \mathbf{u})$ is known at any $(\mathbf{x}, \mathbf{u}) \in V \times S$, so that the value $\nu(\widehat{\mathbf{z}}, \widehat{\mathbf{u}})$ is exact.

If i) and ii) are satisfied, relation (12) is also exact; *assume* that consequently, (12) leads to an approximate source $\varphi_{\infty,j}(\mathbf{x},\mathbf{u})$ such that

iii) $\|q(t_j)-\varphi_{\infty,j}\|_\infty < \varepsilon,\ \ j=0,1,\dots J-1$ where $\|\cdot\|_\infty$ is the norm in $L^\infty(V)$ (we recall that $q(\mathbf{x},t_j)\equiv 0$ and $\varphi_{\infty,j}(\mathbf{x})\equiv 0$ if $\mathbf{x}\notin V_{0i}$). Note that iii) should be satisfied if the family Φ is suitably chosen.

Consider now the first step ($j=0$) of Section 4; starting from (15) (with $n_1(\widehat{\mathbf{z}},\widehat{\mathbf{u}})$ and $\nu_0(\widehat{\mathbf{z}},\widehat{\mathbf{u}})$ both exact), and taking into account assumption iii), we identify $\varphi_{\infty,1}\in\Phi$ such that $n_1(\widehat{\mathbf{z}},\widehat{\mathbf{u}})=\nu_0(\widehat{\mathbf{z}},\widehat{\mathbf{u}})+\widehat{H}\varphi_{\infty,1}$, with

$$\|q(t_0)-\varphi_{\infty,1}\|<\varepsilon \tag{23}$$

Consequently, (9) with $j=1$ and (18) give

$$\begin{aligned}|\tilde{n}_1(\mathbf{x},\mathbf{u})-n_1(\mathbf{x},\mathbf{u})| &\le \tau\|G\|_\infty\|\varphi_{\infty,1}\|_\infty \le \frac{\tau\varepsilon}{1+c\sigma\tau}\\ \|\tilde{n}_1-n_1\|_\infty &\le \frac{\tau\varepsilon}{1+c\sigma\tau}\end{aligned} \tag{24}$$

because definition (4a) also implies that $\|G\|_\infty\le\dfrac{1}{1+c\sigma\tau}$. Then, from (13a) with $j=1$, we obtain that

$$\begin{aligned}|\tilde{\nu}_1(\mathbf{x},\mathbf{u})-\nu_1(\mathbf{x},\mathbf{u})| &\le |(G(\tilde{n}_1-n_1)(\mathbf{x},\mathbf{u})|+\tau|(GK(\tilde{n}_1-n_1)(\mathbf{x},\mathbf{u})| \le\\ &\le \left(\frac{1}{1+c\sigma\tau}+\frac{\tau c\sigma_s}{1+c\sigma\tau}\right)\|\tilde{n}_1-n_1\|_\infty \le \frac{\tau\varepsilon}{1+c\sigma\tau}\end{aligned} \tag{25}$$

where we recall that $\tilde{\nu}_j$ is defined by (13a) with $\tilde{n}_j$ instead of n_j and where we used (24).

Consider then the second step ($j=2$) of Section 4; if we knew the exact $\nu_1(\widehat{\mathbf{z}},\widehat{\mathbf{u}})$, (19) would lead to the element $\varphi_{\infty,2}\in\Phi$ such that $n_2(\widehat{\mathbf{z}},\widehat{\mathbf{u}}) = \nu_1(\widehat{\mathbf{z}},\widehat{\mathbf{u}})+\widehat{H}\varphi_{\infty,2}$, with $\|q(t_1)-\varphi_{\infty,2}\|_\infty<\varepsilon$.

However, we only know the approximate value $\tilde{\nu}_1(\widehat{\mathbf{z}}, \widehat{\mathbf{u}})$ and so we obtain the element $\tilde{\varphi}_{\infty,2} \in \Phi$ such that $n_2(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) = \tilde{\nu}_1(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) + \widehat{H}\tilde{\varphi}_{\infty,2}$. Thus we have

$$\widehat{H}(\tilde{\varphi}_{\infty,2} - \varphi_{\infty,2}) = \nu_1(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}) - \tilde{\nu}_1(\widehat{\mathbf{z}}, \widehat{\mathbf{u}}). \tag{26}$$

For simplicity, we shall now assume that the family Φ is defined as in Remark 4.1.

iv) $\Phi = \Phi_{[0,1]} = \{\varphi_h \colon \varphi_h = (1-h)\varphi_m + h\varphi_M, h \in [0,1]\}$.

Then we have

$$\tilde{\varphi}_{\infty,2} = (1-\tilde{h})\varphi_m + \tilde{h}\varphi_M, \qquad \varphi_{\infty,2} = (1-h)\varphi_m + h\varphi_M,$$

$$\widehat{H}(\tilde{\varphi}_{\infty,2} - \varphi_{\infty,2}) = (\tilde{h} - h)\widehat{H}(\varphi_M - \varphi_m).$$

Note that, since both $\tilde{\varphi}_{\infty,2}$ and $\varphi_{\infty,2}$ belong to Φ, either $\tilde{\varphi}_{\infty,2} > \varphi_{\infty,2}$ or $\tilde{\varphi}_{\infty,2} > \varphi_{\infty,2}$. Suppose, for instance, that $\tilde{\varphi}_{\infty,2} > \varphi_{\infty,2}$, i.e. $\tilde{h} > h$.

Then, (25) and (26) give

$$(\tilde{h} - h)\widehat{H}(\varphi_M - \varphi_m) \leq \frac{\tau\varepsilon}{1 + c\sigma\tau}, \qquad \tilde{h} - h \leq \frac{\tau\varepsilon}{1 + c\sigma\tau} \, \frac{1}{\widehat{H}(\varphi_M - \varphi_m)},$$

$$\|\tilde{\varphi}_{\infty,2} - \varphi_{\infty,2}\| \leq \varepsilon\eta, \quad \text{where} \quad \eta = \frac{\tau}{1 + c\sigma\tau} \, \frac{\|\varphi_M - \varphi_m\|_\infty}{\widehat{H}(\varphi_M - \varphi_m)}.$$

It follows that

$$\|q(t_1) - \tilde{\varphi}_{\infty,2}\|_\infty \leq \|q(t_1) - \varphi_{\infty,2}\|_\infty + \|\varphi_M - \varphi_m\|_\infty \leq \varepsilon(1 + \eta). \tag{27}$$

Iterations of the above procedure leads to the inequality

$$\|q(t_j) - \tilde{\varphi}_{\infty,j+1}\|_\infty \leq \varepsilon(1 + \eta)^j, \qquad j = 0, 1, \ldots, J-1,$$

that gives the error with which the set $\{\varphi_{\infty,1},\ \tilde{\varphi}_{\infty,2},\ \ldots,\ \tilde{\varphi}_{\infty,J}\} \subset \Phi$ approximates the "physical" set of sources $\{q(t_0),\ q(t_1),\ \ldots,\ q(t_{J-1})\}$.

4. The algorithm presented in this paper has been recently implemented by S. Pieraccini et al in [13].

5. Assume, for instance, that the two values $N(\widehat{\mathbf{x}}, \widehat{\mathbf{u}}, t_j)$ and $N(\widehat{\mathbf{x}}, \widehat{\mathbf{u}'}, t_j)$ of the photon distribution function can be measured at $\widehat{\mathbf{x}}$, at each t_j, and corresponding to the two directions $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{u}'}$. This is possible if the interstellar cloud under consideration is seen from the satellite containing the recording instrument under a solid angle which is not "too small".

   Then, a family Φ_1 of source functions ϕ may be chosen as a two-parameter family: $\Phi_1 = \{\varphi\colon \varphi = \varphi_{h,k}, h \in [0,1], k \in [0,1]\}$, where for instance $\varphi_{h,k} = h\varphi_1 + k(1-h)\varphi_2 + (1-k)(1-h)\varphi_3$. Hence, Φ_1 allows a larger choice than the family Φ defined in Remark 4.1.

6. If the measured $N(\widehat{\mathbf{x}}, \widehat{\mathbf{u}}, t)$ is a continuous function of t, then it is not difficult to prove that $\|\widetilde{\varphi}_{\infty,t_{j-1}} - \widetilde{\varphi}_{\infty,t_j}\|_\infty = 0$ if $t_j \to t_{j-1}$, with t_{j-1} given. However, some difficulties arise if we let $J \to \infty$ (i.e. $\tau \to 0_+$) because it can be shown that the corresponding approximated source $\widetilde{\varphi}_\infty(\mathbf{x}, t)$ is such that only the total number of photons arriving at $\mathbf{x}$ during some time interval $[0,\ t^*]$ can be evaluated. In other words, $\widetilde{\varphi}_\infty$ is such that

$$\int_0^{t^*} n(\widehat{\mathbf{x}}, \widehat{\mathbf{u}}, t)\, dt = \int_0^{t^*} N(\widehat{\mathbf{x}}, \widehat{\mathbf{u}}, t)\, dt.$$

   A further paper will be devoted to study this problem.

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